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C*-ALGEBRAS IN RECONSTRUCTION OF MANIFOLDS

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We deal with two dynamical systems associated with a Riemannian manifold with boundary. The first one is a system governed by the scalar wave equation, the second is governed by Maxwells equations. Both of the systems are controlled from the boundary. The inverse problems are to recover the manifold via the relevant measurements at the boundary (inverse data). We show that that the inverse data determine a C*-algebras, whose (topologized) spectra are identical to the manifold. By this, to recover the manifold is to determine a proper algebra from the inverse data, find its spectrum, and provide the spectrum with a Riemannian structure. This paper develops an algebraic version of the boundary control method (M.I.Belishev'1986), which is an approach to inverse problems based on their relations to control theory.

Keywords: inverse problems on manifolds, C*-algebras, boundary control method.

1. Setup

1.1. Acoustics

We deal with a compact C^∞-smooth Riemannian manifold Ω with the boundary Γ, $\dim\Omega = n \geqslant 2$; Δ is the (scalar) Beltrami-Laplace operator on Ω; $\mathcal{H} := L_2(\Omega)$.
Forward problem of acoustics is to find a solution $u = u^f(x,t)$ of the system

$$u_{tt} - \Delta u = 0 \qquad \text{in } (\Omega\backslash\Gamma)\times(0,T) \tag{1.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in } \Omega \tag{1.2}$$

$$u = f \qquad \text{on } \Gamma\times[0,T], \tag{1.3}$$

where $f \in \mathcal{F}^T := L_2(\Gamma\times[0,T])$ is a (given) *boundary control*.

With the system one associates a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$R^T f := \left.\frac{\partial u^f}{\partial \nu}\right|_{\Gamma\times[0,T]}$$

(for smooth enough f), ν is the outward normal to Γ.
Inverse problem is: *given for a fixed* $T > \operatorname{diam}\Omega$ *the operator* R^{2T}*, to recover* Ω.

1.2. Electrodynamics

Let Ω be *oriented*, $\dim\Omega = 3$. The definitions of the vector analysis operations $\wedge$, $\operatorname{curl}, \operatorname{div}$ on a manifold see, e.g., in [9].
Forward problem Find a solution $e = e^f(x,t)$, $h = h^f(x,t)$ of the Maxwell system

$$e_t = \operatorname{curl} h,\ h_t = -\operatorname{curl} e \qquad \text{in } \Omega\times(0,T) \tag{1.4}$$

$$e|_{t=0} = 0,\ h|_{t=0} = 0 \qquad \text{in } \Omega \tag{1.5}$$

$$\nu\wedge e = f \qquad \text{in } \Gamma\times[0,T], \tag{1.6}$$

$f \in \mathcal{F}^T := L_2([0,T]; T\Gamma)$ is a *boundary control* (time-dependent tangent field on Γ).

With the system, one associates a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$R^T f := \nu \wedge h^f \Big|_{\Gamma \times [0,T]}$$

(for smooth enough f).

Inverse problem is: *given for a fixed* $T > \operatorname{diam}\Omega$ *the operator* R^{2T}*, to recover* Ω.

1.3. Nonuiqueness

Let Ω' be such that $\partial\Omega' = \partial\Omega = \Gamma$ and there is an isometry $i : \Omega \to \Omega'$ provided $i|_\Gamma = \mathrm{id}$. Then, for the response operators of the systems (1.1)–(1.3) and (1.4)–(1.6) one has $R'^T = R^T$ for all $T > 0$.

Hence, the map "manifold $\mapsto$ its response operator" in not injective. By this, to determine Ω uniquely is impossible, and we have to clarify the setup of the inverse problems as follows [3]. The only reasonable setup is: *given* R^{2T} *for a fixed* $T > \operatorname{diam}\Omega$*, to construct a Riemannian manifold* $\tilde\Omega$ *such that* $\partial\tilde\Omega = \partial\Omega = \Gamma$ *and* $\tilde R'^{2T} = R^{2T}$.

Philosophical question: From what "material" can such an $\tilde\Omega$ be constructed?

Answer in advance: $\tilde\Omega$ is a spectrum of a relevant C*-algebra determined by R^{2T}.

2. Eikonal algebra in Acoustics

2.1. Reachable sets

Return to the system (1.1)–(1.3).

Controllability For an open $\sigma \subset \Gamma$, define a *reachable set*

$$\mathcal{U}^s_\sigma := \left\{u^f(\,\cdot\,,T) \,|\, \operatorname{supp} f \subset \overline{\sigma} \times [T-s,T]\right\} \subset \mathcal{H} \qquad (0 < s \leqslant T)$$

of delayed controls acting from σ. Denote

- $\Omega^s[\sigma] := \{x \in \Omega \,|\, \operatorname{dist}(x,\sigma) < s\}$ (the metric neighborhood of σ)
- $\mathcal{H}\langle\Omega^s[\sigma]\rangle := \{y \in \mathcal{H} \,|\, \operatorname{supp} y \subset \overline{\Omega^s[\sigma]}\}$ (the subspace of functions supported in $\Omega^s[\sigma]$).

A finiteness of the wave propagation speed in Ω implies $\mathcal{U}^s_\sigma \subset \mathcal{H}\langle\Omega^s[\sigma]\rangle$. The Holngren-John-Tataru uniqueness theorem leads to the relation

$$\overline{\mathcal{U}^s_\sigma} = \mathcal{H}\langle\Omega^s[\sigma]\rangle \tag{2.1}$$

(closure in $\mathcal{H}$), which is referred to as a *local approximate boundary controllability* of the system (1.1)–(1.3) [1]. For $T >$ diam Ω, one has $\overline{\mathcal{U}^T_\sigma} = \mathcal{H}$.

Eikonals Let P^s_σ be the projection in $\mathcal{H}$ onto $\overline{\mathcal{U}^s_\sigma}$. By (2.1) one has

$$P^s_\sigma y = \begin{cases} y & \text{in } \Omega^s[\sigma] \\ 0 & \text{in } \Omega\backslash\Omega^s[\sigma] \end{cases}, \tag{2.2}$$

i.e., P^s_σ cuts off functions on $\Omega^s[\sigma]$. An operator

$$\tau_\sigma := \int_0^T s\, dP^T_\sigma$$

is called an *eikonal*. If $T > \operatorname{diam}\Omega$, then (2.2) implies

$$(\tau_\sigma y)(x) = \operatorname{dist}(x,\sigma)\, y(x), \qquad x \in \Omega\,,$$

i.e., τ_σ is a multiplication by the distant function. It is a bounded self-adjoint operator in $\mathcal{H}$.

2.2. Algebra $\mathfrak{T}$

Recall that a *spectrum* $\widehat{\mathcal{A}}$ of a commutative Banach algebra $\mathcal{A}$ is the set of its maximal ideals endowed with the Gelfand topology [7], [8]. If $\mathcal{A}$ and $\mathcal{B}$ are two isometrically isomorphic algebras (we write $\mathcal{A} \stackrel{\rm isom}{=} \mathcal{B}$), then their spectra are homeomorphic (as topological spaces; we write $\widehat{\mathcal{A}} \stackrel{\rm hom}{=} \widehat{\mathcal{B}}$). For the algebra of real continuous functions $C(\Omega)$, one has $\widehat{C(\Omega)} \stackrel{\rm hom}{=} \Omega$ [7], [8].

For a set $S \subset \mathcal{A}$, by $\vee S$ we denote the minimal norm-closed subalgebra of $\mathcal{A}$, which contains S. Let $\mathfrak{B}(\mathcal{H})$ be the (normed) algebra of bounded operators in $\mathcal{H}$. By $\mathfrak{T} := \vee\{\tau_\sigma \mid \sigma \subset \Gamma\} \subset \mathfrak{B}(\mathcal{H})$ we denote the (sub)algebra generated by eikonals.

Theorem 1. *If* $T > \operatorname{diam}\Omega$ *then* $\mathfrak{T} \stackrel{\rm isom}{=} C(\Omega)$ *and hence* $\widehat{\mathfrak{T}} \stackrel{\rm hom}{=} \widehat{C(\Omega)} \stackrel{\rm hom}{=} \Omega$.

2.3. Solving IP

Connecting operator With the system (1.1)–(1.3) one associates a *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$ defined by the relation

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \left(u^f(\,\cdot\,,T), u^g(\,\cdot\,,T)\right)_{\mathcal{H}}\,, \qquad f,g \in \mathcal{F}^T.$$

It is a positive bounded operator. The following is a key fact of our approach (the Boundary Control method).

Proposition 1. *The operator* C^T *is determined by the response operator* R^{2T} *via a simple and explicit formula [1], [3].*

Isometry U^T By the definitions, the map

$$U^T : \mathcal{U}^T_\Gamma \ni u^f(\,\cdot\,,T) \mapsto (C^T)^{\frac12} f \in \mathcal{F}^T$$

is an isometry. For $T > \operatorname{diam}\,\Omega$, one has $\overline{\mathcal{U}^T_\Gamma} = \mathcal{H}$, and U^T is a unitary operator from $\mathcal{H}$ onto $\overline{(C^T)^{\frac12}\mathcal{F}^T}$.

Let $\tilde P^s_\sigma := U^T P^s_\sigma (U^T)^*$ be the projection in $\mathcal{F}^T$ onto the subspace

$$\overline{\left\{(C^T)^{\frac12} f \mid \operatorname{supp} f \subset \overline{\sigma} \times [T-s, T]\right\}} = U^T \overline{\mathcal{U}^s_\sigma}\,.$$

By Proposition 1, $\tilde P^s_\sigma$ *is determined by the response operator* R^{2T}.

By the latter, the operators

$$\tilde\tau_\sigma := U^T \tau_\sigma (U^T)^* = \int_0^T s\, d\left[U^T P^s_\sigma (U^T)^*\right] = \int_0^T s\, d\,\tilde P^s_\sigma \tag{2.3}$$

are also determined by R^{2T}. We define an algebra $\tilde{\mathfrak{T}} := U^T \mathfrak{T} (U^T)^* \subset \mathfrak{B}\left(\overline{(C^T)^{\frac12}\mathcal{F}^T}\right)$. By the definition, we have

$$\tilde{\mathfrak{T}} = U^T \left[\vee\{\tau_\sigma \mid \sigma \subset \Gamma\}\right] (U^T)^* = \vee\{\tilde\tau_\sigma \mid \sigma \subset \Gamma\}\,. \tag{2.4}$$

By the aforesaid, this algebra and its spectrum $\widehat{\tilde{\mathfrak{T}}} =: \tilde\Omega$ *are determined by the response operator* R^{2T}. Since $\tilde{\mathfrak{T}} \stackrel{\rm isom}{=} \mathfrak{T}$, with regards to Theorem 1 one has

$$\Omega \stackrel{\rm hom}{=} \widehat{\mathfrak{T}} \stackrel{\rm hom}{=} \widehat{\tilde{\mathfrak{T}}} =: \tilde\Omega \tag{2.5}$$

as $T > \operatorname{diam}\Omega$.

Reconstruction The response operator R^{2T} (provided $T > \operatorname{diam}\Omega$) determines the manifold Ω up to a homeomorphism by the following scheme:

$$R^{2T} \overset{\text{Prop 1}}{\Rightarrow} C^T \Rightarrow \overline{\left\{(C^T)^{\frac{1}{2}} f \mid \operatorname{supp} f \subset \overline{\sigma} \times [T-s, T]\right\}}_{\sigma \subset \Gamma} \Rightarrow$$
$$\Rightarrow \{\tilde{P}^s_\sigma \mid \sigma \subset \Gamma\} \overset{(2.3)}{\Rightarrow} \{\tilde{\tau}_\sigma \mid \sigma \subset \Gamma\} \overset{(2.4)}{\Rightarrow} \tilde{\mathfrak{T}} \Rightarrow$$
$$\Rightarrow \hat{\tilde{\mathfrak{T}}} \overset{(2.5)}{=} \tilde{\Omega} \overset{\text{hom}}{=} \Omega\,.$$

Then, one can endow $\tilde{\Omega}$ with a proper Riemannian metric and identify $\partial\tilde{\Omega}$ with Γ (see, e.g., [5]).

As a result, we get a Riemannian manifold $\tilde{\Omega}$, which is isometric to the original (unknown) Ω by construction, and $\tilde{R}^{2T} = R^{2T}$ does hold. The inverse problem for the system (1.1)–(1.3) is solved.

3. Eikonal algebra in Electrodynamics

3.1. Maxwell system

Turn to the system (1.4)–(1.6). The Hilbert space $\vec{L}_2(\Omega)$ of the square-summable vector fields (sections of the tangent bundle $T\Omega$) contains the subspace of curls $\mathcal{C} := \left\{\operatorname{curl} h \,\middle|\, h, \operatorname{curl} h \in \vec{L}_2(\Omega)\right\}$.

Electric reachable sets For an open $\sigma \subset \Gamma$, define

$$\mathcal{E}^s_\sigma := \left\{e^f(\,\cdot\,,T) \mid \operatorname{supp} f \subset \overline{\sigma} \times [T-s,T]\right\} \subset \mathcal{C} \qquad (0 < s \leqslant T)\,.$$

We denote $\mathcal{C}\langle\Omega^s[\sigma]\rangle := \{y \in \mathcal{C} \mid \operatorname{supp} y \subset \overline{\Omega^s[\sigma]}\}$. The finiteness of the electromagnetic wave propagation speed in Ω implies $\mathcal{E}^s_\sigma \subset \mathcal{C}\langle\Omega^s[\sigma]\rangle$.

Controllability The Eller-Isakov-Nakamura-Tataru uniqueness theorem leads to

$$\overline{\mathcal{E}^s_\sigma} = \mathcal{C}\langle\Omega^s[\sigma]\rangle \tag{3.1}$$

(the *local boundary controllability*). For $T >$ diam Ω, one has $\overline{\mathcal{E}^T_\sigma} = \mathcal{C}$.

Projections Let E^s_σ be the projection in $\mathcal{C}$ onto $\overline{\mathcal{E}^s_\sigma}$. This projection acts in more complicated way than its acoustic analog: its action is not reduced to cutting off fields. Moreover, in the general case, for the different σ and σ' the projections E^s_σ and $E^{s'}_{\sigma'}$ *do not commute.*

Eikonals An operator

$$\varepsilon_\sigma := \int_0^T s\, dE^s_\sigma$$

acts in the space $\mathcal{C}$ and is called an *eikonal*. Since $\operatorname{diam}\Omega < \infty$, ε_σ is a bounded positive self-adjoint operator. In the general case, for $\sigma \neq \sigma'$ the eikonals E^T_σ and $E^T_{\sigma'}$ *do not commute.* The following fact plays a key role.

Lemma 1. *(M.N.Demchenko [6]) The representation*

$$(\varepsilon_\sigma y)(x) = \operatorname{dist}(x,\sigma)\, y(x) + \left(K^T y\right)(x), \qquad x \in \Omega$$

holds with a compact operator $K^T : \mathcal{C} \to \vec{L}_2\,(\Omega)$.

3.2. Algebra $\mathfrak{E}$

Let $\mathfrak{B}(\mathcal{C})$ be the (normed) algebra of bounded operators in $\mathcal{C}$. It contains the two-side ideal $\mathfrak{K}(\mathcal{C})$ of compact operators.

We denote by

$$\mathfrak{E} := \vee\{\varepsilon_\sigma \mid \sigma \subset \Gamma\}$$

the algebra generated by (electric) eikonals. Also, we denote $\mathfrak{K}[\mathfrak{E}] := \mathfrak{E} \cap \mathfrak{K}(\mathcal{C})$ and introduce the factor-algebra

$$\dot{\mathfrak{E}} := \mathfrak{E}/\mathfrak{K}[\mathfrak{E}]$$

Theorem 2. *(M.N.Demchenko [6]) The factor-algebra $\dot{\mathfrak{E}}$ is a commutative Banach algebra. The relation $\dot{\mathfrak{E}} \overset{\text{isom}}{=} C(\Omega)$ holds and implies $\widehat{\dot{\mathfrak{E}}} \overset{\text{hom}}{=} \Omega$.*

3.3. Solving IP

Connecting operator A Maxwell *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$ is introduced by the relation

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \left(e^f(\,\cdot\,,T), e^g(\,\cdot\,,T)\right)_{\mathcal{C}}$$

for smooth controls $f, g \in \mathcal{F}^T$ vanishing near $t = 0$ [3]. In contrast to the scalar (acoustic) case, this C^T is an unbounded operator. However, the following principal fact of the BC-method remains valid.

Proposition 2. *The operator C^T is determined by the response operator R^{2T} via a simple and explicit formula [3], [5].*

Isometry U^T By the definitions, the map

$$U^T : \mathcal{E}^T_\Gamma \ni e^f(\,\cdot\,,T) \mapsto (C^T)^{\frac{1}{2}} f \in \mathcal{F}^T$$

is an isometry. For $T > \operatorname{diam}\Omega$, by (3.1) one has $\overline{\mathcal{E}^T_\Gamma} = \mathcal{C}$, and U^T is a unitary operator from $\mathcal{C}$ onto $\overline{(C^T)^{\frac{1}{2}}\mathcal{F}^T} \subset \mathcal{F}^T$.

By Proposition 2, the projection $\tilde{E}^s_\sigma := U^T E^s_\sigma (U^T)^*$ in $\overline{(C^T)^{\frac{1}{2}}\mathcal{F}^T}$ onto the subspace

$$\overline{\left\{(C^T)^{\frac{1}{2}} f \mid \operatorname{supp} f \subset \overline{\sigma} \times [T-s, T]\right\}} = U^T \overline{\mathcal{E}^s_\sigma}$$

is determined by the response operator R^{2T}.

An operator

$$\tilde{\varepsilon}^T_\sigma := U^T \varepsilon^T_\sigma (U^T)^* = \int_0^T s\, d\left[U^T E^s_\sigma (U^T)^*\right] = \int_0^T s\, d\, \tilde{E}^s_\sigma$$

acts in $\overline{(C^T)^{\frac{1}{2}}\mathcal{F}^T}$ and *is determined by the response operator R^{2T}.*

An algebra

$$\tilde{\mathfrak{E}} := U^T \mathfrak{E} (U^T)^* = U^T \left[\vee\{\varepsilon_\sigma \mid \sigma \subset \Gamma\}\right] (U^T)^* = \vee\{\tilde{\varepsilon}_\sigma \mid \sigma \subset \Gamma\}$$

is a subalgebra of $\mathfrak{B}\left(\overline{(C^T)^{\frac{1}{2}}\mathcal{F}^T}\right)$. By the aforesaid, this algebra, the factor-algebra $\dot{\tilde{\mathfrak{E}}} := \tilde{\mathfrak{E}}/\mathfrak{K}[\tilde{\mathfrak{E}}]$ and its spectrum

$$\widehat{\dot{\tilde{\mathfrak{E}}}} =: \tilde{\Omega}$$

are determined by the response operator R^{2T}.

The isometry $\tilde{\mathfrak{C}} \overset{\rm isom}{=} \mathfrak{C}$ implies the isometry of the factors $\dot{\tilde{\mathfrak{C}}} \overset{\rm isom}{=} \dot{\mathfrak{C}}$. Theorem 2 leads to

$$\Omega \overset{\rm hom}{=} \widehat{\dot{\mathfrak{C}}} \overset{\rm hom}{=} \widehat{\dot{\tilde{\mathfrak{C}}}} =: \tilde{\Omega}\,.$$

Reconstruction The response operator R^{2T} (provided $T > \operatorname{diam}\Omega$) determines the manifold Ω up to a homeomorphism by the following scheme:

$$\begin{aligned} &R^{2T} \Rightarrow C^T \Rightarrow \overline{\left\{(C^T)^{\frac{1}{2}} f \,\middle|\, \operatorname{supp} f \subset \overline{\sigma} \times [T-s, T]\right\}}_{\sigma \subset \Gamma} \Rightarrow \\ &\Rightarrow \{\tilde{E}^s_\sigma \,|\, \sigma \subset \Gamma\} \Rightarrow \{\tilde{\varepsilon}_\sigma \,|\, \sigma \subset \Gamma\} \Rightarrow \tilde{\mathfrak{C}} \Rightarrow \dot{\tilde{\mathfrak{C}}} \\ &\Rightarrow \widehat{\dot{\tilde{\mathfrak{C}}}} =: \tilde{\Omega} \overset{\rm hom}{=} \Omega\,. \end{aligned}$$

Then, one can endow $\tilde{\Omega}$ with a proper Riemannian metric and identify $\partial\tilde{\Omega}$ with Γ (see, e.g., [5]).

As a result, we get a Riemannian manifold $\tilde{\Omega}$, which is isometric to the original (unknown) Ω by construction, and $\tilde{R}^{2T} = R^{2T}$ does hold. The inverse problem for the Maxwell system (1.4)–(1.6) is thus solved.

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