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ESSENTIAL AND DISCRETE SPECTRUM OF A THREE-PARTICLE LATTICE HAMILTONIAN WITH NON-LOCAL POTENTIALS

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We consider a model operator (Hamiltonian) H associated with a system of three particles on a d-dimensional lattice that interact via non-local potentials. Here the kernel of non-local interaction operators has rank n with $n \geq 3$. We obtain an analog of the Faddeev equation for the eigenfunctions of H and describe the spectrum of H. It is shown that the essential spectrum of H consists the union of at most $n+1$ bounded closed intervals. We estimate the lower bound of the essential spectrum of H for the case $\mathrm{d}=1$.

Keywords: three-particle lattice Hamiltonian, non-local interaction operators, Hubbard model, Faddeev equation, essential and discrete spectrum.

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1. Introduction

In the physical literature, local potentials, i.e., multiplication operators by a function, are typically used. But the potentials constructed, for example, in pseudo-potential theory [6] turn out to be non-local. Such for a periodic operator are given by the sum of local and a finite dimensional potentials. Non-local separable two-body interactions have often been used in nuclear physics and many-body problems because of the fact that the two-body Schrödinger equation is easily solvable for them, and leads to closed expressions for a large class of such interactions. They have also been used very systematically with Faddeev equations for the three-body problem. Their main feature is that the partial-wave t-matrix has a very simple form, and can be continued off the energy-shell in a straightforward manner, a feature which is most important, as is well known, in nuclear physics, and in the Faddeev equations [11].

Many works are devoted to the investigations of the essential spectrum of the discrete Schrödinger operators with local potentials, see e.g., [2,8]. In particular, in [2] it was proved that the essential spectrum of a three-particle discrete Schrödinger operator is the union of at most finitely many closed intervals even in the case where the corresponding two-particle discrete Schrödinger operator has an infinite number of eigenvalues.

In the present paper, we study the model operator H associated with a system of three particles on a d-dimensional lattice and interacting via non-local potentials, where the role of a two-particle discrete Schrödinger operator played by the Friedrichs model. Usually, such operators are arise in the Hubbard model [7, 9]. It is remarkable that the Hubbard model is currently one of the most intensively studied many-electron models of metal, but very few exact results have been obtained for the spectrum and the wave functions of the crystal described by this model. Hence, it is very interesting to obtain exact results, at least in special cases, for example, in the case of non-local potentials. For this reason, we intend to discuss the case where the kernel of non-local interaction operators (partial integral operators) has rank n with $n \geq 3$. An important problem in the spectral theory of such operators is to describe the essential spectrum and to study the number of eigenvalues located outside the essential spectrum.

The following results are obtained:
(i) We construct an analog of the Faddeev equation for the eigenfunctions of H;
(ii) We describe the location of the essential spectrum of H and show that it is the union of at most $n+1$ bounded closed intervals;
(iii) We find upper bound of the spectrum of H;
(iv) We estimate the lower bound of the essential spectrum of H for the case $\mathrm{d}=1$.

We remark that the results (i) and (ii) has been announced in [16] without proof. This paper is devoted to the detailed proof of the results (i)–(iv).

The organization of the present paper is as follows. Section 1 is an introduction. In Section 2, the model operator H is described as a bounded self-adjoint operator in the Hilbert space. In Section 3, the main results are formulated. In Section 4, the number and location of the eigenvalues of the corresponding Friedrichs model are studied. In Section 5, an analog of the Faddeev equation and its symmetric version for the eigenfunctions of H is obtained. In Section 5, the essential spectrum of H is investigated. In Section 7, the lower bound of the essential spectrum of H is estimated for the case $\mathrm{d}=1$.

2. Three-particle model operator on a lattice

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ be the set of all complex, real, integer and positive integer numbers, respectively.

We consider the discrete Schrödinger operator $\widehat{A} := \widehat{A}_0 - \widehat{K}$ acting in the space $l_2((\mathbb{Z}^{\mathrm{d}})^2)$. The kinetic energy $\widehat{A}_0$ is given by a convolution with a function of the general form:

$$(\widehat{A}_0\widehat{\psi})(s_1,s_2) = \sum_{n_1,n_2\in\mathbb{Z}^{\mathrm{d}}} u_0(s_1-n_1, s_2-n_2)\widehat{\psi}(n_1,n_2),$$

and the potential energy $\widehat{K}$ is defined by:

$$(\widehat{K}\widehat{\psi})(s_1,s_2) = (u_1(s_1)+u_2(s_2))\widehat{\psi}(s_1,s_2).$$

We assume that the functions $u_0(\cdot,\cdot)$ and u_α, $\alpha=1,2$ satisfy the conditions

$$|u_0(s_1,s_2)| \leq C_0 \exp(-a(|s_1|+|s_2|)), \quad a>0;$$

$$|u_\alpha(s_1)| \leq C_\alpha \exp(-b_\alpha|s_1|), \quad b_\alpha>0, \quad \alpha=1,2,$$

where $|s_1| := |s_{11}|+\ldots+|s_{1\mathrm{d}}|$ for $s_1=(s_{11},\ldots,s_{1\mathrm{d}})\in\mathbb{Z}^{\mathrm{d}}$ and C_α, $\alpha=1,2,3$ are constants.

The operator $\widehat{A}$ is a particular case of the lattice model Hamiltonian studied in [10, 18].

Let $\mathbb{T}^{\mathrm{d}}$ be the d-dimensional torus. The operations addition and multiplication by real numbers elements of $\mathbb{T}^{\mathrm{d}} \subset \mathbb{R}^{\mathrm{d}}$ should be regarded as operations on $\mathbb{R}^{\mathrm{d}}$ modulo $(2\pi\mathbb{Z}^1)^{\mathrm{d}}$. For example, if $\mathrm{d}=4$ and

$$a = \left(\frac{\pi}{2}, \frac{\pi}{6}, -\frac{2\pi}{3}, \frac{2\pi}{3}\right),\ b = \left(\frac{2\pi}{3}, -\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{5\pi}{6}\right) \in \mathbb{T}^4,$$

then

$$a+b = \left(-\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{5\pi}{6}, -\frac{\pi}{2}\right),\ 6a = (\pi,\pi,0,0) \in \mathbb{T}^4.$$

Let $L_2((\mathbb{T}^{\mathrm{d}})^{\alpha})$ be the Hilbert space of square integrable (complex) functions defined on $(\mathbb{T}^{\mathrm{d}})^{\alpha}$, $\alpha = 1, 2$ and $\mathcal{F} : l_2((\mathbb{Z}^{\mathrm{d}})^2) \to L_2((\mathbb{T}^{\mathrm{d}})^2)$ be the standard Fourier transformation:

$$(\mathcal{F}\widehat{\psi})(p, q) = \frac{1}{(2\pi)^{\mathrm{d}}} \sum_{n_1, n_2 \in \mathbb{Z}^{\mathrm{d}}} \widehat{\psi}(n_1, n_2) \exp(i[(p, n_1) + (q, n_2)]).$$

Then, (see [18]) the operator:

$$A := \mathcal{F}\widehat{A}\mathcal{F}^{-1} : L_2((\mathbb{T}^{\mathrm{d}})^2) \to L_2((\mathbb{T}^{\mathrm{d}})^2)$$

can be represented as $A := A_0 - K_1 - K_2$, where the operators A_0 and K_α, $\alpha = 1, 2$ are defined by:

$$(A_0 f)(p, q) = k_0(p, q) f(p, q), \quad f \in L_2((\mathbb{T}^{\mathrm{d}})^2);$$

$$(K_1 f)(p, q) = \int_{\mathbb{T}^{\mathrm{d}}} k_1(p - s) f(s, q) ds, \quad (K_2 f)(p, q) = \int_{\mathbb{T}^{\mathrm{d}}} k_2(q - s) f(p, s) ds, \quad f \in L_2((\mathbb{T}^{\mathrm{d}})^2).$$

Here $k_0(\cdot, \cdot)$ and $k_\alpha(\cdot)$ are the Fourier transform of the functions $u_0(\cdot, \cdot)$ and $u_\alpha(\cdot)$, $\alpha = 1, 2$, respectively. Usually, the operator A is called the momentum representation of the discrete operator $\widehat{A}$.

In the Hilbert space $L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$ of square integrable symmetric (complex) functions defined on $(\mathbb{T}^{\mathrm{d}})^2$, we consider the model operator:

$$H := H_0 - V_1 - V_2, \tag{2.1}$$

where H_0 is the multiplication operator by the function $w(\cdot, \cdot)$:

$$(H_0 f)(p, q) = w(p, q) f(p, q)$$

and V_α, $\alpha = 1, 2$ are non-local interaction operators:

$$(V_1 f)(p, q) = \sum_{i=1}^{n} v_i(q) \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f(p, s) ds, \quad (V_2 f)(p, q) = \sum_{i=1}^{n} v_i(p) \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f(s, q) ds.$$

Here, $f \in L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$, $n \in \mathbb{N}$ with $n \geq 3$, the functions $v_i(\cdot)$, $i = 1, \ldots, n$ are real-valued linearly independent continuous functions on $\mathbb{T}^{\mathrm{d}}$ and the function $w(\cdot, \cdot)$ is a real-valued symmetric continuous function on $(\mathbb{T}^{\mathrm{d}})^2$. By definition, the operators V_α, $\alpha = 1, 2$ are partial integral operators with a degenerate kernel of rank n.

Under these assumptions, the operator H is bounded and self-adjoint.

The spectrum, the essential spectrum and the discrete spectrum of a bounded self-adjoint operator will be denoted by $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$, respectively.

Schrödinger operators of the form (2.1), associated with a system of three particles on a lattice, were studied in [1, 3, 5, 14] for the case $n = 1$ and [15] for the case $n = 2$. In [1, 3] the sufficient conditions for the finiteness and infiniteness of the discrete spectrum are found. In [14], the Efimov effect for (2.1) was demonstrated when the parameter function $w(\cdot, \cdot)$ has a special form. In [5] the essential spectrum and the number of eigenvalues of the model (2.1) were studied for the function $w(\cdot, \cdot)$ of the form $w(p, q) = u(p)u(q)$.

3. Statements of the main results

To study the spectral properties of the operator H, we introduce a family of bounded self-adjoint operators (Friedrichs models) $h(p)$, $p \in \mathbb{T}^{\rm d}$, acting on $L_2(\mathbb{T}^{\rm d})$ by the rule:

$$h(p) := h_0(p) - v,$$

where $h_0(p)$ is the multiplication operator by the function $w(p,\cdot)$ on $L_2(\mathbb{T}^{\rm d})$:

$$(h_0(p)f)(q) = w(p,q)f(q)$$

and v is the non-local interaction operator on $L_2(\mathbb{T}^{\rm d})$:

$$(vf)(q) = \sum_{i=1}^{n} v_i(q) \int\limits_{\mathbb{T}^{\rm d}} v_i(s)f(s)ds.$$

The perturbation v of the operator $h_0(p)$ is a self-adjoint operator of rank n. Therefore, in accordance with the Weyl theorem about the invariance of the essential spectrum under the finite rank perturbations, the essential spectrum of the operator $h(p)$ coincides with the essential spectrum of $h_0(p)$. It is evident that $\sigma_{\rm ess}(h_0(p)) = [m(p); M(p)]$, where the numbers $m(p)$ and $M(p)$ are defined by:

$$m(p) := \min_{q \in \mathbb{T}^{\rm d}} w(p,q) \quad \text{and} \quad M(p) := \max_{q \in \mathbb{T}^{\rm d}} w(p,q).$$

This yields $\sigma_{\rm ess}(h(p)) = [m(p); M(p)]$.

We remark that for some $p \in \mathbb{T}^{\rm d}$ the essential spectrum of $h(p)$ may degenerate to the set consisting of the unique point $\{m(p)\}$ and hence we cannot state that the essential spectrum of $h(p)$ is absolutely continuous for any $p \in \mathbb{T}^{\rm d}$. For example, if the function $w(\cdot,\cdot)$ has the form:

$$w(p,q) := \sum_{i=1}^{\rm d} \left[3 - \cos p_i - \cos(p_i + q_i) - \cos q_i\right], \quad q = (q_1, \ldots, q_{\rm d}) \in \mathbb{T}^{\rm d},$$

and $p = \overline{\pi} := (\pi, \ldots, \pi) \in \mathbb{T}^{\rm d}$, then $\sigma_{\rm ess}(h(\overline{\pi})) = \{4{\rm d}\}$.

For any $p \in \mathbb{T}^{\rm d}$, we define the analytic functions in $\mathbb{C} \setminus [m(p); M(p)]$ by:

$$I_{ij}(p\,; z) := \int\limits_{\mathbb{T}^{\rm d}} \frac{v_i(s)v_j(s)ds}{w(p,s) - z}, \quad i,j = 1, \ldots, n;$$

$$\Delta(p\,; z) := \det\left(\delta_{ij} - I_{ij}(p\,; z)\right)_{i,j=1}^{n}, \quad \delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

It is clear that $I_{ij}(p\,; z) = I_{ji}(p\,; z)$ for all $i,j = 1, \ldots, n$. The function $\Delta(p\,; \cdot)$ is called the Fredholm determinant associated with the operator $h(p)$.

Note that for the discrete spectrum of $h(p)$, the equality

$$\sigma_{\rm disc}(h(p)) = \{z \in \mathbb{C} \setminus [m(p); M(p)] : \Delta(p\,; z) = 0\}$$

holds (see Lemma 4.1).

Let us introduce the following notations:

$$m := \min_{p,q \in \mathbb{T}^{\rm d}} w(p,q), \quad M := \max_{p,q \in \mathbb{T}^{\rm d}} w(p,q), \quad \sigma := \bigcup_{p \in \mathbb{T}^{\rm d}} \sigma_{\rm disc}(h(p)), \quad \Sigma := \sigma \cup [m; M];$$

$$L_2^{(n)}(\mathbb{T}^{\rm d}) := \{g = (g_1, \ldots, g_n) : g_i \in L_2(\mathbb{T}^{\rm d}),\ i = 1, \ldots, n\}.$$

For each $z \in \mathbb{C} \setminus [m; M]$, we define the $n \times n$ block operator matrices $A(z)$ and $K(z)$ acting in the Hilbert space $L_2^{(n)}(\mathbb{T}^{\mathrm{d}})$ as:

$$A(z) := (A_{ij}(z))_{i,j=1}^{n}, \quad K(z) := (K_{ij}(z))_{i,j=1}^{n},$$

where the operator $A_{ij}(z)$ is the multiplication operator by the function $\delta_{ij} - I_{ij}(\cdot\,; z)$ and the operator $K_{ij}(z)$ is the integral operator with the kernel:

$$K_{ij}(p, s; z) := \frac{v_j(p)v_i(s)}{w(p, s) - z},$$

(s is the integration variable).

We note that for each $z \in \mathbb{C} \setminus [m; M]$, all entries of $K(z)$ belong to the Hilbert-Schmidt class and therefore, $K(z)$ is a compact operator.

Recall that for each $z \in \mathbb{C} \setminus \Sigma$, the operator $A(z)$ is bounded and invertible (see Lemma 5.1) and for such z we define the operator $T(z) := A^{-1}(z)K(z)$.

Now, we give the main results of the paper.

The following theorem is an analog of the well-known Faddeev's result for the operator H and establishes a connection between eigenvalues of H and $T(z)$.

Theorem 3.1. *The number $z \in \mathbb{C} \setminus \Sigma$ is an eigenvalue of the operator H if and only if the number $\lambda = 1$ is an eigenvalue of the operator $T(z)$. Moreover, the eigenvalues z and 1 have the same multiplicities.*

We point out that the matrix equation $T(z)g = g$, $g \in L_2^{(n)}(\mathbb{T}^{\mathrm{d}})$ is an analog of the Faddeev type system of integral equations for eigenfunctions of the operator H and it plays a crucial role in the analysis of the spectrum of H.

Since for any $z \in \mathbb{C} \setminus \Sigma$ the kernels of the entries of $T(z)$ are continuous functions on $(\mathbb{T}^{\mathrm{d}})^2$, the Fredholm determinant $\Delta(z)$ of the operator $I - T(z)$, where I is the identity operator in $L_2^{(n)}(\mathbb{T}^{\mathrm{d}})$, exists and is a real-analytic function on $\mathbb{C} \setminus \Sigma$.

According to Fredholm's theorem and Theorem 3.1, the number $z \in \mathbb{C} \setminus \Sigma$ is an eigenvalue of H if and only if $\Delta(z) = 0$, that is,

$$\sigma_{\mathrm{disc}}(H) = \{z \in \mathbb{C} \setminus \Sigma : \Delta(z) = 0\}.$$

The following theorem describes the essential spectrum of the operator H.

Theorem 3.2. *For the essential spectrum of H, the equality $\sigma_{\mathrm{ess}}(H) = \Sigma$ holds. Moreover the set $\sigma_{\mathrm{ess}}(H)$ consists no more than $n + 1$ bounded closed intervals and $\max(\sigma_{\mathrm{ess}}(H)) = M$.*

The sets σ and $[m; M]$ are called two- and three-particle branches of the essential spectrum of H, respectively.

The definition of the set σ and the equality,

$$\bigcup_{p \in \mathbb{T}^{\mathrm{d}}} [m(p); M(p)] = [m; M]$$

together with Theorem 3.2, give the following equality:

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{p \in \mathbb{T}^{\mathrm{d}}} \sigma(h(p)). \tag{3.1}$$

Here, the family of operators $h(p)$ have a simpler structure than the operator H. Hence, in many instances, (3.1) provides an effective tool for the description of the essential spectrum.

In [12], the essential spectrum of several classes of discrete Schrödinger operators on the lattice $\mathbb{Z}^{\mathrm{d}}$ was studied by means of the limit operators method. In [13], this method has been applied to study the location of the essential spectrum of electromagnetic Schrödinger operators.

Roughly speaking, the limit operators approach of [13] works as follows. The study of the essential spectrum of unbounded operator is reduced to the study of the essential spectrum of a related bounded operator which belongs a certain Banach space $\mathcal{B}$. With each operator $A \in \mathcal{B}$, there is an associated family A_h of operators, called the limit operators of A, which reflect the behavior of the operator A at infinity. It is shown in [13] that:

$$\sigma_{\text{ess}}(A) = \bigcup \sigma(A_h),$$

where the union is taken over all limit operators A_h of A and mentioned that this identity also holds for operators in the Wiener algebra on $\mathbb{Z}^{\mathrm{d}}$.

4. Estimates for the number of eigenvalues of $h(p)$

In this section we study the number and location of the eigenvalues of $h(p)$. The following lemma describes the relation between the eigenvalues of the operators $h(p)$ and zeros of the function $\Delta(p\,;\cdot)$.

Lemma 4.1. *For any fixed* $p \in \mathbb{T}^{\mathrm{d}}$ *the number* $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$ *is an eigenvalue of* $h(p)$ *if and only if* $\Delta(p\,; z(p)) = 0$.

Proof. Let $p \in \mathbb{T}^{\mathrm{d}}$ be a fixed. Suppose $f_p(\cdot) \in L_2(\mathbb{T}^{\mathrm{d}})$ is an eigenfunction of the operator $h(p)$ associated with the eigenvalue $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$. Then, $f_p(\cdot)$ satisfies the equation:

$$w(p,q)f_p(q) - \sum_{i=1}^{n} v_i(q) \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f_p(s) ds = z(p) f_p(q). \tag{4.1}$$

For any $z(p) \in \mathbb{C} \setminus [m(p); M(p)]$ and $q \in \mathbb{T}^{\mathrm{d}}$ the relation $w(p,q) - z(p) \neq 0$ holds. Then, the equation (4.1) implies that the function $f_p(\cdot)$ can be represented as:

$$f_p(q) = \frac{1}{w(p,q) - z(p)} \sum_{i=1}^{n} C_i v_i(q), \tag{4.2}$$

where

$$C_i := \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f_p(s) ds, \quad i = 1, \dots, n. \tag{4.3}$$

Substituting the expression (4.2) for $f_p(\cdot)$ into the equality (4.3), we conclude that the equation (4.1) has a nontrivial solution if and only if the following system of n linear equations with n unknowns

$$\begin{cases} \sum\limits_{j=1}^{n} (\delta_{1j} - I_{1j}(p\,; z(p)))C_j = 0 \\ \sum\limits_{j=1}^{n} (\delta_{2j} - I_{2j}(p\,; z(p)))C_j = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum\limits_{j=1}^{n} (\delta_{nj} - I_{nj}(p\,; z(p)))C_j = 0 \end{cases}$$

or $n \times n$ matrix equation

$$(\delta_{ij} - I_{ij}(p\,;z(p)))_{i,j=1}^{n} \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = 0$$

has a nontrivial solution $(C_1, \ldots, C_n) \in \mathbb{C}^n$, i.e., if the condition $\Delta(p\,;z(p)) = 0$ is satisfied, where $\mathbb{C}^n$ is the n-th Cartesian power of the set $\mathbb{C}$. Lemma 4.1 is proven. □

For $\lambda \in \mathbb{R}$ and a bounded self-adjoint operator A acting in the Hilbert space $\mathcal{H}$ denoted by $\mathcal{H}_A(\lambda)$, a subspace such that $(Af, f) < \lambda\|f\|$ for any $f \in \mathcal{H}_A(\lambda)$ and set

$$N(\lambda, A) := \sup_{\mathcal{H}_A(\lambda)} \dim\mathcal{H}_A(\lambda).$$

The number $N(\lambda, A)$ is equal to infinity if $\lambda > \max(\sigma_{\mathrm{ess}}(A))$; if $N(\lambda, A)$ is finite, then it is equal to the number of the eigenvalues of A smaller than λ.

The following lemma describes the number and location of the eigenvalues of $h(p)$.

Lemma 4.2. *For any fixed* $p \in \mathbb{T}^{\mathrm{d}}$, *the operator* $h(p)$ *has no more than* n *eigenvalues* (*counting multiplicities*) *lying on the l.h.s. of* $m(p)$ *and has no eigenvalues on the r.h.s. of* $M(p)$.

Proof. Let $p \in \mathbb{T}^{\mathrm{d}}$ be a fixed. Since the operator v is a self-adjoint operator of rank n, applying Theorem 9.3.3 of [4] we obtain:

$$N(m(p), h_0(p)) - n \le N(m(p), h(p)) \le N(m(p), h_0(p)) + n;$$

$$N(-M(p), -h_0(p)) - n \le N(-M(p), -h(p)) \le N(-M(p), -h_0(p)) + n.$$

The equality $\sigma(h_0(p)) = [m(p); M(p)]$ implies that

$$N(m(p), h_0(p)) = N(-M(p), -h_0(p)) = 0.$$

Thus, $N(m(p), h(p)) \le n$.

From the positivity of the operator v, it follows that the assertions:

$$((h(p) - z)f, f) = \int (w(p, s) - z)|f(s)|^2 ds - (vf, f) < 0,$$

hold for any $z > M(p)$ and $f \in L_2(\mathbb{T}^{\mathrm{d}})$. This means that the operator $h(p)$ has no eigenvalues lying on the r.h.s. of $M(p)$, that is, $N(-M(p), -h(p)) = 0$. Lemma 4.2 is proven. □

5. An analog of the Faddeev equation for eigenfunctions of H

In this section, we derive an analog of the Faddeev type system of integral equations for the eigenfunctions, corresponding to the eigenvalues of H, that is, we prove Theorem 3.1. First, we give an additional lemma.

For any fixed $p \in \mathbb{T}^{\mathrm{d}}$ we define the matrix-valued analytic functions in $\mathbb{C} \setminus [m(p); M(p)]$ by

$$A(p\,;\cdot) := (\delta_{ij} - I_{ij}(p\,;\cdot))_{i,j=1}^{n}\,, \quad \Delta_{ij}(p\,;\cdot) := (-1)^{i+j} M_{ij}(p\,;\cdot),$$

where $M_{ij}(p\,;z)$ is the (i,j) minor, i.e., the determinant of the submatrix formed from the original matrix $A(p\,;z)$ by deleting the i-th row and j-th column ($i, j = 1, \ldots, n$).

Lemma 5.1. *For any $z \in \mathbb{C} \setminus \Sigma$, the operator $A(z)$ is bounded and invertible. Moreover, the inverse operator $A^{-1}(z)$ is the multiplication operator by the matrix:*

$$A^{-1}(p\,;z) := \frac{1}{\Delta(p\,;z)} \begin{pmatrix} \Delta_{11}(p\,;z) & \Delta_{21}(p\,;z) & \dots & \Delta_{n1}(p\,;z) \\ \Delta_{12}(p\,;z) & \Delta_{22}(p\,;z) & \dots & \Delta_{n2}(p\,;z) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1n}(p\,;z) & \Delta_{2n}(p\,;z) & \dots & \Delta_{nn}(p\,;z) \end{pmatrix}.$$

Proof. By definition, $A(z)$ is the multiplication operator by the matrix $A(p\,;z)$.

It is clear that for any fixed $z \in \mathbb{C} \setminus [m; M]$, the matrix-valued function $A(\cdot\,;z)$ is continuous on the compact set $\mathbb{T}^{\mathrm{d}}$. This fact yields the boundedness of the operator $A(z)$. Taking into account the equality $\det(A(p\,;z)) = \Delta(p\,;z)$, we obtain that for any $p \in \mathbb{T}^{\mathrm{d}}$ and $z \notin \Sigma$ the inequality $\det(A(p\,;z)) \neq 0$ holds. Therefore, for any for any $p \in \mathbb{T}^{\mathrm{d}}$ and $z \notin \Sigma$ the matrix $A(p\,;z)$ is invertible. Now, using the definition of $A^{-1}(p\,;z)$, one can easily see that for any $z \notin \Sigma$, the operator $A^{-1}(z)$ is the inverse to $A(z)$ and is bounded. Lemma 5.1 is thus proved. □

Proof of Theorem 3.1. Let $z \in \mathbb{C} \setminus \Sigma$ be an eigenvalue of the operator H and $f \in L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$ be the corresponding eigenfunction. Then, the function f satisfies the equation $Hf = zf$ or

$$(w(p,q) - z)f(p,q) - \sum_{i=1}^{n} \Big[v_i(q) \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f(p,s) ds + v_i(p) \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f(s,q) ds \Big] = 0. \tag{5.1}$$

The condition $z \notin [m; M]$ yields that the inequality $w(p,q) - z \neq 0$ holds for all $p, q \in \mathbb{T}^{\mathrm{d}}$. Then, from equation (5.1), we have that the function f has form:

$$f(p,q) = \frac{1}{w(p,q) - z} \sum_{i=1}^{n} \left[v_i(q) g_i(p) + v_i(p) g_i(q) \right], \tag{5.2}$$

where for $i = 1, \dots, n$ the functions $g_i(\cdot)$ are defined by:

$$g_i(p) := \int_{\mathbb{T}^{\mathrm{d}}} v_i(s) f(p,s) ds. \tag{5.3}$$

For any $i, j \in \{1, \dots, n\}$, $p \in \mathbb{T}^{\mathrm{d}}$ and $z \notin [m; M]$, we set

$$\widehat{g}_{ij}(p\,;z) := \int_{\mathbb{T}^{\mathrm{d}}} \frac{v_i(s) g_j(s)}{w(p,s) - z} ds.$$

Substituting the expression (5.2) for f to the equality (5.3), we obtain that the following system of n linear equations with n unknowns:

$$\begin{cases} \sum\limits_{i=1}^{n} (\delta_{1i} - I_{1i}(p\,;z)) g_i(p) = \sum\limits_{j=1}^{n} v_j(p) \widehat{g}_{1j}(p\,;z) \\ \sum\limits_{i=1}^{n} (\delta_{2i} - I_{2i}(p\,;z)) g_i(p) = \sum\limits_{j=1}^{n} v_j(p) \widehat{g}_{2j}(p\,;z) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum\limits_{i=1}^{n} (\delta_{ni} - I_{ni}(p\,;z)) g_i(p) = \sum\limits_{j=1}^{n} v_j(p) \widehat{g}_{nj}(p\,;z) \end{cases}$$

or $n \times n$ matrix equation

$$A(z) g = K(z) g, \quad g = (g_1, \dots, g_n) \in L_2^{(n)}(\mathbb{T}^{\mathrm{d}}) \tag{5.4}$$

has a nontrivial solution if and only if the equation (5.1) has a nontrivial solution and the linear subspaces of solutions of (5.1) and (5.4) have the same dimension.

By Lemma 5.1, for any $z \in \mathbb{C} \setminus \Sigma$, the operator $A(z)$ is invertible and hence, equation (5.4) is equivalent to the following $n \times n$ matrix equation $g = A^{-1}(z)K(z)g$, i.e. the equation $g = T(z)g$ has a nontrivial solution if and only if the equation (5.4) has a nontrivial solution. □

It is easy to see that for any $p \in \mathbb{T}^{\mathrm{d}}$ and $z < \min \Sigma$ the inequality $\Delta(p\,; z) > 0$ holds. This means that the operator $A(z)$ is a strictly positive and hence, there exists its positive square root, which will be denoted by $A^{-\frac{1}{2}}(z)$. So for $z < \min \Sigma$ we define the operator $\widehat{T}(z) := A^{-\frac{1}{2}}(z)K(z)A^{-\frac{1}{2}}(z)$. Then the operator equation $\widehat{T}(z)g = g$ is called the symmetric version of the Faddeev equation for the eigenfunction of the operator H. Analogously to Theorem 3.1 one can prove that the number $z < \min \Sigma$ is an eigenvalue of the operator H if and only if the number 1 is an eigenvalue of $\widehat{T}(z)$.

6. Investigations of the essential spectrum of H

In this section, applying the statements of sections 4 and 5, the Weyl criterion [17] and the theorem on the spectrum of decomposable operators [17] we prove Theorem 3.2.

Denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the norm and scalar product in the corresponding Hilbert spaces.

Proof of Theorem 3.2. We start the proof with the inclusion $\Sigma \subset \sigma_{\text{ess}}(H)$. Since the set Σ has form $\Sigma = \sigma \cup [m; M]$, first we show that $[m; M] \subset \sigma_{\text{ess}}(H)$. Let $z_0 \in [m; M]$ be an arbitrary point. We prove that $z_0 \in \sigma_{\text{ess}}(H)$. To this end, it is convenient to use Weyl criterion [17], i.e. it suffices to construct a sequence of orthonormal functions $\{f_k\} \subset L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$ such that $\|(H - z_0E)f_k\| \to 0$ as $k \to \infty$. Here, E is an identity operator on $L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$.

From continuity of the function $w(\cdot, \cdot)$ on the compact set $(\mathbb{T}^{\mathrm{d}})^2$, it follows that there exists some point $(p_0, q_0) \in (\mathbb{T}^{\mathrm{d}})^2$ such that $z_0 = w(p_0, q_0)$.

For $k \in \mathbb{N}$ we consider the following vicinity of the point $(p_0, q_0) \in (\mathbb{T}^{\mathrm{d}})^2$:

$$W_k := V_k(p_0) \times V_k(q_0),$$

where

$$V_k(p_0) := \Big\{p \in \mathbb{T}^{\mathrm{d}} : \frac{1}{k+1} < |p - p_0| < \frac{1}{k}\Big\},$$

is the punctured neighborhood of the point $p_0 \in \mathbb{T}^{\mathrm{d}}$.

Let $\mu(\Omega)$ be the Lebesgue measure of the set Ω and $\chi_\Omega(\cdot)$ be the characteristic function of the set Ω. We choose the sequence of functions $\{f_k\} \subset L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$ as follows:

$$f_k(p, q) := \frac{1}{\sqrt{\mu(W_k)}}\chi_{W_k}(p, q).$$

It is clear that $\{f_k\}$ is an orthonormal sequence.

For any $k \in \mathbb{N}$, let us consider $(H - z_0E)f_k$ and estimate its norm:

$$\|(H - z_0E)f_k\|^2 \leq 2 \sup_{(p,q)\in W_k} |w(p, q) - z_0|^2 + 8n\,\mu(V_k(p_0)) \sum_{i=1}^{n} \|v_i\|^2 \max_{p\in\mathbb{T}^{\mathrm{d}}} |v_i(p)|^2.$$

From the construction of the set $V_k(p_0)$ and from the continuity of the function $w(\cdot, \cdot)$, it follows $\|(H - z_0E)f_k\| \to 0$ as $k \to \infty$, i.e. $z_0 \in \sigma_{\text{ess}}(H)$. Since the point z_0 is arbitrary, we have $[m; M] \subset \sigma_{\text{ess}}(H)$.

Now, let us prove that $\sigma \subset \sigma_{\text{ess}}(H)$. Taking an arbitrary point $z_1 \in \sigma$, we show that $z_1 \in \sigma_{\text{ess}}(H)$. Two cases are possible: $z_1 \in [m; M]$ or $z_1 \notin [m; M]$. If $z_1 \in [m; M]$, then it is already proven above that $z_1 \in \sigma_{\text{ess}}(H)$. Let $z_1 \notin [m; M]$. Definition of the set σ and Lemma 4.1

imply that there exists a point $p_1 \in \mathbb{T}^{\rm d}$ such that $\Delta(p_1\,;z_1)=0$. Then, the system of n linear homogeneous equations with n unknowns:

$$\begin{cases} \sum\limits_{j=1}^{n}(\delta_{1j}-I_{1j}(p_1\,;z_1))l_j=0 \\ \sum\limits_{j=1}^{n}(\delta_{2j}-I_{2j}(p_1\,;z_1))l_j=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum\limits_{j=1}^{n}(\delta_{nj}-I_{nj}(p_1\,;z_1))l_j=0 \end{cases}$$

or $n\times n$ matrix equation

$$(\delta_{ij}-I_{ij}(p_1\,;z_1))_{i,j=1}^{n}\begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}=0 \qquad (6.1)$$

with respect to $l_1,\dots,l_n$ has infinitely many solutions.

We denote by $l':=(l'_1,\dots,l'_n)\in\mathbb{C}^n$ one of the non-trivial solition of (6.1).

Let us choose a sequence of orthogonal functions $\{\widetilde{f}_k\}$ as follows:

$$\widetilde{f}_k(p,q):=\frac{1}{w(p,q)-z_1}\sum_{i=1}^{n}\left[v_i(q)g_i^{(k)}(p)+v_i(p)g_i^{(k)}(q)\right],$$

where for $i=1,\dots,n$ and $k\in\mathbb{N}$ the function $g_i^{(k)}(\cdot)$ is defined by:

$$g_i^{(k)}(p):=l'_i c_k(p)\chi_{V_k(p_1)}(p)(\mu(V_k(p_1)))^{-1/2}.$$

Here, $\{c_k\}\subset L_2(\mathbb{T}^{\rm d})$ is chosen from the orthogonality condition for $\{\widetilde{f}_k\}$, that is, from the condition:

$$(\widetilde{f}_k,\widetilde{f}_m)=\frac{2}{\sqrt{\mu(V_k(p_1))}\sqrt{\mu(V_m(p_1))}}\sum_{i,j=1}^{n}l'_il'_j\int\limits_{V_k(p_1)}\int\limits_{V_m(p_1)}\frac{c_k(p)c_m(q)v_i(p)v_j(q)}{(w(p,q)-z_1)^2}dpdq=0 \qquad (6.2)$$

for $k\neq m$. The existence of $\{c_k\}$ is a consequence of the following proposition.

Proposition 6.1. *There exists an orthonormal system* $\{c_k\}\subset L_2(\mathbb{T}^{\rm d})$ *satisfying the conditions* $\operatorname{supp} c_k\subset V_k(p_1)$ *and* (6.2).

Proof of Proposition 6.1. We construct the sequence $\{c_k\}$ by the induction method. Suppose that $c_1(p):=\chi_{V_1(p_1)}(p)\left(\sqrt{\mu(V_1(p_1))}\right)^{-1}$. Now, we choose $\tilde{c}_2\in L_2(V_2(p_1))$ so that $\|\tilde{c}_2\|=1$ and $(\tilde{c}_2,\varepsilon_1^{(2)})=0$, where:

$$\varepsilon_1^{(2)}(p):=\chi_{V_2(p_1)}(p)\sum_{i,j=1}^{n}l'_il'_jv_i(p)\int\limits_{\mathbb{T}^{\rm d}}\frac{v_j(q)c_1(q)dq}{(w(p,q)-z_1)^2}.$$

Set $c_2(p):=\tilde{c}_2(p)\chi_{V_1(p_1)}(p)$. We continue this process. Suppose that $c_1(p),\dots,c_k(p)$ are constructed. Then, the function $\tilde{c}_{k+1}(\cdot)\in L_2(V_{k+1}(p_0))$ is chosen so that it is orthogonal to all functions:

$$\varepsilon_m^{(k+1)}(p):=\chi_{V_{k+1}(p_1)}(p)\sum_{i,j=1}^{n}l'_il'_jv_i(p)\int\limits_{\mathbb{T}^{\rm d}}\frac{v_j(q)c_m(q)dq}{(w(p,q)-z_1)^2},\quad m=1,\dots,k$$

and $\|\tilde{c}_{n+1}\| = 1$. Let $c_{k+1}(p) := \tilde{c}_{k+1}(p)\chi_{V_{k+1}(p_1)}(p)$. Thus, we have constructed the orthonormal system of functions $\{c_k\}$ satisfying the assumptions of the proposition. Proposition 6.1 is proved. □

We continue the proof of Theorem 3.2. To estimate the norm of the function $\widetilde{f}_k$ from below, we rewrite it in the form:

$$\widetilde{f}_k(p,q) = \frac{(\mu(V_k(p_1)))^{-1/2}}{w(p,q)-z_1}\left[\chi_{V_k(p_1)}(p)c_k(p)\sum_{i=1}^{n} l'_i v_i(q) + \chi_{V_k(p_1)}(q)c_k(q)\sum_{i=1}^{n} l'_i v_i(p)\right].$$

Then direct calculation shows that

$$\|\widetilde{f}_k\| \geq \frac{M_n}{\sqrt{\mu(V_k(p_1))}}, \quad M_n := \frac{1}{\max\limits_{p,q\in\mathbb{T}^{\mathrm{d}}}|w(p,q)-z_1|}\|\sum_{i=1}^{n} l'_i v_i\|. \tag{6.3}$$

By the assumption the functions $v_i(\cdot)$, $i = 1,\ldots,n$ are linearly independent and hence, we have $\|\sum\limits_{i=1}^{n} l'_i v_i\| > 0$.

Setting $f_k := \widetilde{f}_k/\|\widetilde{f}_k\|$, $k \in \mathbb{N}$, we conclude that the system of functions $\{f_k\}$ is orthonormal.

Now, for $k \in \mathbb{N}$, we consider $(H - z_1 E)f_k$ and estimate its norm as:

$$\|(H - z_1E)f_k\| \leq \|A(z_1)G_k\| + \|K(z_1)G_k\|, \tag{6.4}$$

where the vector function G_k is defined by:

$$G_k := \left(\frac{g_1^{(k)}}{\|\widetilde{f}_k\|},\ldots,\frac{g_n^{(k)}}{\|\widetilde{f}_k\|}\right) \in L_2^{(n)}(\mathbb{T}^{\mathrm{d}}).$$

Note that $\{G_k\} \subset L_2^{(n)}(\mathbb{T}^{\mathrm{d}})$ is a bounded orthogonal system. Indeed, the orthogonality of this system follows from the fact that for any $i = 1,\ldots,n$ and $k \neq m$, the supports of the functions $g_i^{(k)}(\cdot)$ and $g_i^{(m)}(\cdot)$ do not intersect. Taking into account the equality:

$$\|G_k\|^2 = \frac{1}{\|\widetilde{f}_k\|^2}\frac{1}{\mu(V_k(p_1))}\sum_{i=1}^{n} l_i'^2,$$

and the inequality (6.3), we conclude that the system of vector-functions $\{G_k\}$ is uniformly bounded, more exactly, the inequality:

$$\|G_k\|^2 \leq \frac{1}{M_n^2}\sum_{i=1}^{n} l_i'^2,$$

holds for any $k \in \mathbb{N}$.

Since the operator $K(z_1)$ is compact and $\{G_k\}$ is a bounded orthogonal system, we have $\|K(z_1)G_k\| \to 0$ as $k \to \infty$.

Let us now estimate the first summand of (6.4):

$$\|A(z_1)G_k\| \leq \frac{1}{M_n}\sup_{p\in V_k(p_1)}\|A(p\,;z_1)l'\|.$$

Taking into account the equality $A(p_1\,;z_1)l' = 0$ and the continuity of the matrix-valued function $A(\cdot\,;z_1)$, we get the following:

$$\sup_{p\in V_k(p_1)}\|A(p\,;z_1)l'\| \to 0 \quad \text{as} \quad k \to \infty$$

and hence, by (6.4), we have $\|(H - z_1 E) f_k\| \to 0$ as $k \to \infty$. This implies that $z_1 \in \sigma_{\text{ess}}(H)$. Since the point z_1 is arbitrary, we have $\sigma \subset \sigma_{\text{ess}}(H)$. Therefore, we have proved that $\Sigma \subset \sigma_{\text{ess}}(H)$.

Now, we prove the inverse inclusion, i.e. $\sigma_{\text{ess}}(H) \subset \Sigma$. Since for each $z \in \mathbb{C} \setminus \Sigma$, the operator $K(z)$ is compact, $A^{-1}(z)$ is bounded and $\|T(z)\| \to 0$ as $z \to \infty$, the operator $T(z)$ is a compact-operator-valued function on $\mathbb{C}\setminus\Sigma$. Then from the self-adjointness of H and Theorem 3.1, it follows that the operator $(I - T(z))^{-1}$ exists if z is real and has a large absolute value. The analytic Fredholm theorem (see, e.g., Theorem VI.14 in [17]) implies that there is a discrete set $S \subset \mathbb{C}\setminus\Sigma$ such that the function $(I - T(z))^{-1}$ exists and is analytic on $\mathbb{C}\setminus(S \cup \Sigma)$ and is meromorphic on $\mathbb{C} \setminus \Sigma$ with finite-rank residues. This implies that the set $\sigma(H) \setminus \Sigma$ consists of isolated points, and the only possible accumulation points of Σ can be on the boundary. Thus $\sigma(H) \setminus \Sigma \subset \sigma_{\text{disc}}(H) = \sigma(H) \setminus \sigma_{\text{ess}}(H)$. Therefore, the inclusion $\sigma_{\text{ess}}(H) \subset \Sigma$ holds. Finally, we obtain the equality $\sigma_{\text{ess}}(H) = \Sigma$.

By Lemma 4.2 for any $p \in \mathbb{T}^{\mathrm{d}}$, the operator $h(p)$ has no more than n eigenvalues (counted multiplicities) on the l.h.s. of $m(p)$ and has no eigenvalues on the r.h.s. of $M(p)$. Then, by the theorem on the spectrum of decomposable operators [17] and by the definition of the set σ, it follows that the set σ consists of the union of no more than n bounded closed intervals, which are located on the r.h.s. of the point M. Therefore, the set Σ consists of the union of no more than $n + 1$ bounded closed intervals and $\max \Sigma = M$. Theorem 3.2 is completely proved. □

At the end of this section we give information about the upper bound of the spectrum of H. By Theorem 3.2, we have $\max(\sigma_{\text{ess}}(H)) = \max(\sigma(H_0)) = M$. Then, the positivity of the operator $V_1 + V_2$ implies:

$$((H - z)f, f) = ((H_0 - z)f, f) - ((V_1 + V_2)f, f) < 0,$$

for all $z > M$ and $f \in L_2^{\mathrm{s}}((\mathbb{T}^{\mathrm{d}})^2)$, that is, the operator H has no eigenvalues greater than M. This fact, together with Theorem 3.2, gives $\max(\sigma(H)) = M$. Therefore, the eigenvalues of the operator H are located only below the bottom of the three-particle branch of its essential spectrum.

7. The lower bound of the essential spectrum of H. Case $\mathrm{d} = 1$

In this section, we consider the special class of parameter functions $v_i(\cdot)$, $i = 1, \ldots, n$ and $w(\cdot, \cdot)$ to estimate the lower bound of the essential spectrum of H when $\mathrm{d} = 1$.

Let $\mathrm{d} = 1$ and $P_0 \in \mathbb{T}$ be a fixed element. Throughout this section, we always assume that there exists a number $j_0 \in \{1, \ldots, n\}$ such that the function $v_i(\cdot)$ is a P_0-periodic for all $i \in \{1, \ldots, n\} \setminus \{j_0\}$, and the function $v_{j_0}(\cdot)$ is an analytic function on $\mathbb{T}$ satisfying the condition:

$$\int_{\mathbb{T}} v_{j_0}(s) g(s) ds = 0, \tag{7.1}$$

for any P_0 - periodic function $g \in L_2(\mathbb{T})$. In addition, we suppose that:
(i) $w(\cdot, \cdot)$ is a P_0 - periodic function by the second variable;
(ii) $w(\cdot, \cdot)$ is a twice continuously differentiable function on $\mathbb{T}^2$;
(iii) there exists a finite subset $\Lambda \subset \mathbb{T}$ such that the function $w(\cdot, \cdot)$ has non-degenerate minima at the points of $\Lambda \times \Lambda$.

The following example shows that the class of functions $v_i(\cdot)$, $i = 1, \ldots, n$ and $w(\cdot, \cdot)$, satisfying the above mentioned conditions is non empty. We set

$$v_1(x) := c_1 \cos(x), \quad v_i(x) := c_i (\cos(2x))^i, \quad c_i \in \mathbb{R} \setminus \{0\}, \quad i = 2, \ldots, n.$$

Then $j_0 = 1$, the functions $v_i(\cdot)$, $i = 2, \ldots, n$ are π - periodic, i.e. $P_0 = \pi$. If $g \in L_2(\mathbb{T})$ is a π -periodic function, then:

$$\int\limits_{\mathbb{T}} v_1(s)g(s)ds = \int\limits_{\mathbb{T}} v_1(s+\pi)g(s+\pi)ds = -\int\limits_{\mathbb{T}} v_1(s)g(s)ds,$$

which implies the equality (7.1). One can see that the function $w(\cdot,\cdot)$ defined by:

$$w(x,y) := 2\gamma_1 + \gamma_2 - \gamma_1\cos(2x) - \gamma_2\cos(2x+2y) - \gamma_1\cos(2y), \tag{7.2}$$

with $\gamma_1, \gamma_2 > 0$ satisfy the conditions (i)–(iii) with $\Lambda := \{0, \pi\}$.

Let the operator $h_{j_0}(x)$ act in $L_2(\mathbb{T})$ as follows:

$$(h_{j_0}(x)f)(y) = w(x,y)f(y) - v_{j_0}(y)\int\limits_{\mathbb{T}} v_{j_0}(s)f(s)ds.$$

Setting $n = 1$ and $\Delta_{j_0}(x\,;z) := 1 - I_{j_0j_0}(x\,;z)$, from Lemma 4.1, we obtain that:

$$\sigma_{\rm disc}(h_{j_0}(x)) = \{z \in \mathbb{C} \setminus [m(x); M(x)] : \Delta_{j_0}(x\,;z) = 0\}. \tag{7.3}$$

Since, for any fixed $x \in \mathbb{T}$, $i \in \{1,\ldots,n\} \setminus \{j_0\}$ and $z \in \mathbb{C} \setminus [m(x); M(x)]$, the function $v_i(\cdot)(w(x,\cdot) - z)^{-1}$ is a π - periodic continuous function on compact set $\mathbb{T}$, according to the equality (7.3) we obtain:

$$\int\limits_{\mathbb{T}} \frac{v_{j_0}(s)v_i(s)ds}{w(x,s) - z} = 0, \quad i \in \{1,\ldots,n\} \setminus \{j_0\}.$$

Then, the definition of the function $\Delta(\cdot\,;\cdot)$ implies that:

$$\Delta(x\,;z) = \Delta_{j_0}(x\,;z)\, M_{j_0j_0}(x\,;z),$$

where $M_{j_0j_0}(x\,;z)$ is defined in Section 5.

It means that $\sigma_{\rm disc}(h_{j_0}(x)) \subset \sigma_{\rm disc}(h(x))$. Therefore,

$$\min\sigma \le \min \bigcup_{x\in\mathbb{T}} \sigma_{\rm disc}(h_{j_0}(x)).$$

For $\delta > 0$ and $a \in \mathbb{T}$ we set

$$U_\delta(a) := \{x \in \mathbb{T} : |x - a| < \delta\}.$$

Now, we study the discrete spectrum of $h_{j_0}(x)$.

Lemma 7.1. *If $v_{j_0}(x_0) \neq 0$ for some $x_0 \in \Lambda$, then there exists $\delta > 0$ such that for any $x \in U_\delta(x_0)$ the operator $h_{j_0}(x)$ has a unique eigenvalue $z(x)$, lying on the left of $m(x)$.*

Proof. Since the function $w(\cdot,\cdot)$ has non-degenerate minimum at the point $(x_0, x_0) \in \mathbb{T}^2$, by the implicit function theorem there exists $\delta > 0$ and an analytic function $y_0(\cdot)$ on $U_\delta(x_0)$ such that for any $x \in U_\delta(x_0)$, the point $y_0(x)$ is the unique non-degenerate minimum of the function $w(x,\cdot)$ and $y_0(x_0) = x_0$. Therefore, we have $w(x, y_0(x)) = m(x)$ for any $x \in U_\delta(x_0)$.

Let $\widetilde{w}(\cdot,\cdot)$ be the function on $U_\delta(x_0) \times \mathbb{T}$ as:

$$\widetilde{w}(x,y) := w(x, y + y_0(x)) - m(x).$$

Then, for any $x \in U_\delta(x_0)$, the function $\widetilde{w}(x,\cdot)$ has non-degenerate zero minimum at the point $x_0 \in \mathbb{T}$. Now, using the equality

$$\int\limits_{\mathbb{T}} \frac{v_{j_0}^2(s)ds}{w(x,s) - m(x)} = \int\limits_{\mathbb{T}} \frac{v_{j_0}^2(s + y_0(x))ds}{\widetilde{w}(x,s)}, \quad x \in U_\delta(x_0),$$

the continuity of the function $v_{j_0}(\cdot)$, the facts that $v_{j_0}(x_0) \neq 0$ and $y_0(x_0) = x_0$, it is easy to see that:

$$\lim_{z \to m(x)-0} \Delta_{j_0}(x\,; z) = -\infty$$

for all $x \in U_\delta(x_0)$.

Since, for any $x \in \mathbb{T}$, the function $\Delta_{j_0}(x\,; \cdot)$ is continuous and monotonically decreasing on $(-\infty; m(x))$, the equality

$$\lim_{z \to -\infty} \Delta_{j_0}(x\,; z) = 1 \tag{7.4}$$

implies that for any $x \in U_\delta(x_0)$, the function $\Delta_{j_0}(x\,; \cdot)$ has a unique zero $z = z(x)$, lying in $(-\infty; m(x))$. By equality (7.3), the number $z(x)$ is the eigenvalue of $h_{j_0}(x)$. □

Let us give an example for the function $y_0(\cdot)$ mentioned in the proof of Lemma 7.1. To this end, we consider the function $w(\cdot, \cdot)$ of the form (7.2). This function can be written as follows:

$$w(x, y) = \gamma_1 + \gamma_2 + \gamma_1(1 - \cos(2x)) - a(x)\cos(2y) - b(x)\sin(2y), \tag{7.5}$$

where the coefficients $a(x)$ and $b(x)$ are given by:

$$a(x) := \gamma_1 + \gamma_2 \cos(2x), \quad b(x) := -\gamma_2 \sin(2x). \tag{7.6}$$

Then, from the equality (7.5), we obtain following representation for $w(\cdot, \cdot)$:

$$w(x, y) = \gamma_1 + \gamma_2 + \gamma_1(1 - \cos(2x)) - r(x)\cos(2(y - y_0(x))$$

with

$$r(x) := \sqrt{a^2(x) + b^2(x)}, \quad y_0(x) := \arcsin \frac{b(x)}{r(x)}.$$

Taking into account (7.6), we have that the function $y_0(\cdot)$ is an odd regular function and for any $x \in \mathbb{T}$ the point $y_0(x)$ is the minimum point of the function $w(x, \cdot)$.

We note that if $v_{j_0}(x_0) = 0$, then from analyticity of $v_{j_0}(\cdot)$ on $\mathbb{T}$, it follows that there exist positive numbers C_1, C_2 and δ such that the inequalities:

$$C_1|x - x_0|^\theta \leq |v_{j_0}(x)| \leq C_2|x - x_0|^\theta, \quad x \in U_\delta(x_0), \tag{7.7}$$

hold for some $\theta \in \mathbb{N}$. Since the function $w(\cdot, \cdot)$ has non-degenerate minima at the points of $\Lambda \times \Lambda$, there exist $C_1, C_2 > 0$ and $\delta > 0$ such that estimates:

$$C_1(|x-x'|^2+|y-y'|^2) \leq w(x, y)-m \leq C_2(|x-x'|^2+|y-y'|^2),\ (x, y) \in U_\delta(x') \times U_\delta(y'); \tag{7.8}$$

$$w(x, y) - m \geq C_1 \quad (x, y) \notin \Lambda \times \Lambda. \tag{7.9}$$

Hence, if $v_{j_0}(x') = 0$ for all $x' \in \Lambda$, then using the inequalities (7.7), (7.8) and (7.9), one can easily see that for any $x \in \mathbb{T}$ the integral

$$\int_{\mathbb{T}} \frac{v_{j_0}^2(s)ds}{w(x, s) - m},$$

is positive and finite.

For $x' \in \Lambda$, the Lebesgue dominated convergence theorem yields $\Delta_{j_0}(x'\,; m) = \lim\limits_{x \to x'} \Delta_{j_0}(x\,; m)$, and hence, if $v_{j_0}(x') = 0$ for all $x' \in \Lambda$, then the function $\Delta_{j_0}(\cdot\,; m)$ is continuous on $\mathbb{T}$.

Lemma 7.2. *Let* $v_{j_0}(x') = 0$ *for all* $x' \in \Lambda$;
(i) *If* $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) \geq 0$, *then for any* $x \in \mathbb{T}$ *the operator* $h_{j_0}(x)$ *has no eigenvalues, lying on the left of* m;
(ii) *If* $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) < 0$, *then there exists a non empty set* $G_{j_0} \subset \mathbb{T}$ *such that for any* $x \in G_{j_0}$ *the operator* $h_{j_0}(x)$ *has a unique eigenvalue* $z(x)$, *lying on the left of* m.

Proof. First, we recall that if $v_{j_0}(x') = 0$ for all $x' \in \Lambda$, then the function $\Delta_{j_0}(\cdot\,;m)$ is continuous on the compact set $\mathbb{T}$. Two cases are possible: $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) \geq 0$ or $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) < 0$.

Let $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) \geq 0$. Since for any $x \in \mathbb{T}$ the function $\Delta_{j_0}(x\,;\cdot)$ is monotonically decreasing on $(-\infty; m)$ we have:

$$\Delta_{j_0}(x\,;z) > \Delta_{j_0}(x\,;m) \geq \min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) \geq 0,$$

that is, $\Delta_{j_0}(x\,;z) > 0$ for all $x \in \mathbb{T}$ and $z < m$. Therefore, by equality (7.3) for any $x \in \mathbb{T}$, the operator $h_{j_0}(x)$ has no eigenvalues in $(-\infty; m)$.

Now, we suppose that $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) < 0$ and introduce the following subset of $\mathbb{T}$:

$$G_{j_0} := \{x \in \mathbb{T} : \Delta_{j_0}(x\,;m) < 0\}.$$

Since $\Delta_{j_0}(\cdot\,;m)$ is continuous on the compact set $\mathbb{T}$, there exists at least one point $x_0 \in \mathbb{T}$ such that:

$$\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) = \Delta_{j_0}(x_0\,;m),$$

that is, $x_0 \in G_{j_0}$. So, the set G_{j_0} is non empty. It is clear that, if $\max_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) < 0$, then $\Delta_{j_0}(x\,;m) < 0$ for all $x \in \mathbb{T}$ and hence $G_{j_0} = \mathbb{T}$.

Since for any $x \in \mathbb{T}$ the function $\Delta_{j_0}(x\,;\cdot)$ is continuous and monotonically decreasing on $(-\infty; m]$ by the equality (7.4) for any $x \in G_{j_0}$, there exists a unique point $z(x) \in (-\infty; m)$ such that $\Delta_{j_0}(x\,;z(x)) = 0$. By the equality (7.3) for any $x \in G_{j_0}$ the point $z(x)$ is the unique eigenvalue of $h_{j_0}(x)$.

By the construction of G_{j_0}, the inequality $\Delta_{j_0}(x\,;m) \geq 0$ holds for all $x \in \mathbb{T} \setminus G_{j_0}$. In this case, for any $x \in \mathbb{T} \setminus G_{j_0}$, the operator $h_{j_0}(x)$ has no eigenvalues in $(-\infty; m)$. □

We set

$$E_{\min} := \min\{\lambda : \lambda \in \sigma_{\mathrm{ess}}(H)\}.$$

Then, $E_{\min} \in \sigma_{\mathrm{ess}}(H)$ and it is called the lower bound of the essential spectrum of H.

Lemma 7.3. *Let one of the following conditions hold:*
(i) $v_{j_0}(x_0) \neq 0$ *for some* $x_0 \in \Lambda$;
(ii) $v_{j_0}(x') = 0$ *for all* $x' \in \Lambda$ *and* $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) < 0$.

Then $E_{\min} < m$.

Proof. Let $v_{j_0}(x_0) \neq 0$ for some $x_0 \in \Lambda$. Then, by Lemma 7.1 there exists $\delta > 0$ such that for any $x \in U_\delta(x_0)$ the operator $h_{j_0}(x)$ has a unique eigenvalue $z(x)$, lying on the left of $m(x)$. In particular, $z(x_0) < m(x_0)$. Since $m = \min_{x\in\mathbb{T}} m(x) = m(x_0)$, it follows that $\min\sigma \leq z(x_0) < m$, that is, $E_{\min} < m$.

Let $v_{j_0}(x') = 0$ for all $x' \in \Lambda$ and $\min_{x\in\mathbb{T}} \Delta_{j_0}(x\,;m) < 0$. Then, by part (ii) of Lemma 7.2, for any $x \in G_{j_0}$ the operator $h_{j_0}(x)$ has a unique eigenvalue $z(x)$, lying on the left of $m(x)$. Therefore, we obtain $\min\sigma \leq z(x') < m$ for all $x' \in G_{j_0}$, that is, $E_{\min} < m$. □

Notice that if $v_{j_0}(x') = 0$ for all $x' \in \Lambda$ and $\min\limits_{x \in \mathbb{T}} \Delta_{j_0}(x\,; m) \geq 0$, then the location of the bounds $E_{\min}$ and m depends on the zeros of the function $M_{j_0 j_0}(x\,; \cdot)$. If for all $x \in \mathbb{T}$ this function has no zeros, lying on the l.h.s. of m, then $E_{\min} = m$. If for some $x = x_0 \in \mathbb{T}$ this function has at least one zero on $(-\infty; m)$, then $E_{\min} < m$.

We remark that the results of this section are useful when we find the conditions which guarantee the finiteness or infiniteness of the number of the eigenvalues of H, lying below the bottom of its essential spectrum, in the one dimensional case.

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References

[1] Albeverio S., Lakaev S. N., Djumanova R. Kh. The essential and discrete spectrum of a model operator associated to a system of three identical quantum particles. *Rep. Math. Phys.*, **63** (3), P. 359–380 (2009).
[2] Albeverio S., Lakaev S. N., Muminov Z. I. On the structure of the essential spectrum for the three-particle Schrödinger operators on lattices. *Math. Nachr.*, **280** (7), P. 699–716 (2007).
[3] Albeverio S., Lakaev S. N., Muminov Z. I. On the number of eigenvalues of a model operator associated to a system of three-particles on lattices. *Russ. J. Math. Phys.*, **14** (4), P. 377–387 (2007).
[4] Birman M. S., Solomjak M. Z. *Spectral Theory of Self-Adjoint Operators in Hilbert Space*. Dordrecht: D. Reidl P.C., 313 P. (1987).
[5] Eshkabilov Yu. Kh., Kuchkarov R. R. Essential and discrete spectra of the three-particle Schrödinger operator on a lattice. *Theor. Math. Phys.*, **170** (3), P. 341–353 (2012).
[6] Heine V., Cohen M., Weaire D. *The Pseudopotential Concept*. Academic Press, New York–London, 558 P. (1970).
[7] Karpenko B. V., Dyakin V. V., Budrina G. A. Two electrons in Hubbard model. *Fiz., Met., Metalloved.*, **61** (4), P. 702–706 (1986).
[8] Lakaev S. N., Muminov M. É. Essential and discrete spectra of the three-particle Schrödinger operator on a lattices. *Theor. Math. Phys.*, **135** (3), P. 849–871 (2003).
[9] Mattis D. The few-body problem on a lattice. *Rev. Modern Phys.*, **58** (2), P. 361–379 (1986).
[10] Mogilner A. I. Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: problems and results. *Advances in Sov. Math.*, **5**, P. 139–194 (1991).
[11] Newton R. G. *Scattering Theory of Waves and Particles*. Springer-Verlag, New York, 745 P. (1982).
[12] Rabinovich V. S., Roch S. The essential spectrum of Schrödinger operators on lattices. *J. Phys. A: Math. Gen.*, **39**, P. 8377-8394 (2006).
[13] Rabinovich V. S. Essential spectrum of perturbed pseudodifferential operators. Applications to Schrödinger, Klein-Gordon, and Dirac operators. *Russ. J. Math. Phys.*, **12**, P. 62–80 (2005).
[14] Rasulov T. Kh. Asymptotics of the discrete spectrum of a model operator associated with the system of three particles on a lattice. *Theor. Math. Phys.*, **163** (1), P. 429–437 (2010).
[15] Rasulov T. Kh. Essential spectrum of a model operator associated with a three particle system on a lattice. *Theor. Math. Phys.*, **166** (1), P. 81–93 (2011).
[16] Rasulova Z. D. Investigations of the essential spectrum of a model operator associated to a system of three particles on a lattice. *J. Pure and App. Math.: Adv. Appl.*, **11** (1), P. 37–41 (2014).
[17] Reed M., Simon B. *Methods of modern mathematical physics. IV: Analysis of Operators*. Academic Press, New York, 396 P. (1979).
[18] Zhukov Y. V. The Iorio-O'Caroll theorem for an N-particle lattice Hamiltonian. *Theor. Math. Phys.*, **107** (1), P. 478–486 (1996).