NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2013, 4 (4), P. 529–537

EFIMOV'S EFFECT FOR PARTIAL INTEGRAL OPERATORS OF FREDHOLM TYPE

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PACS 02.30.Tb

We study the existence of an infinite number of eigenvalues (the existence of Efimov's effect) for a self-adjoint partial integral operators. We prove a theorem on the necessary and sufficient conditions for the existence of Efimov's effect for the Fredholm type partial integral operators.

Keywords: essential spectrum, discrete spectrum, Efimov's effect, partial integral operator..

1. Introduction

Linear equations and operators involving partial integrals appear in elasticity theory [1–3], continuum mechanics [2, 4–6], aerodynamics [7] and in PDE theory [8, 9]. Self-adjoint partial integral operators arise in the theory of Schrodinger operators [10–13]. Spectral properties of a discrete Schrodinger operator H are tightly connected (see [13,14]) with those of the partial integral operators which participate in the presentation of operator H.

Let Ω_1 and Ω_2 be closed boundary subsets in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, respectively. The partial integral operator (PIO) of Fredholm type in the space $L_p(\Omega_1 \times \Omega_2)$, $p \geq 1$ is an operator of the following form [15]

$$T = T_0 + T_1 + T_2 + K, \tag{1}$$

where operators T_0, T_1, T_2 and K are defined by the following formulas:

$$T_0 f(x,y) = k_0(x,y) f(x,y),$$
$$T_1 f(x,y) = \int_{\Omega_1} k_1(x,s,y) f(s,y) ds,$$
$$T_2 f(x,y) = \int_{\Omega_2} k_2(x,t,y) f(x,t) dt,$$
$$K f(x,y) = \int_{\Omega_1} \int_{\Omega_2} k(x,y;s,t) f(s,t) ds dt.$$

Here k_0, k_1, k_2 and k are given measurable functions on $\Omega_1 \times \Omega_2$, $\Omega_1^2 \times \Omega_2^2$, $\Omega_1 \times \Omega_2$ and $(\Omega_1 \times \Omega_2)^2$, respectively. All integrals have to be understood in the Lebesgue sense.

In 1975, Likhtarnikov and Vitova [16] studied spectral properties of partial integral operators. In [16], the following restrictions were imposed: $k_1(x,s) \in L_2(\Omega_1 \times \Omega_1)$, $k_2(y,t) \in L_2(\Omega_2 \times \Omega_2)$ and $T_0 = K = 0$. In [17] spectral properties of PIO with positive kernels were studied (under the restriction $T_0 = K = 0$). Kalitvin and Zabrejko [18] studied the spectral properties of PIO in the space $L_p, p \geq 1$. Kernels of PIO in all mentioned articles are functions of two variables and $T_0 = 0$. In [19], a full spectral description of self-adjoint

PIO in the space $C([a,b]\times[c,d])$ with continuous kernels was given. Self-adjoint PIO with $T_0 \neq 0$ were first studied in [10], where theorem about essential spectrum was proved. The finiteness and infiniteness of a discrete spectrum of self-adjoint PIO arising in the theory of Schrodinger operators was investigated in [11–13].

In [20], PIOs in some functional spaces were investigated and a number of applications were considered. Some important spectral properties of PIO in the space L_2 are still open. The present paper is dedicated to the mentioned problem.

We study the existence of an infinite number of eigenvalues (the existence Efimov's effect) for a self-adjoint partial integral operators.

2. The notations and necessary information

Let A be a linear self-adjoint operator in the Hilbert space $\mathcal{H}$. Resolvent set, spectrum, essential spectrum and discrete spectrum of the operator A are denoted by ρ, σ, σ_{ess} and σ_{disc}, respectively (see [21]).

We define the numbers

$$\mathcal{E}_{\min}(A) = \inf\{\lambda : \lambda \in \sigma_{ess}(A)\},$$

$$\mathcal{E}_{\max}(A) = \sup\{\lambda : \lambda \in \sigma_{ess}(A)\}.$$

We call the number $\mathcal{E}_{\min}(A)$ ($\mathcal{E}_{\max}(A)$) the lower (the higher) boundary of the essential spectrum of A.

Let $\Omega_1 = [a,b]^{\nu_1}$, $\Omega_2 = [c,d]^{\nu_2}$ and k_0, k_1, k_2 are continuous functions on $\Omega_1\times\Omega_2$, $\Omega_1^2\times\Omega_2$, $\Omega_1\times\Omega_2^2$ respectively, k_0 is real function, $k_1(x,s,y)=\overline{k_1(s,x,y)}, k_2(x,t,y)=\overline{k_2(x,y,t)}$. We define the linear self-adjoint bounded operator H in the Hilbert space $L_2(\Omega_1\times\Omega_2)$ by rule

$$H = T_0 - (T_1 + T_2). \tag{2}$$

We set

$$T = T_1 + T_2.$$

For the essential spectrum of the operators H and T the equalities

$$\sigma_{ess}(T) = \sigma(T_1)\cup\sigma(T_2),$$

$$\sigma_{ess}(H) = \sigma(T_0 - T_1)\cup\sigma(T_0 - T_2) \tag{3}$$

are held (see [22],[10]).

Let $k_1(x,s,y) = \alpha = const > 0, k_2(x,t,y) = \beta = const > 0$ in the model (2). Then at the $\mathcal{E}_{\min}(H) = 0$ the Efimov's effect (the existence infinite numbers of eigenvalues below the lower boundary $\mathcal{E}_{\min}(H)$ of the essential spectrum) in the model (2) was demonstrated by S. Albeverio, S.N.Lakaev, Z.I. Muminov [11] and Rasulov T.Kh. [12].

We study the existence Efimov's effect in the model (2) in the case $\mathcal{E}_{\min}(H) \neq 0$. Consider this problem for the function $k_0(x,y)$ of the form $k_0(x,y) = u(x)u(y)$ and $k_0(x,y) = u(x) + u(y)$.

Let $u(x)$ and $v(y)$ be a continuous nonnegative function on Ω_1 and Ω_2, respectively and suppose $k_1(x,s,y) = k_1(x,s)$, $k_2(x,t,y) = k_2(y,t)$. We define the self-adjoint compact integral operators $Q_1 : L_2(\Omega_1) \to L_2(\Omega_1)$ and $Q_2 : L_2(\Omega_2) \to L_2(\Omega_2)$ by the following equalities

$$Q_1\varphi(x) = \int_{\Omega_1} k_1(x,s)\varphi(s)ds, \quad Q_2\psi(y) = \int_{\Omega_2} k_2(y,t)\psi(t)dt$$

and suppose that $Q_1 \geq \theta, \ \ Q_2 \geq \theta.$

We define by V_1 and V_2 multiplication on the space $L_2(\Omega_1)$ and $L_2(\Omega_2)$ are acting by the following formulas

$$V_1\varphi(x) = u(x)\varphi(x), \quad V_2\psi(y) = v(y)\psi(y).$$

We consider the operators $H_k, k = 1,2$ in the Friedrichs model:

$$H_1 = V_1 - Q_1, \tag{4}$$

$$H_2 = V_2 - Q_2. \tag{5}$$

3. Efimov's effect for PIO

Let $u(x) \geq 0, \ \ x \in \Omega_1, v(y) \geq 0, \ \ y \in \Omega_2$ and $u^{-1}(\{0\}) \neq \emptyset, \ \ v^{-1}(\{0\}) \neq \emptyset.$

Theorem 3.1. *Let $k_0(x,y) = u(x) + v(y)$, $u(x) \geq 0$, $v(y) \geq 0$ and $Q_1 \geq \theta, \ \ Q_2 \geq \theta.$*

(a) the $\sigma_{ess}(H) = \sigma(H_0)$ iff the $\sigma_{disc}(H) = \emptyset$;

(b) if $\sigma_{disc}(H) \neq \emptyset$, then $\sigma_{disc}(H_1) \neq \emptyset$ and $\sigma_{disc}(H_2) \neq \emptyset$;

(c) if $\sigma_{disc}(H) \neq \emptyset$, then $\mathcal{E}_{\min}(H) = \inf\{\lambda : \lambda \in \sigma_{disc}(H_1) \cup \sigma_{disc}(H_2)\}$;

(d) Efimov's effect exists in the model (2) iff Efimov's effect exists in Friedrich's model (4) and $\sigma_{disc}(H_2) \neq \emptyset$ or Efimov's effect exists in Friedrich's model (5) and $\sigma_{disc}(H_1) \neq \emptyset$.

Proof. We define the operator $W = H_1 \otimes E + E \otimes H_2$ on the space $L_2(\Omega_1) \otimes L_2(\Omega_2)$. For the spectrum of the operator W we have [18]

$$\sigma(W) = \sigma(H_1) + \sigma(H_2).$$

But, the operators W and H is unitary equivalent (see [10]), i.e. $W \cong H$. Consequently, that

$$\sigma(H) = \sigma(H_1) + \sigma(H_2). \tag{6}$$

Also, if we define the operators A_1 and A_2 by

$$A_1 = H_1 \otimes E + E \otimes V_2, \quad A_2 = V_1 \otimes E + E \otimes H_2$$

we see $A_1 \cong T_0 - T_1, \ \ A_2 \cong T_0 - T_2$. Thus we have

$$\sigma(T_0 - T_1) = \sigma(H_1) + \sigma(V_2),$$

$$\sigma(T_0 - T_2) = \sigma(V_1) + \sigma(H_2).$$

Then, by the equality (6) for the essential spectrum of the operator H we obtain

$$\sigma_{ess}(H) = (\sigma(H_1) + \sigma(V_2)) \cup (\sigma(V_1) + \sigma(H_2)). \tag{7}$$

On the other hand

$$\sigma_{ess}(H_k) = \sigma(V_k), \quad k = 1,2. \tag{8}$$

Now, from the equalities (6), (7) and (8) it follows proof of theorem 1. □

Corollary 3.1. *Let be $k_0(x,y) = u(x) + v(y)$ and $k_1(x,s,y) = k_1(x,s)$, $k_2(x,t,y) = k_2(y,t)$.*

a) for the existence of Efimov's effect in model (2) *it is necessary, that* $\dim(Ran(Q_1)) = \infty$ *or* $\dim(Ran(Q_2)) = \infty$;

b) if $\dim(Ran(Q_1)) < \infty$ *and* $\dim(Ran(Q_2)) < \infty$, *then Efimov's effect is absent in model* (2).

Suppose, that $k_0(x,y) \geq 0, 0 \in Ran(k_0)$ and $T_k \geq \theta, k = 1,2$. Let $N_0(H)$ be the number of all eigenvalues (with account multiplicity) below the lower boundary of the essential spectrum in model (2), i.e.

$$N_0(H) = \sum_{\lambda \in \sigma_{disc}(H),\ \lambda < \mathcal{E}_{\min}(H)} n_H(\lambda),$$

where $n_H(\lambda)$ – the multiplicity of the eigenvalue λ of the operator H and $N(T)$ is the number of all eigenvalues(with account multiplicity) of the discrete spectrum of operator T, i.e.

$$N(T) = \sum_{\lambda \in \sigma_{disc}(T)} n_T(\lambda).$$

Theorem 3.2. *Let the relation*

$$T_0 \geq (\mathcal{E}_{\min}(H) + \mathcal{E}_{\max}(T)) \cdot E$$

is hold, where E - identical operator. Then

$$N_0(H) \leq N(T).$$

Proof. We have

$$\sigma_{ess}(\mathcal{E}_{\max}(T) \cdot E - T) = \{\xi : \ \xi = \mathcal{E}_{\max}(T) - \lambda, \ \lambda \in \sigma_{ess}(T)\}.$$

Then

$$\sigma_{ess}(\mathcal{E}_{\max}(T) \cdot E - T) \subset [0, \infty)$$

and

$$0 \in \sigma_{ess}(\mathcal{E}_{\max}(T) - T), \ \ N(\mathcal{E}_{\max}(T) \cdot E - T) = N(T).$$

Hence it follows

$$\mathcal{E}_{\min}(T) = \mathcal{E}_{\min}(\mathcal{E}_{\max}(T) \cdot E - T) = 0.$$

Let the inequality

$$T_0 \geq (\mathcal{E}_{\min}(H) + \mathcal{E}_{\max}(T)) \cdot E$$

hold. Consequently, we obtain

$$\mathcal{E}_{\max}(T) \cdot E - T \leq H - \mathcal{E}_{\min}(H) \cdot E$$

Then by lemma 2.1 [23] we have

$$\mu_k(\mathcal{E}_{\max}(T) \cdot E - T) \leq \mu_k(H - \mathcal{E}_{\min}(H) \cdot E), \ \ k \in \{1, 2, \ \ldots, \ \ N(T) + 1\}. \tag{10}$$

where $\{\mu_k(A)\}$ – the set of all eigenvalues of operator A below the lower boundary of the essential spectrum, which was constructed by the mini-max principle. By theorem 2.1 [23] we have

$$\mu_{N(T)+1}(\mathcal{E}_{\max}(T)\cdot E - T) = 0.$$

Therefore, from inequality (10)

$$\mu_{N(T)+k}(H - \mathcal{E}_{\min}(H)\cdot E) = 0, \quad k \in \mathbb{N}\cup\{0\},$$

i.e. the number of negative eigenvalues of operator $H - \mathcal{E}_{\min}(H)\cdot E$ taking into account multiplicity, is no more than $N(T)$. Consequently, the number of eigenvalues of the operator H below the lower boundary $\mathcal{E}_{\min}(H)$ of the essential spectrum, also will be no more than $N(T)$, i.e.

$$N_0(H) \leq N(T).$$ □

Corollary 3.2. *Let be $N(T) < \infty$. If*

$$\mathcal{E}_{\min}(H) + \mathcal{E}_{\max}(T) \leq 0,$$

then Efimov's effect is absent in model (2).

Let $u(x)$ and $v(y)$ be nonnegative continuous functions on Ω_1 and Ω_2, respectively. Suppose, that $u^{-1}(\{0\}) = \{x^{min}\}, v^{-1}(\{0\}) = \{y^{min}\}$.

Theorem 3.3. *Let be $k_0(x,y) = u(x)v(y)$. Then*

(a) the equality $\mathcal{E}_{\min}(H) = -\max\{\|Q_1\|, \|Q_2\|\}$ holds;

(b) if $\dim(Ran(Q_1)) < \infty$ and $\dim(Ran(Q_2)) < \infty$, then Efimov's effect is absent in model (2).

Proof. (a) We define the family $\{W_1(\alpha)\}_{\alpha\in\Omega_2}$ of self-adjoint operators in Fredrich's model on the space $L_2(\Omega_1)$:

$$W_1(\alpha)\varphi(x) = u(x)v(x)\varphi(x) - Q_1\varphi(x).$$

We set

$$\sigma_1 = \{\lambda \in (-\infty, 0): \ \textit{for some } \alpha_0 \in \Omega_2 \ \textit{the number} \ \lambda \ \textit{is eigenvalue of the operator} \ W_1(\alpha_0)\}.$$

Then by the theorem 6 from [24] we have

$$\sigma(T_0 - T_1) = \sigma_0 \cup \sigma_1,$$

where $\sigma_0 = \sigma(T_0)$. However,

$$((T_0 - T_1)f, f) \geq -(T_1 f, f) \geq -\|Q_1\|, \quad f \in L_2(\Omega_1\times\Omega_2),$$

because $\|T_1\| = \|Q_1\|$. Conversely, $W_1(y_{\min}) = -Q_1$, i.e. the number $-\|Q_1\|$ is eigenvalue of the operator $W_1(y_{\min})$. Consequently, $-\|Q_1\| \in \sigma_1$, i.e. $-\|Q_1\| \in \sigma(T_0 - T_1)$. Then we have $\mathcal{E}_{\min}(T_0 - T_1) = \|Q_1\|$. Analogously, for the operator $T_0 - T_2$ we obtain, that $\mathcal{E}_{\min}(T_0 - T_2) = -\|Q_2\|$. Finally, from (3) follows, that $\mathcal{E}_{\min}(H) = -\max\{\|Q_1\|, \|Q_2\|\}$.

(b) By statement (a) of theorem 3.3 we have $\mathcal{E}_{\min}(H) = -\max\{\|Q_1\|, \|Q_2\|\}$. However, $\mathcal{E}_{\max}(T) = \max\{\|Q_1\|, \|Q_2\|\}$ (see [25]) Consequently, the condition of theorem 2 is satisfied. Still, by theorem 3.1 from [25] (also see [18]) we have

$$N(T) = \sum_{\substack{p+q \notin \sigma_{ess}(T), \\ p \in \sigma_{disc}(Q_1), \\ q \in \sigma_{disc}(Q_2)}} n_{Q_1}(p) \cdot n_{Q_2}(q),$$

where $\sigma_{ess}(T) = \sigma(Q_1) \cup \sigma(Q_2)$. Then, from the inequality $\dim(Ran(Q_1)) < \infty$ and $\dim(Ran(Q_2)) < \infty$, we obtain $N(T) < \infty$. Consequently, by theorem 3.2 Efimov's effect is absint in model (2). □

Remark 3.1. *The author's previous work [14] showed the existence of Efimov's effect in the case* $\dim(Ran(Q_1)) = 1$ *and* $\dim(Ran(Q_2)) = \infty$ *in the model* (2) *for the* $\mathcal{E}_{\min}(H) \neq 0$.

4. The examples

Example 4.1. On $\Omega = [0, 1]^\nu$, we consider the functions

$$u(x) = {x_1}^{\alpha_1} \cdot ... \cdot {x_\nu}^{\alpha_\nu}, \;\; v(y) = {y_1}^{\beta_1} \cdot ... \cdot {y_\nu}^{\beta_\nu},$$

where $\alpha_k \geq 0, \beta_j \geq 0, k, j \in \{1, .., \nu\}$.
In the space $L_2(\Omega^2)$, we define the operators

$$T_0 f(x, y) = (u(x) + v(y)) f(x, y),$$
$$T_1 f(x, y) = \int_\Omega exp(|x - s|) f(s, y) ds,$$
$$T_2 f(x, y) = \int_\Omega exp(|y - t|) f(x, t) dt,$$

where

$$|x| = \sqrt{{x_1}^2 + ... + {x_\nu}^2}.$$

If $\alpha_1 + ... + \alpha_\nu > 2\nu$ and $\beta_1 + ... + \beta_\nu > 2\nu$, then Efimov's effect exist for operator (2).

We define subsets $A_n (n \in \mathbb{N})$ and $B_n (n \in \mathbb{N})$ by the following way

$$A_n = \{t \in \Omega : \;\; 0 \leq t_i < \frac{1}{n}, \;\; i = 1, ..., \nu\}, \;\; n \in \mathbb{N},$$

$$B_n = \{t \in \Omega : \;\; 0 \leq t_i < \frac{1}{n+1}, \;\; i = 1, ..., \nu\}, \;\; n \in \mathbb{N}.$$

We have $B_n \subset A_n, \;\; n \in \mathbb{N}$. We set

$$G_n = A_n \backslash B_n, \;\; n \in \mathbb{N}.$$

Then $G_n \cap G_m = \emptyset$ at $n \neq m$ and $G_n \subset O_{\frac{1}{n}}(\theta) \cap \Omega$, where $O_r(\theta)$ – the open ball with radius r in center $\theta \in \mathbb{R}^\nu$($\theta$ – the zero element of the space $\mathbb{R}^\nu$). For the Lebesque measure $\mu(G_n)$ of the set G_n we obtain

$$\mu(G_n) = \mu(A_n) - \mu(B_n) = \frac{1}{n^\nu} - \frac{1}{(n+1)^\nu} > \frac{1}{n^\nu} - \frac{1}{n^\nu + 1} = \frac{1}{n^\nu (n^\nu + 1)} > \frac{1}{2n^{2\nu}}$$

for all $n \in \mathbb{N},\ \ n \geq 2$. On the other hand

$$\sup_{t \in G_n} u(t) = \left(\frac{1}{n}\right)^{\alpha_1 + \ \dots \ + \alpha_\nu}, \ \ n \in \mathbb{N}.$$

Consequently, if $\alpha_1 + \ \dots \ + \alpha_\nu > 2\nu$, then the following inequality

$$\sup_{t \in G_n} u(t) < \mu(G_n) \inf_{t,u \in G_n} k_1(t,u), \ \ n \in \mathbb{N} \backslash \{1\}.$$

holds, i.e. the condition of theorem 4.1 from [26] is satisfied. So, by theorem 4.1 operator $H_1 = V_1 - Q_1$ has an infinite number of negative eigenvalues.

Analogously, we show that, at the $\beta_1 + \ \dots \ + \beta_\nu > 2\nu$ the operator $H_2 = V_2 - Q_2$ has infinite number of negative eigenvalues.

Therefore, by the theorem 3.1 $\mathcal{E}_{\min}(H) \neq 0$ and Efimov's effect exists for the PIO H.

Remark 4.1. The statement of theorem 4.1 from [26] holds for the set $\Omega = [0,1]^\nu$ for arbitrary $\nu \in \mathbb{N}$. In work [26] proof was given for the simple case $\nu = 1$. The proof of theorem 4.1 [26] for the case $\nu \geq 2$ is analogous to the case $\nu = 1$.

Example 4.2. We consider the sequence $p_0 = 0,\ \ p_1 = 1/2,\ \ p_n = p_{n-1} + 1/2^n,\ \ n \in \mathbb{N}$. We set

$$q_n = \frac{p_n - p_{n-1}}{2}, \ \ n \in \mathbb{N}.$$

On $[0,1]$, we define the function

$$u(x) = \begin{cases} 0, & \text{if } x \in [0,1/2], \\ u_0(x), & \text{if } x \notin [0,1/2], \end{cases}$$

where $u_0(x) = \sum\limits_{n \in \mathbb{N}} \delta_n r_n(x)$,

$$r_\kappa(x) = \begin{cases} \dfrac{p_\kappa - x}{p_\kappa - q_{\kappa+1}}, & \text{if } x \in [p_\kappa, q_{\kappa+1}], \\[2ex] \dfrac{p_{\kappa+1} - x}{p_{\kappa+1} - q_{\kappa+1}}, & \text{if } x \in [q_{\kappa+1}, p_{\kappa+1}], \\[2ex] 0, & \text{if } x \notin [p_\kappa, p_{\kappa+1}], \end{cases}$$

$$\delta_1 = 1, \quad \delta_n \leq \left(\frac{\sqrt{2}}{3}\right)^n, \quad n \geq 2.$$

In the space $L_2[0,1]$, we consider the sequence of orthonormal functions

$$\varphi_n(y) = 2^{(n+1)/2} \sin \xi_n(y), \ \ n \in \mathbb{N},$$

where

$$\xi_\kappa(y) = \begin{cases} \dfrac{\pi}{p_\kappa - p_{\kappa-1}}(y - p_{\kappa-1}), & \text{if } y \in [p_{\kappa-1}, p_\kappa], \\ 0, & \text{if } y \notin [p_{\kappa-1}, p_\kappa]. \end{cases}$$

We define the kernel

$$k_2(y,t) = \sum_{n\in\mathbb{N}} \left(\frac{2}{3}\right)^n \varphi_n(y)\varphi_n(t). \tag{11}$$

Series (11) converges uniformly in the square $[0,1]^2$. Hence, the integral operator Q_2, defined by its kernel $k_2(y,t)$, is self-adjoint and positive in $L_2[0,1]$. It is clear, that $\dim(Ran(Q_1)) = 1$ and $\dim(Ran(Q_2)) = \infty$.

In the space $L_2([0,1]^2)$, we consider the model

$$H = H_0 - (\gamma T_1 + T_2), \ \ \gamma \geq \frac{2}{3}, \tag{12}$$

where

$$T_0 f(x,,y) = u(x)u(y)f(x,y),$$
$$T_1 f(x,y) = \int_0^1 f(s,y)ds,$$
$$T_2 f(x,y) = \int_0^1 k_2(y,t)f(x,t)dt.$$

Then

$$\mathcal{E}_{\min}(H) = \mathcal{E}_{\min}(H_0 - (\gamma T_1 + T_2))$$

and there exists Efimov's effect for operator (12) below the lower boundary $\mathcal{E}_{\min}(H) = -\gamma$ of the essential spectrum [14].

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