NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2013, 4 (6), P. 747–759

WEYL FUNCTION FOR SUM OF OPERATORS TENSOR PRODUCTS

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PACS 03.65 Nk

The boundary triplets approach is applied to the construction of self-adjoint extensions of the operator having the form $S = A \otimes I_T + I_A \otimes T$ where the operator A is symmetric and the operator T is bounded and self-adjoint. The formula for the γ-field and the Weyl function corresponding the the boundary triplet Π_S is obtained in terms of the γ-field and the Weyl function corresponding to the boundary triplet Π_A.

Keywords: operator extension, Weyl function, boundary triplet.

1. Introduction

The spectral theory of differential operators is very important for mathematics and has many applications in quantum physics (see, e.g., [1]. The theory of self-adjoint operators and especially of self-adjoint extensions of symmetric operators occupies a special place in the operator theory [2]. In many interesting problems of quantum physics (like the interaction of photons with electrons) the operators take on the form of the sum of tensor products [3], [4]. From general position, the extensions are usually described in terms of so-called boundary triplets [5]. Up to now, there is no boundary triplets method for obtaining all self-adjoint extensions of such an operator.

In particular, we consider a closed densely defined symmetric operator

$$S = A \otimes I_T + I_A \otimes T \tag{1.1}$$

where A is a closed densely defined symmetric operator on the separable Hilbert space $\mathfrak{H}_A$ and T is a bounded self-adjoint operator acting on the separable infinite dimensional Hilbert space $\mathfrak{H}_T$. Notice that the deficiency indices of S are infinite even if A has finite deficiency indices.

Our aim is to describe all self-adjoint extensions of S using the boundary triplet approach. More precisely, assuming that $\Pi_A = \{\mathcal{H}_A, \Gamma_0^A, \Gamma_1^A\}$ is a boundary triplet for A^* we construct a boundary triplet $\Pi_S = \{\mathcal{H}_S, \Gamma_0^S, \Gamma_1^S\}$ for S^*. In addition, using the γ-field $\gamma_A(\cdot)$ and the Weyl function $M_A(\cdot)$ of the boundary triplet Π_A we express the γ-field $\gamma_S(\cdot)$ and Weyl function $M_S(\cdot)$ of Π_S.

The present note generalizes results of [6]. In [6] on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}_+, \mathcal{H})$ the operator

$$\begin{gathered}(Sf)(x) = -\frac{d^2}{dt^2} f(t) + Tf(t),\\ f \in \mathrm{dom}\,(S) := \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}) : f(0) = f'(0) = 0\}.\end{gathered} \tag{1.2}$$

was considered where T is a bounded self-adjoint operator. One easily checks that the operator (1.2) has the form (1.1) where A acts on $L^2(\mathbb{R}_+)$ and is given by

$$(Af)(t) = -\frac{d^2}{dt^2}f(t), \quad f \in \mathrm{dom}\,(A) := \{W^{2,2}(\mathbb{R}_+) : f(0) = f'(0) = 0\}.$$

In [6] it was verified that $\Pi_S = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$

$$\Gamma_0 f := f(0), \quad \Gamma_1 f = f'(0), \quad f \in \mathrm{dom}\,(S^*) = W^{2,2}(\mathbb{R}_+, \mathcal{H}).$$

defines a boundary triplet for S^*. The corresponding Weyl function is given by $M_S(z) = i\sqrt{z - T}$, $z \in \mathbb{C}_\pm$.

Notation. Let $\mathfrak{H}$ and $\mathcal{H}$ be separable Hilbert spaces. The set of bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ is denoted by $[\mathfrak{H}_1, \mathfrak{H}_2]$; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. By $\mathfrak{S}_p(\mathfrak{H})$, $p \in (0, \infty]$, we denote the Schatten-v.Neumann ideals of compact operators on $\mathfrak{H}$; in particular, $\mathfrak{S}_\infty(\mathfrak{H})$ denotes the ideal of compact operators in $\mathfrak{H}$.

By $\mathrm{dom}\,(T)$, $\mathrm{ran}\,(T)$ and $\sigma(T)$ we denote the domain, range and spectrum of the operator T, respectively. The symbols $\sigma_p(\cdot)$, $\sigma_c(\cdot)$ and $\sigma_r(\cdot)$ stand for the point, continuous and residual spectrum of a linear operator. Recall that $z \in \sigma_c(H)$ if $\ker\,(H - z) = \{0\}$ and $\mathrm{ran}\,(H - z) \neq \overline{\mathrm{ran}\,(H - z)} = \mathfrak{H}$; $z \in \sigma_r(H)$ if $\ker\,(H - z) = \{0\}$ and $\overline{\mathrm{ran}\,(H - z)} \neq \mathfrak{H}$.

2. Preliminaries

2.1. Linear relations

A linear relation Θ in $\mathcal{H}$ is a closed linear subspace of $\mathcal{H} \oplus \mathcal{H}$. The set of all linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. Denote also by $\mathcal{C}(\mathcal{H})$ the set of all closed linear (not necessarily densely defined) operators in $\mathcal{H}$. Identifying each operator $T \in \mathcal{C}(\mathcal{H})$ with its graph $\mathrm{gr}\,(T)$ we regard $\mathcal{C}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$.

The role of the set $\widetilde{\mathcal{C}}(\mathcal{H})$ in extension theory becomes clear from Proposition 2.3. However, its role in the operator theory is substantially motivated by the following circumstances: in contrast to $\mathcal{C}(\mathcal{H})$, the set $\widetilde{\mathcal{C}}(\mathcal{H})$ is closed with respect to taking inverse and adjoint relations Θ^{-1} and Θ^*. The latter are given by: $\Theta^{-1} = \{\{g, f\} : \{f, g\} \in \Theta\}$ and

$$\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } \begin{pmatrix} h \\ h' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is called symmetric if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

2.2. Boundary triplets and proper extensions

Let us briefly recall some basic facts regarding boundary triplets. Let S be a densely defined closed symmetric operator with equal deficiency indices $n_\pm(S) := \dim(\mathfrak{N}_{\pm \mathrm{i}})$, $\mathfrak{N}_z := \ker\,(S^* - z)$, $z \in \mathbb{C}_\pm$, acting on some separable Hilbert space $\mathfrak{H}$.

Definition 2.1.
(i) *A closed extension* $\widetilde{S}$ *of* S *is called proper if* $\mathrm{dom}\,(S) \subset \mathrm{dom}\,(\widetilde{S}) \subset \mathrm{dom}\,(S^*)$.
(ii) *Two proper extensions* $\widetilde{S}'$, $\widetilde{S}$ *are called disjoint if* $\mathrm{dom}\,(\widetilde{S}') \cap \mathrm{dom}\,(\widetilde{S}) = \mathrm{dom}\,(S)$ *and transversal if in addition* $\mathrm{dom}\,(\widetilde{S}') + \mathrm{dom}\,(\widetilde{S}) = \mathrm{dom}\,(S^*)$.

We denote by $\mathrm{Ext_S}$ the set of all proper extensions of S completed by the non-proper extensions S and S^* is denoted. Any self-adjoint or maximal dissipative (accumulative) extension is proper.

Definition 2.2 ([7]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}\,(S^*) \to \mathcal{H}$ are linear mappings, is called a boundary triplet for S^* if the "abstract Green's identity"

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathrm{dom}\,(S^*). \tag{2.1}$$

is satisfied and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}\,(S^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective, i.e. $\mathrm{ran}\,(\Gamma) = \mathcal{H} \oplus \mathcal{H}$. $\Diamond$

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* always exists whenever $n_+(S) = n_-(S)$. Note also that $n_\pm(S) = \dim(\mathcal{H})$ and $\ker\,(\Gamma_0) \cap \ker\,(\Gamma_1) = \mathrm{dom}\,(S)$.

In general, the linear maps $\Gamma_j : \mathfrak{H} \longrightarrow \mathcal{H}$, $j = 0, 1$, are neither bounded nor closed. However, equipping the domain $\mathrm{dom}\,(S^*)$ with the graph norm

$$\|f\|^2_{S^*} := \|S^*\|^2 + \|f\|^2, \qquad f \in \mathrm{dom}\,(S^*), \tag{2.2}$$

one gets a Hilbert space, which is denoted by $\mathfrak{H}_+(S^*)$, and regarding the maps $\Gamma_j : \mathfrak{H} \longrightarrow \mathcal{H}$, $j = 0, 1$, as acting from $\mathfrak{H}_+(S^*)$ into $\mathcal{H}$ it turns out that that the operators $\Gamma_j : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}$, $j = 0, 1$, are bounded. In the following work we denote the operator $\Gamma_j : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}$ by $\widehat{\Gamma}_j : \mathfrak{H}_+(S^*) \longrightarrow \mathcal{H}$, $j = 0, 1$. From surjectivity it follows that $\mathrm{ran}\,(\widehat{\Gamma}) = \mathcal{H} \oplus \mathcal{H}$, where $\widehat{\Gamma} := (\widehat{\Gamma}_1, \widehat{\Gamma}_1)$. Notice that the abstract Green's identity (2.1) can be written as

$$(\widehat{S}^* f, g) - (f, \widehat{S}^* g) = (\widehat{\Gamma}_1 f, \widehat{\Gamma}_0 g) - (\widehat{\Gamma}_0 f, \widehat{\Gamma}_1 g), \quad f, g \in \mathrm{dom}\,(S^*). \tag{2.3}$$

where $\widehat{S}^*$ denotes the operator S^* regarded as acting from $\mathfrak{H}_+(S^*)$ into $\mathfrak{H}$.

With any boundary triplet Π one associates two canonical self-adjoint extensions $S_j := S^* \upharpoonright \ker\,(\Gamma_j)$, $j \in \{0, 1\}$. Conversely, for any extension $S_0 = S_0^* \in \mathrm{Ext_S}$ there exists a (non-unique) boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* such that $S_0 := S^* \upharpoonright \ker\,(\Gamma_0)$.

Using the concept of boundary triplets one can parameterize all proper extensions of A in the following way.

Proposition 2.3 ([8, 9]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. Then the mapping*

$$\mathrm{Ext_S} \ni \widetilde{\mathrm{S}} \to \Gamma \mathrm{dom}\,(\widetilde{\mathrm{S}}) = (\Gamma_0 \mathrm{f}, \Gamma_1 \mathrm{f})^\top : \ \mathrm{f} \in \mathrm{dom}\,(\widetilde{\mathrm{S}})\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.4}$$

establishes a bijective correspondence between the sets $\mathrm{Ext_S}$ *and* $\widetilde{\mathcal{C}}(\mathcal{H})$. *We write* $\widetilde{S} = S_\Theta$ *if* $\widetilde{S}$ *corresponds to* Θ *by* (2.4). *Moreover, the following holds:*

(i) $S^*_\Theta = S_{\Theta^*}$, *in particular,* $S^*_\Theta = S_\Theta$ *if and only if* $\Theta^* = \Theta$.

(ii) S_Θ *is symmetric (self-adjoint) if and only if* Θ *is symmetric (self-adjoint).*

(iii) *The extensions* S_Θ *and* S_0 *are disjoint (transversal) if and only if there is a closed (bounded) operator* B *such that* $\Theta = \mathrm{gr}\,(B)$. *In this case* (2.4) *takes the form*

$$S_\Theta := S_{\mathrm{gr}\,(B)} = S^* \upharpoonright \ker\,(\Gamma_1 - B\Gamma_0). \tag{2.5}$$

In particular, $S_j := S^* \upharpoonright \ker\,(\Gamma_j) = S_{\Theta_j}$, $j \in \{0, 1\}$, where $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\} = \mathrm{gr}\,(\mathbb{O})$ where $\mathbb{O}$ denotes the zero operator in $\mathcal{H}$. Note also that $\widetilde{\mathcal{C}}(\mathcal{H})$ contains the trivial linear relations $\{0\} \times \{0\}$ and $\mathcal{H} \times \mathcal{H}$ parameterizing the extensions S and S^*, respectively, for any boundary triplet Π.

2.3. γ-field and Weyl function

It is well known that Weyl functions are important tools in the direct and inverse spectral theory of Sturm-Liouville operators. In [8, 11] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator S with $n_+(S) = n_-(S) \leqslant \infty$. Following [8], we briefly recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.4 ([8,11])**.** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* and $S_0 = S^* \upharpoonright \ker(\Gamma_0)$. The operator valued functions $\gamma(\cdot) : \rho(S_0) \to [\mathcal{H}, \mathcal{H}]$ and $M(\cdot) : \rho(S_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := \big(\Gamma_0 \upharpoonright \mathfrak{N}_z\big)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(S_0), \tag{2.6}$$

are called the γ-field and the Weyl function, respectively, corresponding to the boundary triplet Π.

Clearly, the Weyl function can equivalently be defined by

$$M(z)\Gamma_0 f_z = \Gamma_1 f_z, \qquad f_z \in \mathfrak{N}_z, \quad z \in \rho(S_0).$$

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.6) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(S_0)$ and the following relations

$$\gamma(z) = \big(I + (z - \zeta)(S_0 - z)^{-1}\big)\gamma(\zeta), \qquad z, \zeta \in \rho(S_0), \tag{2.7}$$

and

$$M(z) - M(\zeta)^* = (z - \overline{\zeta})\gamma(\zeta)^*\gamma(z), \qquad z, \zeta \in \rho(S_0), \tag{2.8}$$

hold. Identity (2.8) yields that $M(\cdot)$ is $[\mathcal{H}]$-valued Nevanlinna function ($M(\cdot) \in R[\mathcal{H}]$), i.e. $M(\cdot)$ is $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C}_\pm$ satisfying

$$M(z) = M(\overline{z})^* \qquad \text{and} \qquad \frac{\mathrm{Im(M(z))}}{\mathrm{Im(z)}} \geqslant 0, \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

It also follows from (2.8) that $0 \in \rho(\mathrm{Im(M(z))})$ for all $z \in \mathbb{C}_\pm$.

2.4. Krein-type formula for resolvents

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*, $M(\cdot)$ and $\gamma(\cdot)$ the corresponding Weyl function and γ-field, respectively. For any proper (not necessarily self-adjoint) extension $\widetilde{S}_\Theta \in \mathrm{Ext_S}$ with non-empty resolvent set $\rho(\widetilde{S}_\Theta)$ the following Krein-type formula holds (cf. [8, 11, 12])

$$(S_\Theta - z)^{-1} - (S_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\overline{z}), \quad z \in \rho(S_0) \cap \rho(S_\Theta). \tag{2.9}$$

Formula (2.9) extends the known Krein formula for canonical resolvents to the case of any $S_\Theta \in \mathrm{Ext_S}$ with $\rho(S_\Theta) \neq \emptyset$. Moreover, due to relations (2.4), (2.5) and (2.6) formula (2.9) is connected with the boundary triplet Π. We emphasize, that this connection makes it possible to apply the Krein-type formula (2.9) to boundary value problems.

2.5. Operator spectral integrals

Let us recall some useful facts regarding operator spectral integrals. We follow in essentially [10, Section I.5.1].

Definition 2.5. Let $E(\cdot)$ be a spectral measure defined on the Borel sets $\mathcal{B}$ of the real axis $\mathbb{R}$. Let us assume that the support $\operatorname{supp}(E)$ is a bounded set, i.e. $\operatorname{supp}(E) \subset [a,b)$, $-\infty < a < b < \infty$. Further, let $G(\cdot) : [a,b) \longrightarrow \mathcal{B}(\mathfrak{H})$ be a Borel measurable function. Let $\mathfrak{Z}$ be a partition of the interval $[a,b)$ of the form $[a,b) = [\lambda_0, \lambda_1) \cup [\lambda_1, \lambda_2) \cup \cdots \cup [\lambda_{n-1}, \lambda_n)$ where $\lambda_0 = a$ and $\lambda_n = b$, and put $\Delta_m := [\lambda_{m-1}, \lambda_m)$, $m = 1, \ldots, n$. Thus $[a,b) = \bigcup_{m=1}^n \Delta_m$ and the intervals Δ_m are pairwise disjoint. Let $|\mathfrak{Z}| := \max_m |\Delta_m|$ and let

$$F_{\mathfrak{Z}}(G) := \sum_{m=1}^{n} G(x_m)E(\Delta_m), \qquad x_m \in \Delta_m.$$

If there is an operator $F_0 \in \mathcal{B}(\mathfrak{H})$ such that $\lim_{|\mathfrak{Z}|\to 0} \|F_{\mathfrak{Z}}(G) - F_0\| = 0$ independent of $\mathfrak{Z}$ and $\{x_m\}$, then F_0 is called the operator spectral integral of $G(\cdot)$ with respect to $E(\cdot)$ and is denoted by

$$F_0 = \int_a^b G(\lambda) dE(\lambda).$$

Remark 2.6. Similarly the operator spectral integral $\int_a^b dE(\lambda)G(\lambda)$ can be defined as above.

If $f(\cdot) : [a,b) \longrightarrow \mathfrak{H}$ is a Borel measurable function, then the vector spectral integral $\int_a^b dE(\lambda)f(\lambda)$ can be defined similarly.

Let us indicate some properties of the operator spectral integral.

(i) If $G(\lambda) := g(\lambda)I$ where $g(\cdot) \in C([a,b])$, then $\int_a^b G(\lambda)dE(\lambda)$ exists and coincides with scalar spectral integral $\int_a^b g(\lambda)dE(\lambda)$.

(ii) If $\int_a^b G(\lambda)dE(\lambda)$ exists and $h(\cdot) \in C([a,b])$, then also $\int_a^b h(\lambda)G(\lambda)dE(\lambda)$ exists and one has

$$\int_a^b h(\lambda)G(\lambda)dE(\lambda) = \int_a^b G(\lambda)dE(\lambda) \int_a^b h(\lambda)dE(\lambda).$$

Proposition 2.7 (Proposition I.5.1.2 of [10])**.** *Let $G(\cdot)$ be defined on $[a,b)$ and assume the existence of the derivative $G'(\lambda)$ with respect to the operator norm on $[a,b)$. Further, let $G'(\cdot)$ be Bochner integrable on $[a,b)$ and assume that $A(\lambda) = A(a) + \int_a^\lambda G'(x)dx$. Then $\int_a^b G(\lambda)dE(\lambda)$ exists and the estimate*

$$\left\| \int_a^b G(\lambda)dE(\lambda) \right\| \leqslant \|G(a)\| + \int_a^b \|G'(\lambda)\| d\lambda$$

is valid.

Similar existence theorems can be proven for the other types of spectral integrals. For instance the vector spectral integral exists if $f(\cdot)$ is strongly continuous, strongly differentiable on $[a,b]$ and if $f'(\cdot)$ is also strongly continuous. In particular, the operator and vector spectral integrals exist if the integrands $G(\cdot)$ and $f(\cdot)$ are holomorphic.

3. Main results

Let A be a closed symmetric operator with equal deficiency indices acting in the separable Hilbert space $\mathfrak{H}_A$ and let T be a bounded self-adjoint operator acting in the separable Hilbert space $\mathcal{H}_T$. We consider the operator $S = A \otimes I_T + I_A \otimes T$. To define the operator S we first consider the operator $A \otimes I_T$. The operator $A \otimes I_T$ is defined as the closure of the operator $A \odot I_I$ defined by

$$\mathrm{dom}\,(A \odot I_T) := \left\{ f = \sum_{k=1}^{r} g_k \otimes h_k : g_k \in \mathrm{dom}\,(A), \quad h_k \in \mathcal{H}_T, \quad r \in \mathbb{N} \right\}$$

and

$$(A \odot I_T)f = \sum_{k=1}^{r} A g_k \otimes h_k, \quad f \in \mathrm{dom}\,(A \odot I_T).$$

One can easily check that $A \odot I_T$ is a densely defined symmetric operator which yields that $A \otimes I_T$ is a densely defined closed symmetric operator. By $\mathfrak{H}_+(A)$ we denote Hilbert space which is obtained equipping the domain $\mathrm{dom}\,(A)$ with the graph norm of A, cf. (2.2). $\mathrm{dom}\,(A \otimes I_T) = \mathfrak{H}_+(A) \otimes \mathcal{H}_T$. By Proposition 7.26 of [2] we have $(A \otimes I_T)^* = A^* \otimes I_T$. Its domain is given by $\mathrm{dom}\,(A^* \otimes I_T) = \mathfrak{H}_+(A^*) \otimes \mathcal{H}_T$.

Similarly, the operator $I_A \otimes T$ can be defined. $I_A \otimes T$ is found to be a bounded self-adjoint operator with norm $\|T\|$. The operator $S := A \otimes I_T + I_A \otimes T$ is a well-defined closed symmetric operator with domain $\mathrm{dom}\,(A \otimes I_T)$. Notice that

$$S = \overline{A \odot I_T + I_A \odot T} = \overline{A \odot I_T} + I_A \otimes T.$$

Its adjoint is given $S^* = A^* \otimes I_T + I_A \otimes T$.

Let $\widehat{\Gamma}_j := \widehat{\Gamma}^A_j \otimes I_T : \mathfrak{H}_+(A^*) \otimes \mathcal{H}_T \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_T$, $j = 0,1$. Since $\mathrm{ran}\,(\widehat{\Gamma}^A) = \mathcal{H}_A \oplus \mathcal{H}_A$ we have $\mathrm{ran}\,(\widehat{\Gamma}) = (\mathcal{H}_A \otimes \mathcal{H}_T) \oplus (\mathcal{H}_A \otimes \mathcal{H}_T)$ where $\widehat{\Gamma} := (\widehat{\Gamma}_0, \widehat{\Gamma}_1)$. Let us consider the embedding operator $J : \mathfrak{H}_+(A^*) \otimes \mathcal{H}_T \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_T$. We introduce the operator $\Gamma_j : \mathrm{dom}\,(A^* \otimes I_T) \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_T$ by setting

$$\Gamma_j J \widehat{f} := \widehat{\Gamma}_j \widehat{f}, \quad \widehat{f} \in \mathfrak{H}_+(A^*) \otimes \mathcal{H}_T, \quad j = 0,1. \tag{3.1}$$

Notice that $\mathrm{ran}\,(J) = \mathrm{dom}\,(A^* \otimes I_T)$. Since $\mathrm{ran}\,(\widehat{\Gamma}) = (\mathcal{H}_A \otimes \mathcal{H}_T) \oplus (\mathcal{H}_A \otimes \mathcal{H}_T)$ we get $\mathrm{ran}\,(\Gamma) = (\mathcal{H}_A \otimes \mathcal{H}_T) \oplus (\mathcal{H}_A \otimes \mathcal{H}_T)$ where $\Gamma = (\Gamma_0, \Gamma_1)$. Let us introduce the triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ where $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_T$ and Γ_j are given by (3.1).

Proposition 3.1. *If $\Pi_A = \{\mathcal{H}_A, \Gamma^A_0, \Gamma^A_1\}$ is a boundary triplet for A^*, then $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^**

Proof. First, we are going to show that Π is a boundary triplet for $(A \otimes I_I)^* = A^* \otimes I_T$. The surjectivity of $\Gamma = (\Gamma_0, \Gamma_1)$ was already shown above. Next, we check that the "Green's identity" holds Let $g_k, g'_k \in \mathfrak{H}_+(A^*)$, $h_k, h'_k \in \mathcal{H}_T$ so that $f = \sum_{k=1}^{N} g_k \otimes h_k$ and $f' = \sum_{j=1}^{M} g'_j \otimes h'_j$. We have

$$\left((A^* \otimes I_T) J \sum_{k=1}^{N} g_k \otimes h_k, J \sum_{j=1}^{M} g'_j \otimes h'_j\right) - \left(J \sum_{k=1}^{N} g_k \otimes h_k, J(A^* \otimes I_T) \sum_{j=1}^{M} g'_j \otimes h'_j\right)$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{M} (h_k, h'_j) \left[(A^* J_{A^*} g_k, J_{A^*} g'_j) - (J_{A^*} g_k, A^* J_{A^*} g'_j)\right]$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{M} (h_k, h'_j) \left[(\Gamma^A_1 J_{A^*} g_k, \Gamma^A_0 J_{A^*} g'_j) - (\Gamma^A_0 J_{A^*} g_k, \Gamma^A_1 J_{A^*} g'_j)\right]$$

where $J_{A^*} : \mathfrak{H}_+(A^*) \longrightarrow \mathfrak{H}_A$ is the embeding operator. Similarly we get

$$\Big(\Gamma_1 J \sum_{k=1}^N g_k \otimes h_k, \Gamma_0 J \sum_{j=1}^M g'_j \otimes h'_j\Big) - \Big(\Gamma_0 J \sum_{k=1}^N g_k \otimes h_k, \Gamma_1 J \sum_{j=1}^M g'_j \otimes h'_j\Big)$$
$$= \sum_{k=1}^N \sum_{j=1}^M (h_k, h'_j) \left[(\Gamma_1^A J_{A^*} g_k, \Gamma_0^A J_{A^*} g'_j) - (\Gamma_0^A J_{A^*} g_k, \Gamma_1^A J_{A^*} g'_j)\right].$$

Hence we get

$$\Big((A^* \otimes I_T) J \sum_{k=1}^N g_k \otimes h_k, J \sum_{j=1}^M g'_j \otimes h'_j\Big) - \Big(J \sum_{k=1}^N g_k \otimes h_k, J(A^* \otimes I_T) \sum_{j=1}^M g'_j \otimes h'_j\Big)$$
$$= \Big(\Gamma_1 J \sum_{k=1}^N g_k \otimes h_k, \Gamma_0 J \sum_{j=1}^M g'_j \otimes h'_j\Big) - \Big(\Gamma_0 J \sum_{k=1}^N g_k \otimes h_k, \Gamma_1 J \sum_{j=1}^M g'_j \otimes h'_j\Big)$$

which yields

$$\Big((A^* \otimes I_T) J \sum_{k=1}^N g_k \otimes h_k, J \sum_{j=1}^M g'_j \otimes h'_j\Big) - \Big(J \sum_{k=1}^N g_k \otimes h_k, (A^* \otimes I_T) J \sum_{j=1}^M g'_j \otimes h'_j\Big)$$
$$= \Big(\widehat{\Gamma}_1 \sum_{k=1}^N g_k \otimes h_k, \widehat{\Gamma}_0 \sum_{j=1}^M g'_j \otimes h'_j\Big) - \Big(\widehat{\Gamma}_0 \sum_{k=1}^N g_k \otimes h_k, \widehat{\Gamma}_1 \sum_{j=1}^M g'_j \otimes h'_j\Big)$$

Since elements of the form $f = \sum_{k=1}^N g_k \otimes h_k$ and $f' = \sum_{j=1}^M g'_j \otimes h'_j$ are dense in $\mathfrak{H}_+(A^*)$ the equality can be closed which gives

$$\Big((A^* \otimes I_T) J f, J f'\Big) - \Big(J f, (A^* \otimes I_T) J f'\Big) = \Big(\widehat{\Gamma}_1 f, \widehat{\Gamma}_0 f'\Big) - \Big(\widehat{\Gamma}_0 f, \widehat{\Gamma}_1 f'\Big)$$

for $f, f' \in \mathfrak{H}_+(A^*) \otimes \mathcal{H}_T$ which immediately yields the abstract Green's identity for $A^* \otimes I_T$. Hence Π is a boundary triplet $A^* \otimes I_T$. Since $T_A \otimes T$ is a bounded self-adjoint operator one proves that Π is a boundary for S^*. Indeed, since $\mathrm{dom}\,(A^* \otimes I_T) = \mathrm{dom}\,(S^*)$ one immediately verifies the abstract Green's identity and $\Gamma \mathrm{dom}\,(A^* \otimes I_T) = \Gamma \mathrm{dom}\,(S^*)$ shows the surjectivity. □

Let us also mention that $S_0 := S^* \upharpoonright \ker(\Gamma_0^S)$ admits the representation

$$S_0 = A_0 \otimes I_{\mathcal{H}_T} + I_{\mathcal{H}_A} \otimes T. \tag{3.2}$$

Let $E_T(\lambda)$, $\lambda \in \mathbb{R}$, be the spectral measure of the self-adjoint operator T. Obviously,

$$\widehat{E}_T(\lambda) := I_A \otimes E_T(\lambda), \quad \lambda \in \mathbb{R},$$

defines a spectral measure on $\mathcal{H}_A \otimes \mathcal{H}_T$.

Proposition 3.2. *Let Π_A be a boundary triplet for A^* with γ-field $\gamma_A(z)$. If Π_S is the boundary triplet of Proposition 3.1 of S^*, then the γ-field $\gamma_S(\cdot)$ of Π_S admits the representation*

$$\gamma_S(z) = \int_a^b d\widehat{E}_T(\lambda)\, \gamma_A(z-\lambda) \otimes I_{\mathcal{H}_T} = \int_a^b \gamma_A(z-\lambda) \otimes I_{\mathcal{H}_T}\, d\widehat{E}_T(\lambda) \tag{3.3}$$

$z \in \mathbb{C}_\pm$ where $\sigma(T) \subset [a,b)$.

Proof. We set $G(\lambda) := \gamma_A(z-\lambda)\otimes I_T$, $\lambda \in [a,b)$. From (2.7) we get

$$G'(\lambda) = (A_0-\zeta)(A_0-z+\lambda)^{-2}\gamma_A(\zeta)\otimes I_T, \qquad \lambda \in \mathbb{R}.$$

Since $\int_a^b \|G'(\lambda)\|\,d\lambda < \infty$ the operator spectral integral

$$D(z) := \int_{\mathbb{R}} \gamma_A(z-\lambda)\otimes I_{\mathcal{H}_T}\, d\widehat{E}_T(\lambda) \tag{3.4}$$

exists by Proposition 2.7. We will show that $\operatorname{ran}(D(z)) \subseteq \mathfrak{H}_+(S^*-z)$. Let $\mathfrak{Z}$ be a partition of $[a,b)$ and let us consider the Riemann sum

$$D_{\mathfrak{Z}}(z) := \sum_{k=1}^{n} \gamma_A(z-\lambda_k)\otimes I_T\,\widehat{E}_T(\Delta_k), \qquad \lambda_k \in \Delta_k. \tag{3.5}$$

For every $z \in \mathbb{C}_\pm$ one has $\lim_{|\mathfrak{Z}|\to 0}\|D_{\mathfrak{Z}}(z)-D(z)\| = 0$. Obviously, for each $\mathfrak{Z}$ we have $D_{\mathfrak{Z}} f \in \mathfrak{H}_+(S^*)$, $f \in \mathfrak{H}$. Let us estimate the operator norm of $(\gamma_A(z-\lambda_k)\otimes I_T)\,\widehat{E}_T(\Delta_k)$ with respect to the Hilbert space $\mathfrak{H}_+(S^*-z)$. Obviously we have

$$\begin{aligned}&(S^*-z)(\gamma_A(z-\lambda_k)\otimes I_T)\,\widehat{E}_T(\Delta_k)\\ &\quad = (A^*-z)\gamma_A(z-\lambda_k)\otimes E_T(\Delta_k) + \gamma_A(z-\lambda_k)\otimes TE_T(\Delta_k).\end{aligned}$$

which yields

$$(S^*-z)(\gamma_A(z-\lambda_k)\otimes I_T)\,\widehat{E}_T(\Delta_k) = \gamma_A(z-\lambda_k)\otimes (TE_T(\Delta_k)-\lambda_k E_T(\Delta_k))$$

Hence we find

$$\|(S^*-z)(\gamma_A(z-\lambda_k)\otimes I_T)\,\widehat{E}_T(\Delta_k)\| \leqslant \|\gamma_A(z-\lambda_k)\|\,\|TE_T(\Delta_k)-\lambda_k E_T(\Delta_k)\|.$$

Since $\|TE_T(\Delta_k)-\lambda_k E_T(\Delta_k)\| \leqslant |\Delta_k|$, where $|\cdot|$ is the Lebesgue measure of the set Δ_k, we find

$$\|(S^*-z)(\gamma_A(z-\lambda_k)\otimes I_T)\,\widehat{E}_T(\Delta_k)\| \leqslant \|\gamma_A(z-\lambda_k)\|\,|\Delta_k|.$$

Using that $C_{\gamma_A}(z) := \sup_{\lambda\in[a,b)}\|\gamma_A(z-\lambda)\| < \infty$ we immediately get the estimate

$$\|(S^*-z)D_{\mathfrak{Z}}(z)\| \leqslant C_{\gamma_A}(z)(b-a), \qquad z \in \mathbb{C}_\pm. \tag{3.6}$$

In particular we get $\|(S^*-z)D(z)\| \leqslant C_{\gamma_A}(z)(b-a)$, $z \in \mathbb{C}_\pm$. Let us show that the integral $D(z)$ also exists in the strong sense in $\mathfrak{H}_+(S^*-z)$.

$$\begin{aligned}
&(S^*-z)D_{\mathfrak{Z}}(z)g\otimes h = ((A^*-z)\otimes I_T)D_{\mathfrak{Z}}(z)g\otimes h + (I_A\otimes T)D_{\mathfrak{Z}}(z)g\otimes h\\
&= \quad ((A^*-z)\otimes I_T)\sum_{k=1}^{n}\gamma_A(z-\lambda_k)g\otimes E_T(\Delta_k)h + (I_A\otimes T)\sum_{k=1}^{n}\gamma_A(z-\lambda_k)g\otimes E_T(\Delta_k)h\\
&= \quad \sum_{k=1}^{n} -\lambda_k\gamma_A(z-\lambda_k)g\otimes E_T(\Delta_k)h + \sum_{k=1}^{n}\gamma_A(z-\lambda_k)g\otimes TE_T(\Delta_k)h\\
&= \quad \sum_{k=1}^{n}\gamma_A(z-\lambda_k)g\otimes (TE_T(\Delta_k)-\lambda_k E_T(\Delta_k))h\\
&= \quad \Big(\sum_{k=1}^{n}\gamma_A(z-\lambda_k)g\otimes (TE_T(\Delta_k)-\lambda_k E_T(\Delta_k))h\Big).
\end{aligned}$$

Hence

$$\|(S^*-z)D_{\mathfrak{Z}}(z)g\otimes h\| = \sum_{k=1}^{n}\|\gamma_A(z-\lambda_k)g\otimes (TE_T(\Delta_k)-\lambda_k E_T(\Delta_k))h\|,$$

we have

$$\|(S^* - z)D_{\mathfrak{Z}}(z)g \otimes h\| \leqslant \sum_{k=1}^{n} \|\gamma_A(z-\lambda_k)g\| \, \|(TE_T(\Delta_k) - \lambda_k E_T(\Delta_k))h\|. \tag{3.7}$$

Finally we obtain

$$\|(S^* - z)D_{\mathfrak{Z}}(z)g \otimes h\| \leqslant \|h\| \sum_{k=1}^{n} \|\gamma_A(z-\lambda_k)g\| \, |\Delta_k| \, . \tag{3.8}$$

Let $\mathfrak{Z}'$ be a refinement of $\mathfrak{Z}$, that $\mathfrak{Z}' = \{\Delta'_{k'}\}_{k'=1}^{n'}$ where for each k' there is always a k such that $\Delta'_{k'} \subseteq \Delta_k$. This yields the estimate

$$\|(S^* - z)(D_{\mathfrak{Z}}(z) - D_{\mathfrak{Z}'}(z))g \otimes h\| \leqslant \|h\| \sum_{k'=1}^{n'} \|(\gamma_A(z-\lambda'_{k'}) - \gamma_A(z-\lambda_k))g\| \, |\Delta'_{k'}| \, . \tag{3.9}$$

where $\lambda'_{k'} \in \Delta_{k'} \subseteq \Delta_k \ni \lambda_k$. Hence $|\lambda'_{k'} - \lambda_k| \leqslant |\Delta_k| \leqslant |\mathfrak{Z}|$. Using (2.7) we find

$$\|\gamma_A(z-\lambda'_{k'})g - \gamma_A(z-\lambda_k)\| \leqslant \frac{1}{\mathrm{Im(z)}} \sup_{\lambda \in [a,b)} \left\| \frac{S_0 - \zeta}{S_0 - z + \lambda} \right\| \, \|\gamma_A(\zeta)\| \, |\mathfrak{Z}|$$

which yields the estimate

$$\|(S^* - z)(D_{\mathfrak{Z}}(z) - D_{\mathfrak{Z}'}(z))g \otimes h\| \leqslant (b-a)\|h\| \, \|g\| \frac{1}{\mathrm{Im(z)}} \sup_{\lambda \in [a,b)} \left\| \frac{S_0 - \zeta}{S_0 - z + \lambda} \right\| \, \|\gamma_A(\zeta)\| \, |\mathfrak{Z}| \, . \tag{3.10}$$

Hence the Riemann sums $D_{\mathfrak{Z}}(z)$ converge strongly in $\mathfrak{H}_+(S^* - z)$ as $|\mathfrak{Z}| \to 0$. Since the Hilbert spaces $\mathfrak{H}_+(S^*)$ and $\mathfrak{H}_+(S^* - z)$, $z \in \mathbb{C}_\pm$, are isomorph the Riemann sums converge strongly in $\mathfrak{H}_+(S^*)$.

It remains to show that $(S^* - z)D(z) = 0$. Recall that

$$\begin{aligned}(S^* - z)D_{\mathfrak{Z}}(z)g \otimes h &= \sum_{k=1}^{n} ((A^* - z)\gamma_A(z-\lambda_k)g \otimes E_T(\Delta_k)h + \gamma_A(z-\lambda_k)g \otimes TE_T(\Delta_k)h) \\ &= \sum_{k=1}^{n} \gamma_A(z-\lambda_k)g \otimes (TE_T(\Delta_k) - \lambda_k E_T(\Delta_k))h.\end{aligned}$$

For instance,

$$\|(S^* - z)D_{\mathfrak{Z}}(z)g \otimes h\| = \sum_{k=1}^{n} \|\gamma_A(z-\lambda_k)g \otimes (TE_T(\Delta_k) - \lambda_k E_T(\Delta_k))h\| \leqslant$$
$$\sum_{k=1}^{n} \|\gamma_A(z-\lambda_k)g\| \|(T-\lambda_k)E_T(\Delta_k)h\|$$

To the degree that $\|\gamma_A(z-\lambda_k)\|$ is bounded, we have $\|(S^* - z)D_{\mathfrak{Z}}(z)g \otimes h\| \to 0$ as $|\mathfrak{Z}| \to 0$ for any $g \otimes h$. For the element of the form $f = \sum_{k=1}^{n} g_k \otimes h_k$ obviously the same result holds. Then, we use that the set of $f = \sum_{k=1}^{n} g_k \otimes h_k$ is dense in $\mathfrak{H}_+(S^*)$ □

Proposition 3.3. *Let Π_A be a boundary triplet for A^* with Weyl function $M_A(\cdot)$. If Π_S is the boundary triplet of Proposition 3.1 of S^*, then the Weyl function $M_S(\cdot)$ of Π_S admits the representation*

$$M_S(z) = \int_a^b d\widehat{E}_T(\lambda)\ M_A(z-\lambda) \otimes I_{\mathcal{H}_T} = \int_a^b M_A(z-\lambda) \otimes I_{\mathcal{H}_T}\ d\widehat{E}_T(\lambda), \tag{3.11}$$

$z \in \mathbb{C}_\pm$ *where* $\sigma(T) \subset [a,b)$. *In particular, if* $n_\pm(A) = 1$, *then* $M_A(\cdot)$ *is scalar,* $\mathcal{H}_S = \mathcal{H}_T$ *and*

$$M_S(z) = M_A(z-T), \qquad z \in \mathbb{C}_\pm. \tag{3.12}$$

Proof. We set $G(\lambda) := M_A(z-\lambda) \otimes I_T$, $\lambda \in [a,b)$. From (2.7), (2.8) we get

$$G'(\lambda) = -\gamma_A(\zeta)^*\gamma_A(z-\lambda) + (z-\lambda-\overline{\zeta})\gamma_A(\zeta)^*(A_0-\zeta)(A_0-z+\lambda)^{-2}\gamma_A(\zeta) \otimes I_T, \qquad \lambda \in \mathbb{R}. \tag{3.13}$$

Since $\int_a^b \|G'(\lambda)\|\, d\lambda < \infty$ the operator spectral integral

$$D(z) := \int_{\mathbb{R}} \gamma_A(z-\lambda) \otimes I_{\mathcal{H}_T}\ d\widehat{E}_T(\lambda) \tag{3.14}$$

exists by Proposition 2.7.

Analogously to Proposition 3.2, we can prove that the integral exists in the strong sense in $\mathfrak{H}_+(S^*-z)$ and in $\mathfrak{H}_+(S^*)$, as the spaces are isomorph.

Let us note $D_3(z) = \sum_{k=1}^{n} \gamma_A(z-\lambda_k) \otimes I_T\widehat{E}_T(\Delta_k)$. Then,

$$\Gamma_1^S D_3(z) = \Gamma_1^S \sum_{k=1}^{n} \gamma_A(z-\lambda_k) \otimes I_T\widehat{E}_T(\Delta_k) =$$

$$\sum_{k=1}^{n} \Gamma_1^A\gamma_A(z-\lambda_k) \otimes E_T(\Delta_k) = \sum_{k=1}^{n} M_A(z-\lambda_k) \otimes E_T(\Delta_k) = L_3(z)$$

As far as $L_3(z)$ and $D_3(z)$ converge in a strong sense in $\mathfrak{H}_+(S^*)$ and Γ_0^S is bounded in $\mathfrak{H}_+(S^*)$, we get the estimate. □

Note: In case T has pure point spectrum, the formula (3.11) becomes simpler

$$M_S(z) = \sum_{\lambda} M_A(z-\lambda) \otimes \xi_\lambda, \tag{3.15}$$

where ξ_λ is an eigenvector of T, corresponding to λ

4. Example 1

In this section we will describe a simple example. Let's consider the symmetric operator $A = -\frac{d^2}{dx^2}$ with the domain

$$\mathrm{dom}\,(A) = \{f \in W_2^2(0;+\infty) : f(0) = f'(0) = 0\}$$

in the Hilbert space $L^2(\mathbb{R})$. Notice that $n_\pm(A) = 1$. Let's consider the following bounded self-adjoint operator

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

acting on $\mathcal{H}_T = \mathbb{C}^2$. We introduce the operator $S = A \otimes I_T + I_A \otimes T$ defined in $\mathcal{H}_A \otimes H_T$. Our goal is to get the γ-field and the Weyl function corresponding to H in terms of γ-field and the Weyl function, corresponding to A, using the results described above.

Obviously, operator A^* has the deficiency indices $(1;1)$, so it's deficiency subspace is one-dimensional. Let us calculate the boundary form of the operator A^*. Integrating by parts, we get:

$$(A^*f,g)-(f,A^*g)=-\int\limits_0^{+\infty} f''\bar{g}dx=-f'\bar{g}|_0^{+\infty}+f\bar{g}'|_0^{+\infty}-\int\limits_0^{+\infty} f\bar{g}''dx.$$

So,

$$(A^*f,g)-(f,A^*g)=-f'\bar{g}|_0^{+\infty}+f\bar{g}'|_0^{+\infty}.$$

Recall that an element f from the domain of the adjoint operator also satisfies the condition $f(+\infty), f'(+\infty)=0$. Hence, we have:

$$(A^*f,g)-(f,A^*g)=-f(0)\bar{g}'(0)+f'(0)\bar{g}(0).$$

Now we can obtain the boundary operators, corresponding to A^*:

$$\Gamma_0^A f=f(0), \qquad \Gamma_1^A f=f'(0)$$

Recalling the result of Proposition 3.1, we introduce the boundary operators for H^*:

$$\Gamma_0^S f=f(0)\otimes I, \Gamma_1^S f=f'(0)\otimes I.$$

Let us calculate the γ-field, corresponding to A^*. The deficiency element of the operator A^*, corresponding to the point z, has the form: $e^{i\sqrt{z}x}$ (we choose the branch of the square root in such a way that $\Im\sqrt{z}>0$). Applying Γ_0^A, we have:

$$\Gamma_0^A e^{i\sqrt{z}x}=1$$

so that

$$\gamma_A(z)=1.$$

Let us describe the γ-field, corresponding to S^*. As far as T is self-adjoint, the spectral decomposition holds:

$$T=P_1-P_2,$$

where P_1 and P_2 are the projectors onto the invariant subspaces of the operator T, corresponding to the eigenvalues 1 and -1, respectively. The projectors have the following forms:

$$P_1=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, \quad P_2=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}.$$

Using the result of Proposition (3.2), we have:

$$\gamma_S(z)=\gamma_A(z-1)\otimes P_1+\gamma_A(z+1)\otimes P_2=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

The corresponding Weyl function is obviously as follows:

$$M_A(z)=\Gamma_1^A\gamma_A(z)=i\sqrt{z}.$$

Using the result of Proposition (3.3), we have:

$$M_S(z)=M_A(z-1)\otimes P_1+M_A(z+1)\otimes P_2=\begin{pmatrix} i\sqrt{z-1} & 0\\ 0 & i\sqrt{z+1}\end{pmatrix}.$$

5. Example 2

In this example we consider an operator $S = A \otimes I_T + I_A \otimes B$ defined in $\mathcal{H}_A \otimes \mathcal{H}_T$, $\mathcal{H}_A = L_2(a, b), \mathcal{H}_T = \mathbb{C}^2$. Let us take the symmetric operator A as negative Laplacian $A = -\frac{d^2}{dx^2}$ with the domain $\mathrm{dom}\,(A) = \{\phi \in W^{2,2}[a; b] | \phi(a) = \phi(b) = \phi'(a) = \phi'(b) = 0\}$ and a self-adjoint operator T be the same as in previous example. Let us obtain the γ-field for S.

The boundary operators for A^* are:

$$\widehat{\Gamma}_0 f = \begin{pmatrix} f'(b) \\ f(a) \end{pmatrix}, \quad \widehat{\Gamma}_1 f = \begin{pmatrix} f(b) \\ f'(a) \end{pmatrix}$$

Then, the boundary operators for the operator S^* are:

$$\Gamma_0 f = \begin{pmatrix} f'(b) \\ f(a) \end{pmatrix} \otimes I, \quad \Gamma_1 f = \begin{pmatrix} f(b) \\ f'(a) \end{pmatrix} \otimes I \tag{5.1}$$

Due to the fact that the deficiency elements of A corresponding to the point z are

$$e^{i\sqrt{z}x}, e^{-i\sqrt{z}x}, \tag{5.2}$$

we obtain the γ-field $\gamma_A(z)$ for A^* in the form

$$\gamma_A(z) = \frac{-i}{2\sqrt{z}\cos(\sqrt{z}(b-a))} \begin{pmatrix} e^{-i\sqrt{z}a} & i\sqrt{z}e^{-i\sqrt{z}b} \\ -e^{i\sqrt{z}a} & i\sqrt{z}e^{i\sqrt{z}b} \end{pmatrix} \tag{5.3}$$

So, using the result of **Proposition 3.3.**, we have:

$$\gamma_S(z) = \gamma_A(z-1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \gamma_A(z+1) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.4}$$

Direct calculation of the Weyl function for A^* gives us

$$M_A(z) = \frac{1}{\sqrt{z}\cos(\sqrt{z}(b-a))} \cdot \begin{pmatrix} \sin(\sqrt{z}(b-a)) & \sqrt{z} \\ \sqrt{z} & \sqrt{z}\sin\sqrt{z}(b-a) \end{pmatrix}. \tag{5.5}$$

Then,

$$M_S(z) = M_A(z-1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + M_A(z+1) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.6}$$

6. Concluding remarks

In this paper we considered the γ-field and the Weyl function corresponding to the boundary triplet Π_S for the operator $S = A \otimes I_T + I_A \otimes T$ where the operator A is symmetric and the operator T is bounded and self-adjoint. We obtained the formulas in terms of the γ-field and the Weyl function corresponding to the boundary triplet Π_A. The result can be immediately applied to the scattering theory due to the relation between the Weyl function and the scattering matrix (see, e.g., [13]). There is an interesting question about the case when the operator T is unbounded (it is well known that this case has many specific features [14]). We will present the corresponding result in the next paper.

Acknowledgments

Supported by Federal Targeted Program "Scientific and Educational Human Resources for Innovation-Driven Russia" (contract 16.740.11.0030), grant 11-08-00267 of Russian Foundation for Basic Researches and grants of the President of Russia (state contract 14.124.13.2045-MK and grant MK-1493.2013.1), by Program of Development of Leading Russian Universities. A part of this work was made during visits of IYP and AAB to Berlin. We thank WIAS Berlin for kind hospitality.

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