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DIFFUSION AND LAPLACIAN TRANSPORT FOR ABSORBING DOMAINS

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We study (stationary) Laplacian transport by the Dirichlet-to-Neumann formalism. Our results concern a *formal* solution of the *geometrically* inverse problem for localisation and reconstruction of the form of absorbing domains. Here, we restrict our analysis to the one- and two-dimensional cases. We show that the last case can be studied by the conformal mapping technique. To illustrate this, we scrutinize the constant boundary conditions and analyze a numeric example.

Keywords: Laplacian transport, Dirichlet-to-Neumann operators, Conformal mapping.

1. Introduction

1. Is is known (see e.g. [8]) that the problem of determining a *conductivity matrix* field $\gamma(p) = [\gamma_{i,j}(p)]_{i,j=1}^{d}$, for p in a bounded open domain $\Omega \subset \mathbb{R}^d$, is related to "measuring" the elliptic *Dirichlet-to-Neumann* map for the associated conductivity equation. Notice that the solution to this problem has numerous practical applications in various domains: geophysics, electrochemistry etc. It is also an important diagnostic tool in medicine, e.g. in the *electrical impedance tomography*; the tissue in the human body is an example of highly anisotropic conductor [1].

Assuming there are no current sources or sinks, the potential $v(p),\ p \in \Omega$, for a given voltage $f(\omega),\ \omega \in \partial\Omega$, on the (smooth) boundary $\partial\Omega$ of Ω is a solution of the Dirichlet problem:

$$(\mathbf{P1}) \qquad \begin{cases} \mathrm{div}(\gamma\nabla v) = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases}$$

Then, the corresponding (**P1**) Dirichlet-to-Neumann map (operator) $\Lambda_{\gamma,\partial\Omega}$ is (*formally*) defined by [16]

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f/\partial\nu_\gamma := \nu\cdot\gamma\ \nabla v_f\ |_{\partial\Omega}\ . \tag{1.1}$$

Here, ν is the unit *outer-normal* vector to the boundary at $\omega \in \partial\Omega$ and the function $v := v_f$ is a solution of the Dirichlet problem (**P1**).

The Dirichlet-to-Neumann operator (1.1) is also called the *voltage-to-current* map, since the function $\Lambda_{\gamma,\partial\Omega} f$ gives the induced current flux trough the boundary $\partial\Omega$. The key (*inverse*) problem is whether one can determine the conductivity matrix γ by knowing the electrical boundary measurements, i.e. the corresponding Dirichlet-to-Neumann operator. In general, this operator does not determine the matrix γ uniquely, see e.g. [4].

The main question in this context is to find sufficient conditions insuring that the inverse problem is uniquely soluble.

2. The problem of electrical current flux in the form (**P1**) is an example of so-called *diffusive* Laplacian transport [17]. Besides the voltage-to-current problem, the motivation to study of this kind of transport comes for instance from the transfer across *biological membranes*, see e.g. [13], [3].

Let some “species” of concentration $C(p)$, $x \in \mathbb{R}^d$, diffuse stationary in the *isotropic* bulk ($\gamma = I$) from a (distant) source localised on the closed boundary $\partial\Omega$ towards a *semipermeable* compact interface ∂B of the *cell* $B \subset \Omega$, where they disappear at a given rate $W \geq 0$. Then, the *steady* field of concentrations (Laplacian transport with a diffusion coefficient $D \geq 0$) obeys the set of equations:

$$(\mathbf{P2})^* \quad \begin{cases} \Delta C = 0, \ p \in \Omega \setminus \overline{B} \ , \\ C \mid_{\partial\Omega} (p) = C_0, \text{ a constant concentration at the source } \partial\Omega \ , \\ -D \ \partial_\nu C \mid_{\partial B} (\omega) = W \ (C - C^*) \mid_{\partial B} (\omega), \text{ on the interface } \omega \in \partial B \end{cases}$$

Usually, one assumes that $C(p) = C^* \geq 0$, $p \in \overline{B}$, is a constant concentration of the ”species” inside the cell $\overline{B}$.

This example motivates the following abstract *stationary* diffusive Laplacian transport problem with *absorption* on the surface ∂B:

$$(\mathbf{P2}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , (u(p) = Const \, , \ p \in \overline{B}), \\ u \mid_{\partial\Omega} (p) = f(p), \quad p \in \partial\Omega \ , \\ (\alpha \ u + \partial_\nu u) \mid_{\partial B} (\omega) = h(\omega), \quad \omega \in \partial B \ . \end{cases}$$

This is the Dirichlet problem for the domain $\Omega \supset B$ with the Dirichlet-Neumann (or Robin [6]) boundary conditions on the absorbing surface ∂B. Varying α between $\alpha = 0$ and $\alpha = +\infty$ one recovers respectively the Neumann and the Dirichlet boundary conditions.

Now, similar to (1.1), we can associate with the problem **(P2)** a Dirichlet-to-Neumann operator

$$\Lambda_{\gamma=I,\partial\Omega} : f \mapsto \partial_\nu u_f \mid_{\partial\Omega} =: g \ . \tag{1.2}$$

Domain $\mathrm{dom}(\Lambda_{I,\partial\Omega})$ belongs to a certain *Sobolev* space of functions on the boundary $\partial\Omega$, which contains $u_f := U_f^{(\alpha,h)}$, the solutions of the problem **(P2)** for given f and for the Robin boundary condition on ∂B fixed by α and h.

Then, there are at least two (in fact related) *geometrical* inverse problems that are of interest:

(a) Given the Dirichlet data f and the corresponding (measured) Neumann data g (1.2) on the accessible *outer* boundary $\partial\Omega$, to reconstruct the shape of the interior

boundary ∂B.
(b) A simpler inverse problem concerns the localisation of the domain (cell) B of a *given shape* and fixed parameters α and h.

3. The aim of the present paper is to study the above problems (a) and (b) in the framework of application outlined in the problem **(P2)*** and to work out the corresponding formalism based on the Dirichlet-to-Neumann operators.

In Section 2.1 we formulate the mathematical setup of the these problems, and we consider uniqueness of the forward boundary value problem **(P2)** solution. There, we illustrate our strategy by an explicit example of one-dimensional inverse problem for $\Omega \subset \mathbb{R}^1$ and $B = (a, b)$.

Our main results (Section 3) concern the two-dimensional case, when the compact $\Omega \subset \mathbb{R}^2$. Notice that there are three points that need particular attention. The first is that the problems **(P2)*** and **(P2)** are formulated for *non-simply* connected domains $\Omega \setminus \overline{B}$. The second point concerns the peculiarity of the combination of Dirichlet and Robin boundary conditions. As a third point, one has to mention that the *geometrically* inverse problem is *poorly formulated*.

The present paper first presents the *formal* solution for the case when $\alpha = +\infty$, i.e. the Dirichlet boundary conditions $u \mid_{\partial B} (\omega) = 0$, $\omega \in \partial B$. For this case, our approach is motivated by important papers [9], [12]. Here we refine their results in the framework of the Dirichlet-to-Neumann formalism and add certain observations in the case of a fixed geometry of domains B and Ω following [2].

In Section 4 we consider an explicit example and give numerical calculation for constant external boundary conditions $f = 1$ to illustrate abstract results for $\alpha = +\infty$.

For finite $\alpha \geq 0$ and $h = 0$ we restrict the discussion to a few remarks, (Section 5) as a more thorough investigation will be presented in future publications. The same concerns our formal scheme for $d = 2$, since the corresponding inverse problem is *ill-posed*.

The case $d = 1$ allows explicit calculations and serves to illustrate of our main ideas, whereas, for solution of the inverse Problems, i.e. for $d = 2$, we use a method of conformal mappings for harmonic functions in doubly connected domains $\Omega \setminus B$.

2. Setup of the Problems and Uniqueness

1. Below, we suppose that Ω and $B \subset \Omega$ be open bounded domains in $\mathbb{R}^d$ with C^2-smooth disjoint boundaries $\partial\Omega$ and ∂B, that is $\partial(\Omega \setminus \overline{B}) = \partial\Omega \cup \partial B$ and $\partial\Omega \cap \partial B = \emptyset$.

Then, the unit *outer-normal* to the boundary $\partial(\Omega\setminus\overline{B})$ vector-field $\nu(p)_{p\in\partial(\Omega\setminus B)}$ is well-defined, and we consider the normal derivative in **(P2)** as the *interior* limit:

$$(\partial_\nu u) \mid_{\partial B} (\omega) := \lim_{p\to\omega} \nu(\omega) \cdot (\nabla u)(p) \ , \ p \in \Omega \setminus \overline{B} \ . \tag{2.1}$$

The existence of the limit (2.1) as well as the restriction $u\mid_{\partial B}(\omega) := \lim_{p\to\omega} u(p)$ is insured since u has to be a harmonic solution of the problem **(P2)** for C^2-smooth boundaries $\partial(\Omega \setminus \overline{B})$, [15].

Now, we introduce some indispensable standard notations and definitions [5]. Let $\mathcal{H}$ be Hilbert space $L^2(M)$ on domain $M \subset \mathbb{R}^d$ and $\partial\mathcal{H} := L^2(\partial M)$ denote the corresponding *boundary space*. By $W_2^s(M)$, we denote the Sobolev space of $L^2(M)$-functions, whose s-derivatives are also in $L^2(M)$, and similar, $W_2^s(\partial M)$ is the Sobolev space of $L^2(\partial M)$-functions on the C^2-smooth boundary ∂M.

Proposition 2.1. *Let $f, h \in W_2^{1/2}(\partial\Omega)$ for C^2-smooth boundaries $\partial(\Omega \setminus B)$. If $\alpha \geq 0$, then the Dirichlet-Robin problem* **(P2)** *has a unique (harmonic) solution in domain $\Omega \setminus \overline{B}$.*

Proof. For existence, we refer to [15]. To prove the uniqueness, we consider the problem **(P2)** for $f = 0$ and $h = 0$. Then, by the Gauss-Ostrogradsky theorem, the corresponding solution u yields:

$$\int_{\Omega\setminus\overline{B}} dp\ (\nabla\overline{u(p)} \cdot \nabla u)(p)) = \int_{\Omega\setminus\overline{B}} dp\ \operatorname{div}(\overline{u(p)}\ (\nabla u)(p)) =$$
$$\int_{\partial B} d\sigma(\omega)\ \overline{u(\omega)}\ (\partial_\nu u)(\omega) = -\alpha \int_{\partial B} d\sigma(\omega)\ |u(\omega)|^2 \leq 0\ . \tag{2.2}$$

The estimate (2.2) implies that $u(x \in \Omega \setminus \overline{B}) = Const.$ Hence, by the Robin boundary conditions, $(\alpha\, u)\mid_{\partial B}(\omega) = 0$, and by virtue of $u\mid_{\partial\Omega}(p) = f(x \in \partial\Omega) = 0$, we obtain that for $\alpha \geq 0$ the harmonic function $u(p) = 0$ for $x \in \Omega \setminus \overline{B}$. □

The next statement is key for the analysis of inverse geometrical problems (a) and (b). Since we use it below in the case $\mathbb{R}^2$, our formulation is two-dimensional.

Proposition 2.2. *Consider two problems* **(P2)** *corresponding to a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary $\partial\Omega$ and to two subsets B_1 and B_2 with the same smoothness of the boundaries ∂B_1, ∂B_2. If for solutions $u_{f,h}^{(1)}$, $u_{f,h}^{(2)}$ of these problems one has*

$$\partial_\nu u_{f,h}^{(1)}\mid_{\partial\Omega} = \partial_\nu u_{f,h}^{(2)}\mid_{\partial\Omega}\ , \tag{2.3}$$

then $\partial B_1 = \partial B_2$.

Proof. By virtue of $u_{f,h}^{(1)}\mid_{\partial\Omega} = u_{f,h}^{(2)}\mid_{\partial\Omega} = f$ and by condition (2.3), the problem **(P2)** has two solutions for identical external (on $\partial\Omega$) and internal (on ∂B_1 and ∂B_2) Robin boundary conditions. Then, by the standard arguments based on the Holmgren uniqueness theorem [14] for harmonic functions on $\mathbb{R}^2$, one obtains that $\partial B_1 = \partial B_2$. □

2. We finish this section by a simple illustration of the explicit solution of the Inverse Problems (a) and (b) in the one-dimensional case. Motivated by the Laplace transport **(P2)*** we consider the case: $f = c_0$, $h = \alpha\, c^*$, and $\alpha = W/D \geq 0$, for

$\Omega := (-R, R) \subset \mathbb{R}^1$ and $B := (a, b)$:

$$(\mathbf{P_{d=1}}) \qquad \begin{cases} \Delta u = 0,\ x \in (-R,R) \setminus [a,b]\ , \\ u\,|_{\partial\Omega}\ (x = \mp R) = f(\mp R) =: c_{\mp}, \\ (\alpha\ u + \partial_\nu u)\ |_{\partial[a,b]}\ (a) = (\alpha\ u + \partial_\nu u)\ |_{\partial[a,b]}\ (b) = \alpha\, c^*\ , \end{cases}$$

where $R > 0$ and $-R < a < b < R$.

The solution of the problem (a) is straightforward, since in the one-dimensional case, the shape of absorbing cell is trivial: it is the interval $B := (a, b)$.

Now notice that a general solution of the problem $(\mathbf{P_{d=1}})$ is a combination of linear functions supported in domain $\Omega := (-R, R) \setminus [a, b]$ and a constant c^* in the interval $[a, b]$:

$$-R < x < a: \qquad u(x) = -\frac{c_- - c^*}{(R+a) + \alpha^{-1}}(R + x) + c_-\ , \tag{2.4}$$

$$a \le x \le b: \qquad u(x) = c^*\ ,$$

$$b < x < R: \qquad u(x) = -\frac{c_+ - c^*}{(R-b) + \alpha^{-1}}(R - x) + c_+\ . \tag{2.5}$$

Given Dirichlet data c_0 on the boundary $\partial\Omega$ and *measuring* on this boundary the Neumann data in the form of the flux currents:

$$j_- := -\partial_\nu u\ |_{\partial\Omega}\ (x = -R) = \frac{c_- - c^*}{(R+a) + \alpha^{-1}}$$

$$j_+ := -\partial_\nu u\ |_{\partial\Omega}\ (x = +R) = -\frac{c_+ - c^*}{(R-b) + \alpha^{-1}},$$

one can explicitly solve both problems (a) and (b).

In the one-dimensional case the *shape* of the cell is defined by its *size*: $(b - a)$, whereas localization is fixed by the points:

$$a = (c_- - c^*)/j_- - R - \alpha^{-1}\ ,$$

$$b = (c_+ - c^*)/j_+ + R + \alpha^{-1}\ .$$

3. Two-Dimensional Inverse Problem: Conformal Mapping and the Shape of ∂B

1. The relevance of the conformal mapping in the study of the boundary value problems for harmonic functions (solutions of the Laplace equation) is well-known, see e.g. [7] (Ch.III), or [11] (Ch.13).

Recall that if the complex function $w : z \mapsto \mathbb{C}$ is holomorphic in the open domain $\{\Omega \subset \mathbb{C} : z = x + i\,y \in \Omega\}$, then by the Cauchy-Riemann conditions the functions $u(x, y) := (\mathrm{Re}\, w)(x, y)$ and $v(x, y) := (\mathrm{Im}\, w)(x, y)$ are harmonic in Ω. Here, $w(z) = u(x, y) + i\, v(x, y)$.

Remark 3.1. *There is an elementary inverse problem of the complex analysis : given a harmonic function $u(x, y)$ in Ω to construct in this domain the harmonic function $v(x, y)$ (harmonic conjugate to u) such that the complex function $w =$*

$u + i\,v$ is holomorphic. In fact, one finds the harmonic conjugate from the Cauchy-Riemann conditions,

$$\partial_x u = \partial_y v \quad , \quad \partial_y u = -\partial_x v \ , \tag{3.1}$$

since for a given u this is a system of partial differential equations for v. Notice that for a simply connected domain Ω, the solution of this system always exists and it is unique up to a constant, whereas in non-simply connected domains the harmonic conjugate may not be a single-valued function. Conversely, in any simply connected subset $\Omega_0 \subset \Omega$, one can select a single-valued branch of this function. Consequently this means a selection of the single-valued branch of the total complex function w.

Application of conformal mappings to the analysis of harmonic functions and the Laplace equation are based on the following observations:

Proposition 3.2. *Let $\zeta : z \mapsto \zeta(z)$ be a conformal mapping $\zeta(z) : N \to M$ by a holomorphic function $\zeta(z) = \xi(x,y) + i\eta(x,y)$. If the function $\widetilde{u}(\xi,\eta)$ is harmonic in M, then the composition*

$$u(x,y) := (\widetilde{u} \circ \zeta)(x,y) = \widetilde{u}(\xi(x,y), \eta(x,y)) \ , \tag{3.2}$$

is a harmonic function of x, y in N.

In particular one obtains:

$$(\Delta_z u)(x,y) = |\partial_z \zeta(z)|^2 \ (\Delta_\zeta \widetilde{u})(\xi(x,y), \eta(x,y)) \ . \tag{3.3}$$

(Here we explicitly distinguish Laplacians in different coordinates, $\Delta_z := \partial_x^2 + \partial_y^2$ and $\Delta_\zeta := \partial_\xi^2 + \partial_\eta^2$, but we ignore these subindexes below, in order to avoid any confusion.) Notice that this statement is based only on a straightforward application of the Cauchy-Riemann conditions for the mapping $\zeta(z)$, i.e. it *does not* assume the existence of a harmonic conjugate neither for $\widetilde{u}$, nor for u. Although, for a simply connected $N_0 \subset N$, one can show that every harmonic function is a *real part* of a branch of holomorphic in N_0 function.

The second observation is related to the Dirichlet-to-Neumann formalism and makes clear the importance of the notion of the *harmonic conjugate* function, [7], Ch.III.

Proposition 3.3. *Let Ω be open simply connected bounded domain in $\mathbb{R}^2$ with a C^2-smooth boundary $\partial\Omega$. Then the solution of the Neumann problem*

$$(\mathbf{P_N}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , \\ \partial_\nu u \mid_{\partial\Omega} (p) = g(p), \quad p \in \partial\Omega \ , \end{cases}$$

reduces to the Dirichet problem for the function v, which is harmonic conjugate to the function u.

To make this evident, notice first that the normal derivative here is defined in the sense of (2.1). Let the boundary $\partial\Omega$ be parameterized by the natural parameter of its arc-length: $\partial\Omega = \{\Gamma(\tau) \in \mathbb{C}\}_{\tau\in[0,l)}$. Then, the Cauchy-Riemann conditions (3.1) imply that

$$\partial_\tau v \mid_{\partial\Omega} (p) = \partial_\nu u \mid_{\partial\Omega} (p) = g(p). \tag{3.4}$$

Since by integration along the contour Γ, one obtains

$$v(p_1) = v(p_0) + \int_{\tau_0}^{\tau_1} d\tau\ \partial_\tau v(\Gamma(\tau)) = v(p_0) + \int_{\tau_0}^{\tau_1} ds\ g(\Gamma(\tau)) =: f(p_1)\ ,$$

the solution of $(\mathbf{P_N})$ is equivalent to the Dirichlet problem $(\mathbf{P_D})$ for v and the boundary conditions f.

2. To outline the main steps in reconstructing the unknown boundary ∂B, we consider first the problem **(P2)** for the Dirichlet case $\alpha = +\infty$:

$$(\mathbf{P^{\infty}_{d=2}}) \qquad \begin{cases} \Delta u = 0,\ p \in \Omega \setminus \overline{B}\ , \\ u \mid_{\partial\Omega} (p) = f(p),\quad p \in \partial\Omega\ , \\ u \mid_{\partial B} (\omega) = 0,\quad \omega \in \partial B\ . \end{cases}$$

It is well-known, see e.g. [7], [10], that the doubly connected bounded domain $\Omega \setminus \overline{B}$ is the image of a conformal mapping of an annulus

$$A_B := \{z \in \mathbb{C} : 0 < \rho_B < |z| < 1\} \tag{3.5}$$

produced by a bijective holomorphic function $\zeta(z)$. This function maps boundaries to boundaries: $\zeta : C_{\rho_B} \to \partial B$ and $\zeta : C_{r=1} \to \partial\Omega$.

(i) The first step is to find the trace $\zeta \mid_{C_1}$ of the unknown function $\zeta(z)$ on the external unit circle $C_{r=1}$.

(ii) Then the next step is to reconstruct the function $\zeta(z)$ in the whole annulus A_B, which solves the geometrical inverse problem (see Introduction 1.2 (a)) by tracing the boundary ∂B as the limit of ζ from inside: $\partial B = \{\zeta(z)\} \mid_{z\to C_{\rho_B}} := \zeta(C_{\rho_B})$.

(i) Let external boundary in the problem $(\mathbf{P^{\infty}_{d=2}})$ be parameterized by the natural parameter of its arc-length: $\partial\Omega = \{\Gamma(\tau) \in \mathbb{C}\}_{\tau\in[0,l)}$. Then the trace of the conformal mapping $\zeta : C_1 \to \partial\Omega$ defines by the equation:

$$\zeta(e^{i\phi}) = \Gamma(\tau)\quad ,\quad \text{for } \phi \in [0, 2\pi)\ , \tag{3.6}$$

with the condition $\zeta(e^{i\phi}) \mid_{\phi=0} = \Gamma(0)$, a bijective function $\phi : \tau \mapsto \phi(\tau) \in [0, 2\pi)$.

Therefore, to calculate the trace of the function $\zeta(z)$ on the external unit circle $C_{r=1}$ is equivalent to finding a solution $\phi(\tau)$ of (3.6), or the corresponding inverse function $\tau(\phi)$.

To this end, let u_f be a solution of the problem $(\mathbf{P^{\infty}_{d=2}})$. Then, by Proposition 3.2, the function $\widetilde{u}_{\widetilde{f}} := u_f \circ \zeta$ is harmonic in the annulus A_B and is a solution of the Dirichlet problem

$$(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta\widetilde{u} = 0,\ p \in A_B\ , \\ \widetilde{u} \mid_{C_1} (p) = \widetilde{f}(p)\ ,\quad p \in C_1\ , \\ \widetilde{u} \mid_{C_{\rho_B}} (\omega) = 0,\quad \omega \in C_{\rho_B}\ . \end{cases}$$

Here $\widetilde{f}(p) = (f \circ \zeta)(p) = f(\zeta(p)) = f(\xi(x, y), \eta(x, y))$ and $p = (x, y) \in C_1$.

Consider the solution u_f of the Dirichlet problem $(\mathbf{P}^{\infty}_{\mathbf{d=2}})$. Then, the Dirichlet-to-Neumann operator $\Lambda_{\partial\Omega}$ for the external boundary $\partial\Omega$ is defined similarly to (1.2):

$$\Lambda_{\partial\Omega} f = \partial_\nu u_f \mid_{\partial\Omega} =: g \ . \tag{3.7}$$

Let v_f be harmonic conjugate to u_f. Then by (3.4) we obtain that for external boundary $\partial\Omega$

$$\partial_\tau v_f \mid_{\partial\Omega} (\tau) = \partial_\tau v_f(\Gamma(\tau)) = \partial_\nu u_f(\Gamma(\tau)) = (\Lambda_{\partial\Omega} f)(\Gamma(\tau)) = \\ (\Lambda_{\partial\Omega} f)(\zeta(e^{i\phi(\tau)})) = (\Lambda_{\partial\Omega} f \circ \zeta)(e^{i\phi(\tau)}) \ . \tag{3.8}$$

With conformal mapping ζ, the relation (3.8) can be rewritten as:

$$\partial_\tau v_f(\Gamma(\tau)) = \partial_\tau v_f(\zeta(e^{i\phi(\tau)})) = \partial_\phi (v_f \circ \zeta)(e^{i\phi(\tau)}))\partial_\tau \phi(\tau) \ . \tag{3.9}$$

Since $\widetilde{u}_{\widetilde{f}} := u_f \circ \zeta$ and $\widetilde{v}_{\widetilde{f}} := v_f \circ \zeta$, see $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$, by (3.4), we obtain

$$\partial_\phi (v_f \circ \zeta)(e^{i\phi})) = \partial_\phi \widetilde{v}_{\widetilde{f}}(\phi) = \partial_\nu \widetilde{u}_{\widetilde{f}} \mid_{C_1} (\phi) = \Lambda_{C_1}(f \circ \zeta)(e^{i\phi}) \ , \tag{3.10}$$

with a usual convention about the normal derivative $\partial_\nu(\cdot) \mid_{C_1}$ on the unit circle C_1. Here, $\Lambda_{C_1} : \widetilde{f} \mapsto \partial_\nu \widetilde{u}_{\widetilde{f}} \mid_{C_1}$ is the Dirichlet-to-Neumann operator corresponding to the problem $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$.

Relations (3.8)-(3.10) yield the following differential equation for $\phi = \phi(\tau)$:

$$\partial_\tau \phi = \frac{(\Lambda_{\partial\Omega} f \circ \zeta)(e^{i\phi})}{\Lambda_{C_1}(f \circ \zeta)(e^{i\phi})} \ . \tag{3.11}$$

For a given boundary Γ, the solution $\phi(\tau)$ of equation (3.11) gives a trace of the function $\zeta(z)$ on the circle C_1. Indeed, by (3.6), we obtain that on C_1 it is defined by:

$$\zeta(e^{i\phi}) = \Gamma(\tau(\phi)) \quad , \quad \text{for } \phi \in [0, 2\pi) \ , \tag{3.12}$$

where $\tau(\phi)$ is the function, which is inverse to $\phi(\tau)$.

3. Hence, for a fixed boundary Γ, one can in principle find the trace $\zeta(z) \mid_{C_1}$ using the scheme outlined above. To this end, let $\widetilde{f} \in W^1_2(C_1)$, where we identify C_1 with $[0, 2\pi]$, see problem $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$. Then, the solution of this problem takes the form:

$$\widetilde{u}_{\widetilde{f}}(\rho, \phi) = a_0 \ln \rho + b_0 + \\ \sum_{n=1}^{\infty} \left[(a_n \rho^n + b_n \rho^{-n}) \cos n\phi + (c_n \rho^n + d_n \rho^{-n}) \sin n\phi \right] \ , \tag{3.13}$$

The coefficients in expansion (3.13) are equal to the following:

$$a_n = \frac{\widetilde{f}_{1,n}}{(1 - \rho_B^{2n})} \ , \ b_n = -\frac{\rho_B^{2n} \widetilde{f}_{1,n}}{(1 - \rho_B^{2n})} \ , \ a_0 = -\frac{\widetilde{f}_{1,0}}{\ln \rho_B} \ , \quad b_0 = \widetilde{f}_{1,0} \ , \tag{3.14}$$

$$c_n = \frac{\widetilde{f}_{2,n}}{(1 - \rho_B^{2n})} \ , \ d_n = -\frac{\rho_B^{2n} \widetilde{f}_{2,n}}{(1 - \rho_B^{2n})} \ . \tag{3.15}$$

They are related to the Fourier series coefficients for $\widetilde{f}(\phi)$:

$$\widetilde{f}_{1,0} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \widetilde{f}(\phi),\ \widetilde{f}_{1,n} = \frac{1}{\pi}\int_0^{2\pi} d\phi \widetilde{f}(\phi)\cos n\phi,\ \widetilde{f}_{2,n} = \frac{1}{\pi}\int_0^{2\pi} d\phi \widetilde{f}(\phi)\sin n\phi.$$

Then, the corresponding Dirichlet-to-Neumann operator (3.10) acts as a bounded operator from $W_2^1(C_1)$ to $L^2(C_1)$:

$$\Lambda_{C_1}\widetilde{f}(\phi) = \partial_\nu \widetilde{u}_{\widetilde{f}} \mid_{C_1}(\phi) = \tag{3.16}$$
$$-\frac{\widetilde{f}_{1,0}}{\ln \rho_B} + \sum_{n=1}^{\infty} n\left[(a_n - b_n)\cos n\phi + (c_n - d_n)\sin n\phi\right] \ .$$

By (3.10) and (3.16), we obtain the identity:

$$\int_0^{2\pi} d\phi \Lambda_{C_1}\widetilde{f}(\phi) = -\frac{1}{\ln \rho_B}\int_0^{2\pi} d\phi \widetilde{f}(\phi) \ ,$$

which implies by (3.8)-(3.10) that the radius of the *internal* circle is defined as

$$\rho_B = \exp\left\{-\left(\int_0^{2\pi} d\phi\,(f\circ\zeta)(e^{i\phi})\right)\left(\int_{\partial\Omega} d\tau\,\partial_\tau\phi(\tau)\partial_\phi(v_f\circ\zeta)(e^{i\phi(\tau)})\right)^{-1}\right\}$$
$$= \exp\left\{-\left(\int_0^{2\pi} d\phi\,(f\circ\zeta)(e^{i\phi})\right)\left(\int_{\partial\Omega} d\tau\,\partial_\nu u_f(\Gamma(\tau))\right)^{-1}\right\} \ . \tag{3.17}$$

Relation (3.17) allows the calculation of ρ_B if one knows the trace $\zeta(z)\mid_{C_1}$, but by (3.12), we have $\zeta(e^{i\phi(\tau)}) = \Gamma(\tau)$, the first equation to solve is (3.11). Notice that by definition $\partial\Omega = \{\Gamma(\tau) \in \mathbb{C}\}_{\tau\in[0,l)}$ and by (3.6),(3.11) one notes this constraint:

$$l = \int_0^{2\pi} d\phi\,\frac{\Lambda_{C_{\rho_B},C_1}(f\circ\zeta)(e^{i\phi})}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta)(e^{i\phi})} \ , \tag{3.18}$$

as well as that the solution $\tau(\phi)$ of (3.11) must be a 2π-periodic function of ϕ. Here, we explicitly recall the second boundary dependence for the both Dirichlet-to-Neumann operators: $\Lambda_{C_1} = \Lambda_{C_{\rho_B},C_{\rho=1}}$ and $\Lambda_{\partial\Omega} = \Lambda_{\partial B,\partial\Omega}$.

Example 3.4. *We illustrate the above by a trivial example of the round Dirichlet absorbing cell. Let boundaries $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ be two concentric circles with radius r_B, which the only unknown parameter that should be defined as a solution of the inverse geometrical problem. Following our scheme, the domain $\Omega\setminus\overline{B}$ is the image of a conformal mapping of an annulus*

$$A_B := \{z \in \mathbb{C} : 0 < \rho_B < |z| < 1\} \tag{3.19}$$

produced by a bijective holomorphic function $\zeta(z)$. This function maps boundaries to other boundaries: $\zeta : C_{\rho_B} \to \partial B$ and $\zeta : C_{r=1} \to \partial\Omega$.

By virtue of the rotational symmetry, one can try to solve this problem for ∂B via $(\mathbf{P}^{\infty}_{\mathbf{d=2}})$ with boundary conditions $u\mid_{\partial\Omega}(p) = f$ independent of $\arg(p)$. Then solution of the direct problem $(\mathbf{P}^{\infty}_{\mathbf{d=2}})$ is given by the $n = 0$ version of (3.13): $u_f(\rho,\phi) = a\ \ln\rho + b$ for $r_B < \rho < R$. Taking into account boundary conditions *one finds* a *and* b *and the explicit form of the corresponding Dirichlet-to-Neumann operator:*

$$\Lambda_{\partial B,\partial\Omega} : f \mapsto \partial_\nu u_f \,|_{C_R} = \frac{1}{R\,(\ln R - \ln r_B)}\, f\ . \tag{3.20}$$

(*Note that our example is so simple that the one-measure of the "voltage-current" data* $\{f, j := \Lambda_{\partial B,\partial\Omega} f\}$ *is enough to uniquely define the operator* $\Lambda_{\partial B,\partial\Omega}$ *that solves the problem of* r_B *explicitly.*)

Since the conformal mapping for the exterior boundaries gives $\zeta(e^{i\phi}) = R\ e^{i\phi}$*, for* $p \in C_1$ (*trace* $\zeta\ |_{C_1}$)*, one gets* $\widetilde{f}(p) := (f \circ \zeta)(p) = f(\zeta(e^{i\phi})) = f(Re^{i\phi}) = f$*. Then, by (3.13), the Dirichlet-to-Neumann operator for the problem* $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$ *has the form:*

$$\Lambda_{C_{\rho_B},C_{\rho=1}} : \widetilde{f} \mapsto \partial_\nu \widetilde{u}_{\widetilde{f}} \,|_{C_1} = -\ \frac{1}{\ln \rho_B}\ \widetilde{f}\ . \tag{3.21}$$

Then, by (3.20), we get for the numerator in (3.11):

$$(\Lambda_{\partial B,\partial\Omega} f \circ \zeta) = \frac{1}{R\,(\ln R - \ln r_B)}\ f \circ \zeta = \left\{R\ \ln \frac{R}{r_B}\right\}^{-1} \widetilde{f}, \tag{3.22}$$

and by (3.21) one obtains for denominator in (3.11):

$$\Lambda_{C_{\rho_B},C_{\rho=1}}(f \circ \zeta) = -\ \frac{1}{\ln \rho_B}\ \widetilde{f}\ . \tag{3.23}$$

Inserting (3.22) and (3.23) into (3.17) (or into (3.18), where $l = 2\pi R$*) we obtain that* $\rho_B = r_B/R$*, i.e. for internal boundaries, the conformal mapping gives:* $\zeta(\rho_B e^{i\phi}) = r_B e^{i\phi} = R\ \rho_B e^{i\phi}$*. This implies that the mapping is* $\zeta(z) = R\ z$ (*see (ii)*)*, and also the evident final result about the form of the boundary* ∂B *as the trace of* $\zeta(z)$ *on the* C_{ρ_B}*.*

4. This example shows that $\tau(\phi)$ is a 2π-periodic extension of the *linear* function

$$\tau_0(\phi) := \frac{l}{2\pi}\,\phi\ ,\quad \phi \in [0, 2\pi)\ . \tag{3.24}$$

The result is a simple linear form of the corresponding conformal mapping. Any deviation from concentric domains $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ makes the function $\tau(\phi)$ non-linear, but still obeying condition (3.18).

A less trivial application of the scheme presented above is the example of non-concentric domains $\partial\Omega = C_R$ and $\partial B = C_{r_B}$. In this case the conformal mapping ζ is *a priori* known: it is the Möbius transformation, and one can proceed with this trial ζ along the same line of reasoning as in Example 3.4, see [2]. Illustration of the inverse geometrical problem solution needs a complete application of the above formalism, since now, one has to solve two coupled equations (3.11) and (3.17) with condition (3.18). (ii) We rewrite these equations (incorporating the constraint (3.18)) in the following form:

$$\rho_B = \exp\left\{-\left(\int_0^{2\pi} d\phi\,(f\circ\zeta)(e^{i\phi})\right)\left(\int_{\partial\Omega} d\tau\,\partial_\nu u_f(\Gamma(\tau))\right)^{-1}\right\}, \quad (3.25)$$

$$\partial_\phi \tau = \frac{l}{2\pi} + \frac{\Lambda_{C_{\rho_B},C_1}(f\circ\zeta)}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta)} - \frac{1}{2\pi}\int_0^{2\pi} d\phi\,\frac{\Lambda_{C_{\rho_B},C_1}(f\circ\zeta)(e^{i\phi})}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta)(e^{i\phi})} \;. \quad (3.26)$$

Notice that by (3.22) and (3.23) for concentric domains $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ the last two terms in (3.26) cancel. Therefore, one can consider this case as the *zero-order* approximation $\tau = \tau_0(\phi)$ for the solution of (3.26) with $\zeta = \zeta_0(z) := z$ and $\rho_B = \rho_0 := r_B/R$. This observation implies that one can consider equations (3.25) and (3.26), together with relations $\zeta_n(e^{i\phi}) = \Gamma(\tau_n(\phi))$, see (3.12), as a non-linear iterative scheme to obtain ρ_B and the function $\tau(\phi)$ (or $\zeta(z)$), cf [9]:

$$\rho_n = \exp\left\{-\left[\int_0^{2\pi} d\phi (f\circ\zeta_n)(e^{i\phi})\right]\left[\int_{\partial\Omega} d\tau \partial_\nu u_f(\Gamma(\tau))\right]^{-1}\right\}, \quad (3.27)$$

$$\partial_\phi \tau_{n+1} = \frac{l}{2\pi} + \frac{\Lambda_{C_{\rho_n},C_1}(f\circ\zeta_n)}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta_n)} - \frac{1}{2\pi}\int_0^{2\pi} d\phi\,\frac{\Lambda_{C_{\rho_n},C_1}(f\circ\zeta_n)(e^{i\phi})}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta_n)(e^{i\phi})}, \quad (3.28)$$

$$\zeta_n(e^{i\phi}) = \Gamma(\tau_n(\phi)) \;. \quad (3.29)$$

Remark 3.5. *Suppose that for $n\to\infty$ the iterations converge: $\rho_n \to \rho_B$, $\tau_n(\phi)\to\tau(\phi)$ and for given Γ: $\zeta_n(z)\to\zeta(z)$. Then, the function $\Gamma(\tau(\phi))$ can be presented as the Fourier series:*

$$\Gamma(\tau(\phi)) = \sum_{s\in\mathbb{Z}} \gamma_s e^{is\phi} \;. \quad (3.30)$$

Since $\Gamma(\tau(\phi))$ is the image of the external boundary C_1 by the seeking function $\zeta(z)$, the coefficients γ_s are the same as in the Laurent series for this function in the annulus A_B:

$$\zeta(z) = \sum_{s\in\mathbb{Z}} \gamma_s z^s \;. \quad (3.31)$$

Now, the final step is to observe that the unknown internal boundary ∂B coincides with the conformal image $\{\Gamma_{\partial B}(\phi)\}_{0\le\phi<2\pi} = \zeta(C_{\rho_B})$ of the internal A_B-circle C_{ρ_B} with the radius $\rho_B < 1$ calculated by iterations (3.27):

$$\Gamma_{\partial B}(\phi) = \sum_{s\in\mathbb{Z}} (\rho_B)^s\;\gamma_s e^{is\phi} \;. \quad (3.32)$$

The relation (3.32) *formally* solves the inverse geometrical problem for Dirichlet boundary conditions on the unknown contour $\partial B = \{\Gamma_{\partial B}(\phi)\}_{0\le\phi<2\pi}$.

4. Constant boundary conditions

1.1 Problem $(P_{f_\pm=1,0})$**.** Below we suppose that Ω and $B\subset\Omega$ are open bounded domains in $\mathbb{R}^2$ with C^2-smooth disjoint boundaries $\partial\Omega$ and ∂B, that is, $\partial(\Omega\setminus\overline{B}) = \partial\Omega\cup\partial B$ and $\partial\Omega\cap\partial B = \emptyset$.

The *unknown internal boundary* ∂B should be found from the solution u of the Dirichlet problem:

$$(\mathbf{P_{f_\pm=1,0}}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , \\ u \mid_{\partial\Omega} (p) = f_+(p) = 1, \quad p \in \partial\Omega \ , \\ u \mid_{\partial B} (p) = f_-(p) = 0, \quad p \in \partial B \ , \end{cases}$$

with help of the *given* (measured) Neumann data: $g(p) = \partial_\nu u \mid_{\partial\Omega} (p)$, *exterior* normal derivative on the external boundary $p \in \partial\Omega$.

Remark 4.1. *Notice that one can always find a conformal mapping that transforms domain Ω into a unit disc. Therefore, we put for simplicity $\Omega = D_{r=1}$, the unit disc, i.e. $\partial\Omega = C_1$, is the unit circle.*

Remark 4.2. *Since below we use a conformal map approach to the localization of the internal boundary ∂B, we identify the $\mathbb{R}^2$-points $p = (x, y)$ with those of the complex plane $\mathbb{C}$ by: $p \mapsto z(p) := x + iy \in \mathbb{C}$. Then it is known, see e.g. [7], that the harmonic function solving $(\mathbf{P_{f_\pm=1,0}})$ can be viewed as the real part of a holomorphic in domain $\Omega \setminus \overline{B}$ function $\widehat{u}(z)$, i.e., $u(p) = \mathrm{Re}\ \widehat{u}(z(p))$. We put $\widehat{u}(z) = u(x, y) + iv(x, y)$, where $v(x, y)$ is harmonic conjugate to $u(x, y)$, [7]. Recall that for a doubly-connected domain, the function $\widehat{u}(z)$ may be multi-valued. Then, we consider for $\widehat{u}(z)$ only one (principle) branch.*

Remark 4.3. *Recall that in polar coordinates $z = re^{i\phi} \in \mathbb{C}$ the measured Neumann data g on C_1 take the form:*

$$\begin{aligned} g(\phi) &= e_r \cdot \nabla u \mid_{z \in C_1} = (\cos\phi \ \partial_x u + \sin\phi \ \partial_y u) \mid_{z\in C_1} = \\ &= \partial_r u(r\cos\phi, r\sin\phi) \mid_{r=1} \ . \end{aligned} \tag{4.1}$$

We also recall that the Cauchy-Riemann conditions in these coordinates can be written as:

$$\partial_r u = \frac{1}{r} \ \partial_\phi v \quad , \quad \frac{1}{r} \ \partial_\phi u = - \partial_r v \ . \tag{4.2}$$

1.2 Problem $(P^*_{f_\pm=1,0})$**.** Let the holomorphic function $w : z = (x+iy) \mapsto (w_1+iw_2)$ Map the doubly-connected bounded domain $D_1 \setminus \overline{B} \subset \mathbb{C}$ into annulus

$$A_B := \{w \in \mathbb{C} : 0 < \rho_B < |w| < 1\} \ . \tag{4.3}$$

This function maps boundaries to other boundaries: $w : \partial B \to C_{\rho_B}$ and $w : \partial\Omega = C_1 \to C_1$ and define the function $U(w_1, w_2)$ by

$$u(x, y) = (U \circ w)(x, y) = U(w_1(x, y), w_2(x, y)) \ . \tag{4.4}$$

Then, the problem $(P_{f_\pm=1,0})$ transfers into

$$(\mathbf{P^*_{f_\pm=1,0}}) \qquad \begin{cases} \Delta U = 0, \ p \in D_1 \setminus \overline{D_{\rho_B}} \ , \\ U \mid_{C_1} (p) = 1, \quad p \in C_1 \ , \\ U \mid_{C_{\rho_B}} (p) = 0, \quad p \in C_{\rho_B} \ , \end{cases}$$

with the exterior normal derivative:

$$\partial_\nu U \mid_{z\in C_1} (w(z)) = \left(\frac{1}{|w'(z)|}\, g(z)\right)\Big|_{z\in C_1} . \tag{4.5}$$

Notice that the value of the normal derivative (4.5) is B-dependent via conformal mapping w.

1.3 Solution of the Problem $(P^*_{f_\pm=1,0})$**.** For the general solution, one easily finds a representation in the (complex) polar coordinates $w = \rho e i\varphi$:

$$U(\rho,\varphi) = a + b\ln\rho + \sum_{n\in\mathbb{Z}\setminus 0} \left(a_n\rho^n e^{in\varphi} + b_n\rho^{-n}e^{-in\varphi}\right) ,$$

which is simply the standard Fourier-series representation. By virtue of the boundary conditions, we obtain:

$$a = 1 \ , \ b = -\frac{1}{\ln\rho_B} \ , \ a_n = b_n = 0 \ .$$

Then, Consequently, we get for the solution the explicit form:

$$U(w_1,w_2) = U(\rho,\varphi) = \frac{\ln(\rho/\rho_B)}{\ln(1/\rho_B)} = \frac{1}{\ln(1/\rho_B)}\ \ln\frac{|w|}{\rho_B} , \tag{4.6}$$

and the corresponding B-dependent normal derivative on the external boundary C_1, cf. (4.5):

$$\partial_\nu U \mid_{C_1} (w) = \partial_\rho U(\rho,\varphi) \mid_{\rho=1} = \frac{1}{\ln(1/\rho_B)} . \tag{4.7}$$

Notice that in contrast to the Problem $(P_{f_\pm=1,0})$, the Neumann data (4.7) for the Problem $(P^*_{f_\pm=1,0})$ are *isotropic* and they depend on B only via radius ρ_B.

It is clear that to proceed with localization of the internal boundary ∂B, one has to find the conformal mapping $w(z)$. The relations (4.5) and (4.7) yield the functional equation:

$$\frac{1}{\ln(1/\rho_B)} = \left(\frac{1}{|w'(z)|}\, g(z)\right)\Big|_{z\in C_1} \tag{4.8}$$

for w. This equation is insufficient, since it is localized only on the boundary C_1. To overcome this difficulty, we use complex extensions of $(P_{f_\pm=1,0})$ and $(P^*_{f_\pm=1,0})$ indicated in Remark 4.2.

2.1 Complex extension. Let us define the complex extension of (4.6) by

$$\widehat{U}(w = w_1 + iw_2) := \frac{1}{\ln(1/\rho_B)} \ln\frac{w}{\rho_B} = (U + iV)(w) , \tag{4.9}$$

where $V = \arg w$ is the harmonic conjugate to $U = \ln|w|$ and corresponds to the principle branch of the logarithm. Hence, one can similarly introduce the function

$$\widehat{u}(z) := \widehat{U}(w(z)) = (u + iv)(z) = \frac{1}{\ln(1/\rho_B)} \ln\frac{w(z)}{\rho_B} , \tag{4.10}$$

where v is the harmonic conjugate to u.

2.2 Complex extension and the Problem $(P_{f_\pm=1,0})$**.** By (4.10), one gets

$$u(x,y) = \mathrm{Re}\ \widehat{u}(z) = \frac{1}{\ln(1/\rho_B)} \ln \frac{|w(z)|}{\rho_B} \ .$$

Let $z = re^{i\phi}$. Then by virtue of (4.1), (4.10) and

$$\partial_r \widehat{u}(z) = (\partial_r u + i\partial_r v)(z) = \widehat{u}^{'}(z)\, e^{i\phi} = \frac{1}{\ln(1/\rho_B)} \frac{w^{'}(z)}{w(z)}\ e^{i\phi} \ , \tag{4.11}$$

we obtain the following equation:

$$\partial_r u \mid_{C_1} = \mathrm{Re} \left\{ \frac{1}{\ln(1/\rho_B)} \frac{w^{'}(e^{i\phi})}{w(e^{i\phi})}\ e^{i\phi} \right\} = g(\phi) \ . \tag{4.12}$$

Notice that the Cauchy-Riemann conditions (4.2) implies:

$$\partial_r v(z = re^{i\phi}) = -\frac{1}{r}\ \partial_\phi u(re^{i\phi}) = -\frac{1}{r\ \ln(1/\rho_B)}\ \partial_\phi \ln |w(re^{i\phi})| \ . \tag{4.13}$$

Since for $r = 1$, we have $|w(e^{i\phi})| = 1$, one gets $\partial_r v(z) \mid_{C_1} = 0$, i.e. the condition Re in (4.12) is superfluous as soon as we stick to the external boundary C_1:

$$\frac{1}{\ln(1/\rho_B)} \frac{w^{'}(e^{i\phi})}{w(e^{i\phi})}\ e^{i\phi} = g(\phi) \ . \tag{4.14}$$

2.3 Solution for conformal mapping $w(z)$**.** Motivated by (4.14), we define a continuation of (4.12) from the external boundary C_1 into domain $\Omega \setminus \overline{B}$. To this end, we introduce a holomorphic in $\Omega\setminus\overline{B}$ function F with the corresponding Laurent series:

$$F(z) := \frac{1}{\ln(1/\rho_B)} \frac{w^{'}(z)}{w(z)}\ z = F_0 + \sum_{n=1}^{\infty}(F_n\ z^n + F_{-n}\ z^{-n}) \ . \tag{4.15}$$

Then by periodicity of g and by (4.14), (4.33) we obtain the relation

$$g(\phi) = \sum_{n\in\mathbb{Z}} g_n e^{in\phi} = F(z = e^{i\phi}) \ , \tag{4.16}$$

which implies $F_n = g_n$ and $\overline{g_n} = g_{-n}$, for $n \in \mathbb{Z}$, as well as equation

$$\frac{1}{\ln(1/\rho_B)} \frac{w^{'}(z)}{w(z)}\ z = g_0 + \sum_{n=1}^{\infty}(g_n z^n + g_{-n} z^{-n}) \ . \tag{4.17}$$

Therefore, one has

$$\partial_z \ln w(z) = \ln(1/\rho_B) \left[\frac{g_0}{z} + \sum_{n=1}^{\infty}(g_n z^{n-1} + g_{-n} z^{-n-1}) \right] \ . \tag{4.18}$$

Hence, we obtain:

$$w(z) = w_0\ z^{g_0 \ln(1/\rho_B)} \exp \left[\ln(1/\rho_B) \sum_{n=1}^{\infty}(g_n z^n - g_{-n} z^{-n})/n \right] \ . \tag{4.19}$$

Since $w : C_1 \to C_1$, one obviously gets

$$w(e^{i\phi}) = e^{i\varphi(\phi)} \quad \text{and} \quad w(e^{i(\phi+2\pi)}) = e^{i\varphi(\phi+2\pi)} = e^{i\varphi(\phi)} \ , \tag{4.20}$$

which implies that $g_0 \ln(1/\rho_B) = 1$ and

$$\rho_B = e^{-1/g_0} \ , \tag{4.21}$$

i.e., we *must* put $g_0 > 0$. Notice that $|w(e^{i\phi})| = 1$ and (4.21) yield $|w_0| = 1$, which we can choose to be real. Therefore, one finally obtains for the conformal mapping w the expression:

$$w(z) = z \ \exp\left[(1/g_0)\sum_{n=1}^{\infty}(g_n z^n - g_{-n}z^{-n})/n\right] \ , \tag{4.22}$$

which is completely defined by the measured Neumann data $g(p)$ on the external boundary C_1.

Remark 4.4. *In spite of the obvious remark:* $\partial_\phi |w(e^{i\phi})| = 0$*, which we used to establish (4.14), the derivative* $\partial_\phi w(e^{i\phi}) = e^{i\varphi(\phi)}\, \partial_\phi \varphi(\phi) \neq 0$*. This means that* $\varphi(\phi)$ *is a nontrivial periodic function on* C_1 *, see (4.20).*

3.1 Inverse conformal mapping. According to our construction (see 1.2), the inverse function $z(w)$ maps C_{ρ_B} into the contour ∂B, i.e. formally $\partial B = \{z(w = \rho_B e^{i\varphi})\}_{\varphi\in[0,2\pi)}$.

Note that by using (4.33), we can introduce the holomorphic function:

$$G(w) := F(z(w))^{-1} = \ln(1/\rho_B)\ \frac{z'(w)}{z(w)}\ w = G_0 + \sum_{n=1}^{\infty}(G_n w^n + G_{-n}w^{-n}) \ , \tag{4.23}$$

where the last sum is the corresponding Laurent series. Hence, following the same line of reasoning as in Section 2, we obtain:

$$z(w) = z_0\ w^{G_0/\ln(1/\rho_B)} \exp\left[(\ln(1/\rho_B))^{-1}\sum_{n=1}^{\infty}(G_n w^n - G_{-n}w^{-n})/n\right] \ . \tag{4.24}$$

Notice that on the circle C_1 the function $z(w = e^{i\varphi})$ is periodic. Then, the same is true for G. By arguments similar to those in Section 2, this function has the Fourier coefficients satisfying the same properties as g_n in (4.16), i.e. by (4.23) one gets:

$$G_n = \overline{G_{-n}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\ G(e^{i\varphi})\ e^{-in\varphi} \ . \tag{4.25}$$

3.2 Localization of ∂B. Since $z : C_1 \to C_1$, then similar to Section 2, the representation (4.24) for this periodic function implies that we can choose $z_0 = 1$ and that $G_0/\ln(1/\rho_B) = 1$, or $G_0 = 1/g_0$. By virtue of (4.16) and (4.23) the other coefficients are given by

$$G_m = \frac{1}{2\pi i}\int_{C_1} dw\ \frac{1}{w^{m+1}}\ \frac{1}{F(z(w))} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \frac{e^{i\phi}}{g(\phi)}\frac{w'(e^{i\phi})}{w^{m+1}(e^{i\phi})} \ . \tag{4.26}$$

Since the conformal mapping w has been already calculated in (4.22) for given Neumann data g, formulae (4.26) solve the problem of inversion $z(w)$, see (4.24).

Hence in the cases $f_+ = 1$ and $f_- = 0$, the position of unknown boundary ∂B is defined for a given Neumann data g as a set:

$$\partial B = \{z(w = \rho_B e^{i\varphi})\}_{\varphi\in[0,2\pi)} \ , \tag{4.27}$$

which is uniquely defined by (4.24),(4.26) and auxiliary radius $\rho_B = e^{-1/g_0}$.

3.3 Existence and uniqueness. Notice that existence and uniqueness of the solution (4.27) follow from the explicit construction in the above subsection 3.2. This statement is not *unconditional*. The first necessary condition is:

(i) $g_0 > 0$, see (4.21).

Another restriction follows directly from the $f_\pm$-boundary conditions for the Problem $(P_{f_\pm=1,0})$:

(ii) $g(\phi) > 0$, see (4.5) and (4.7).

(iii) A more subtle constraint for the given Neumann data $g(\phi)$ follows from the conditions insuring the invertibility of the conformal mapping w. We study this restriction first for the particular example in the next subsection 4.1.

4.1 Let $g_0 > 0$ and $g_1 > 0$. By (4.22) one gets

$$w(z) = z \ \exp\left[(g_1/g_0)(z - z^{-1})\right] \ , \tag{4.28}$$

but our aim is to inverse the function $w(z)$, i.e. to find (4.24) and then to calculate the unknown boundary ∂B (4.27).

It is worth noting that despite $|w(z = e^{i\phi})| = 1$, the conformal mapping (4.28) acts nontrivially on C_1 since, see (4.20):

$$w(e^{i\phi}) = e^{i\phi}\exp\left[2i(g_1/g_0)\sin\phi\right] = e^{i\varphi(\phi)} \ . \tag{4.29}$$

Equation (4.29) yields for the function $\varphi(\phi)$ the expression:

$$\varphi(\phi) = \phi + 2(g_1/g_0)\sin\phi \ . \tag{4.30}$$

4.2 Notice first that the general conditions on $g(\phi)$ imply: $g_0 > 0$ and $g_0 > 2g_1$, see (i) and (ii). For example, the importance of $g_0 > 2g_1$ is directly related to *monotonicity* of the function (4.30).

A more delicate condition (iii) requires that $w : \partial B \to C_{\rho_B}$ and in particular:

$$w(z = r(\phi))\ |_{\phi=0} = r(\phi) \ \exp\left[(g_1/g_0)(r(\phi) - r(\phi)^{-1})\right] |_{\phi=0} = \rho_B \tag{4.31}$$

$$w(z = -r(\phi))\ |_{\phi=\pi} = -r(\phi) \ \exp\left[(g_1/g_0)(-r(\phi) + r(\phi)^{-1})\right] |_{\phi=\pi} = -\rho_B \tag{4.32}$$

Notice that for given $g_0 > 0$ and $g_0 > 2g_1$, the solution of (4.31) for $r(\phi = 0)$ always exists and is unique. Whereas for $r(\phi = \pi)$, this is not true. Indeed, for any $r < 1$, the function defined by the left side of (4.32):

$$F_\varepsilon(r) := r \ \exp\left[\varepsilon(-r + r^{-1})\right] > 0 \ \ , \ \ \varepsilon := g_1/g_0 < 1/2 \ , \tag{4.33}$$

is monotonously increasing, for increasing ε. Hence, there is a critical value ε_{cr} : $0 < \varepsilon_{cr} < 1/2$, corresponding to condition

$$\min_{r\leq 1} F_{\varepsilon_{cr}}(r) = \rho_B \ , \tag{4.34}$$

and there are no solutions $r(\phi = \pi) < 1$ of (4.32) for $\varepsilon > \varepsilon_{cr}$. Let $g_0 = 1$. Then, one obtains from (4.34) the equation for ε_{cr} in the form:

$$\ln[(1 - \sqrt{1 - 4\varepsilon^2})/2\varepsilon] + \sqrt{1 - 4\varepsilon^2} + 1 = 0 \ . \qquad (4.35)$$

Equation (4.35) implies that a solution for $r(\phi = \pi)$ does not exist, when $1/2 > g_1$, but $g_1 > g_{cr} = 0,13796148...$. This means that for $g_1 > g_{cr}$, the conformal map w is not invertible, i.e. the image ∂B is not correctly defined.

We illustrate this evolution of conformal mapping and the form of the internal absorbing boundary ∂B as a function of g_1 for $g_0 = 1$ by Figures 1-5.

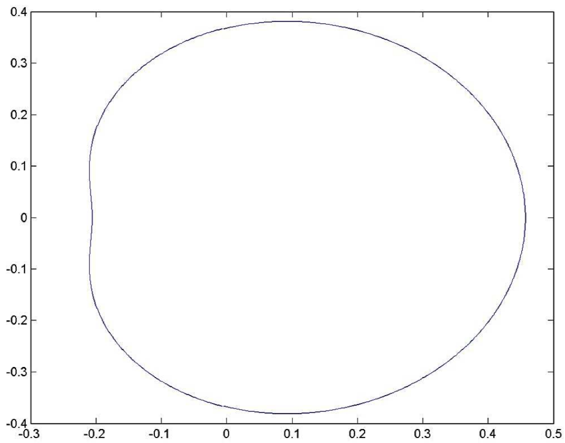

FIG. 1. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,125 < g_{cr}$

On the last two figures, one observes that the boundary ∂B is not closed because of small gaps for $\varphi(\phi = \pi) = \pi$, see (4.30). This is a numerical indication that the conformal map w is not invertible for $g_1 > g_{cr}$.

5. Concluding remarks

1. First, we comment the case $\alpha = 0$, i.e. the Neumann boundary conditions on the absorbing cell ∂B, see **(P2)**. Then, $(\mathbf{P}^{\infty}_{\mathbf{d=2}})$ is transformed into the following problem:

$$(\mathbf{P}^{\alpha=\mathbf{0}}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , \\ u \mid_{\partial\Omega} (p) = f(p), \quad p \in \partial\Omega \ , \\ \partial_\nu u \mid_{\partial B} (\omega) = g(\omega), \quad \omega \in \partial B \ . \end{cases}$$

To map domain $\Omega \setminus \overline{B}$ onto annulus (4.3), we use the same holomorphic function $\zeta(z)$. Since conformal mappings preserve angles, the corresponding problem

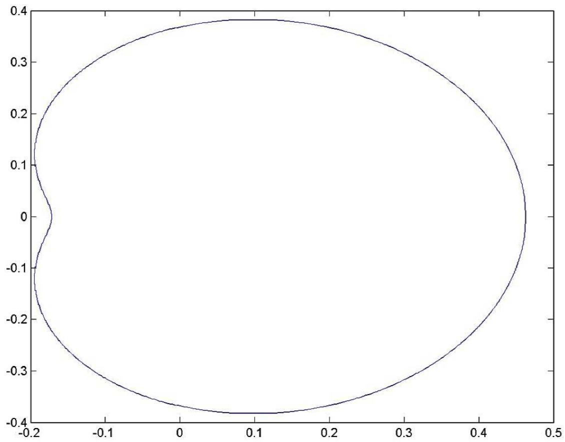

FIG. 2. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,135 < g_{cr}$

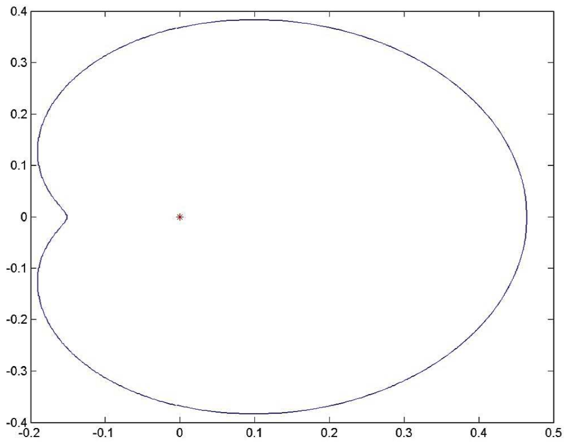

FIG. 3. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,13796148 < g_{cr}$

assumes the form:

$$(\widetilde{\mathbf{P}}^{\alpha=\mathbf{0}}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta \widetilde{u} = 0, \ p \in A_B \ , \\ \widetilde{u}\mid_{C_1}(p) = \widetilde{f}(p) \ , \quad p \in C_1 \ , \\ \partial_\nu \widetilde{u}\mid_{C_{\rho_B}}(\omega) = |\partial_z \zeta(\omega)| \widetilde{g}(\omega), \quad \omega \in C_{\rho_B} \ . \end{cases}$$

Here $\partial_\nu(\cdot)\mid_{C_{\rho_B}}$ is the external *normal* derivative at the point $\omega \in C_{\rho_B} = \zeta(\partial B)$ for a value proportional to $\widetilde{g}(\omega) = (g \circ \zeta)(\omega)$.

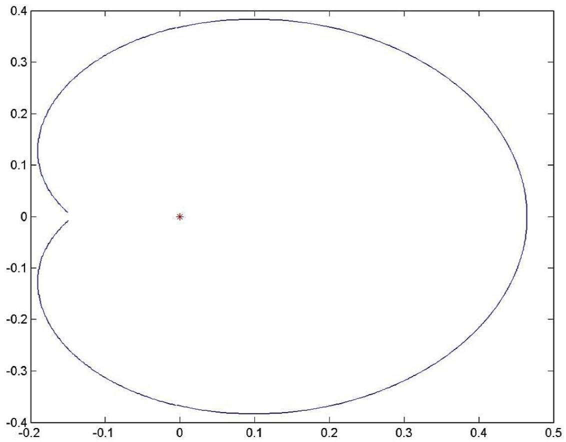

FIG. 4. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,13815648 > g_{cr}$

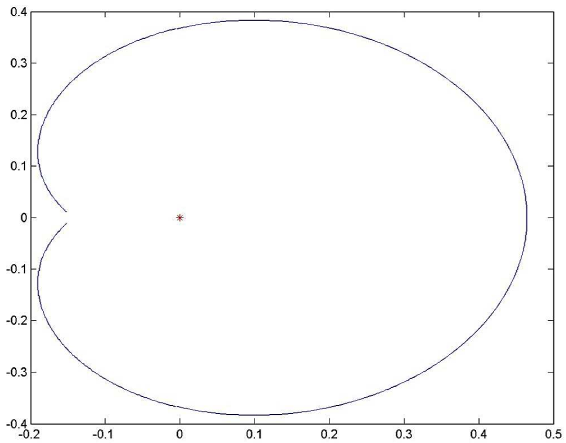

FIG. 5. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,13824948 > g_{cr}$

It is clear now that our scheme must be considerably modified (simplified), since the actual boundary conditions depend on an *unknown* conformal mapping ζ. Note that this cannot be aided by Proposition 3.3 to reduce the Neumann boundary condition to Dirichlet, since our domain is not simply connected. The external data for solving the inverse geometrical problem correspond to $\widetilde{f}(p)$, so we prefer to simplify the conditions on the cell surface ∂B and set $g = 0$, which excludes the annoying dependence of the Neumann boundary conditions on derivative $\partial_z \zeta$.

2. Consider the problem $(\widetilde{\mathbf{P}}^{\alpha=0}_{\mathbf{d=2}})$ for $\widetilde{g}=0$.

$$(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta\widetilde{u}=0,\ p\in A_B\ , \\ \widetilde{u}\,|_{C_1}(p)=\widetilde{f}(p)\ ,\quad p\in C_1\ , \\ \partial_\nu\widetilde{u}\,|_{C_{\rho_B}}(\omega)=0\ ,\quad \omega\in C_{\rho_B}\ . \end{cases}$$

Example 5.1. *As above (see Example 3.4) we first illustrate a possible strategy to solve $(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}})$ by a simple example of the round Neumann absorbing cell.*

Let boundaries $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ be two concentric circles with radius r_B, which is the only unknown parameter that should be defined as a solution of the inverse geometrical problem. Moreover, since $\zeta : C_{\rho_B} \to \partial B = C_{r_B}$ and $\zeta : C_{r=1} \to \partial\Omega = C_R$, we find this conformal mapping coincides with the same linear mapping, $\zeta(z) = R\,z$, as in Example 3.4, i.e. $\rho_B = r_B/R$.

Notice that the constant external condition $\widetilde{f}(p) = (f\circ\zeta)(p) = f(Re^{i\phi}) = f$, $p\in C_1$, implies a trivial constant solution $u_f = \widetilde{u}_f = f$. Therefore, we consider the one-mode boundary condition defined by $\widetilde{f}(e^{i\phi}) = (f\circ\zeta)(e^{i\phi}) = f(Re^{i\phi}) = f(\phi) := \mathrm{f}\cos\phi$. Then by general solution (3.13) in annulus one obtains for the Dirichlet-to-Neumann operator, $(\mathbf{P}^{\alpha=0}_{\mathbf{d=2}})$ with $g=0$:

$$\Lambda_{\partial B,\partial\Omega} : f(\phi) \mapsto \partial_\nu u_f\,|_{C_R} = \frac{R^2-r_B^2}{R\,(R^2+r_B^2)} f(\phi)\ . \tag{5.1}$$

Similarly one obtains for for the problem $(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}})$:

$$\Lambda_{C_{\rho_B},C_{\rho=1}} : f(\phi) \mapsto \partial_\nu\widetilde{u}_{\widetilde{f}}\,|_{C_1}= \frac{1-\rho_B^2}{(1+\rho_B^2)}\ f(\phi)\ . \tag{5.2}$$

By virtue of $\rho_B = r_B/R$, (5.1) and (5.2) imply that relations (3.17) and (3.18), where $l = 2\pi R$, are valid with solution (3.24): $\tau_0(\phi) := (l/2\pi)\,\phi\ ,\ \phi\in[0,2\pi)\ .$

This example shows that following along verbatim through the arguments of Section 3.4, one obtains the same iterative scheme (3.27)-(3.29), but with Dirichlet-to-Neumann operators that are defined by the Neumann problems $(\mathbf{P}^{\alpha=0}_{\mathbf{d=2}})$ and $(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}})$. Example 5.1 gives the zero-order approximation for solution. **3.** Recall that the aim of present note is to advocate a *formal* solution of some $d = 2$ inverse geometrical problems, see e.g. Remark 3.5. Since the error in calculations of the coefficients $\{\gamma_s\}_{s\in\mathbb{Z}}$, see (3.30), can be exponentially amplified in expression (3.32) for the boundary ∂B, it is clear that the problem is ill-posed, i.e. it demands further analysis.

We plan to return to numerical implementations of this formal iterative scheme elsewhere. The cut-offs and regularizations, as well as their possible generalizations to Robin boundary conditions need to be studied.

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