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ON THE NUMBER OF EIGENVALUES OF THE FAMILY OF OPERATOR MATRICES

M. I. Muminov[1], T. H. Rasulov[2]

[1]Universiti Teknologi Malaysia, Faculty of Science, Department of Mathematical Sciences, 81310 UTM Johor Bahru, Malaysia

[2]Bukhara State University, Faculty of Physics and Mathematics, 11 M. Ikbol str., Bukhara, 200100, Uzbekistan

[1]mmuminov@mail.ru, [2]rth@mail.ru

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We consider the family of operator matrices $H(K)$, $K \in \mathbb{T}^3 := (-\pi;\pi]^3$ acting in the direct sum of zero-, one- and two-particle subspaces of the bosonic Fock space. We find a finite set $\Lambda \subset \mathbb{T}^3$ to establish the existence of infinitely many eigenvalues of $H(K)$ for all $K \in \Lambda$ when the associated Friedrichs model has a zero energy resonance. It is found that for every $K \in \Lambda$, the number $N(K,z)$ of eigenvalues of $H(K)$ lying on the left of z, $z < 0$, satisfies the asymptotic relation $\lim\limits_{z \to -0} N(K,z)|\log|z||^{-1} = \mathcal{U}_0$ with $0 < \mathcal{U}_0 < \infty$, independently on the cardinality of Λ. Moreover, we show that for any $K \in \Lambda$ the operator $H(K)$ has a finite number of negative eigenvalues if the associated Friedrichs model has a zero eigenvalue or a zero is the regular type point for positive definite Friedrichs model.

Keywords: operator matrix, bosonic Fock space, annihilation and creation operators, Friedrichs model, essential spectrum, asymptotics.

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1. Introduction

One of the important problems in the spectral theory of Schrödinger operators and operator matrices in Fock space is to study the finiteness or infiniteness (Efimov's effect) of the number of eigenvalues located outside the essential spectrum. The Efimov effect for the three-particle continuous Schrödinger operator has been discussed in [5]. A rigorous mathematical proof of the existence of this effect was originally carried out by Yafaev [14] and then many works devoted to this subject, see for example [12, 13].

It was shown in [1,2,7] that for the three-particle discrete Schrödinger operator $H_\mu(K)$, the Efimov effect exists only for the zero value of the three-particle quasi-momentum ($K = 0$) and for some value $\mu = \mu_0 > 0$ of the interaction energy of two particles. Moreover, the operator $H_\mu(K)$ has only a finite number of eigenvalues for all sufficiently small nonzero values of K and $\mu > 0$. An asymptote analogous to [12, 13] was obtained in [1, 2] for the number of eigenvalues of $H_\mu(K)$.

In all above mentioned papers devoted to the Efimov effect, the systems where the number of quasi-particles is fixed have been considered. In solid-state physics theory [10], quantum field theory [6] and statistical physics [9] some important problems arise where the number of quasi-particles is finite, but not fixed.

In the present note, we consider the family of 3×3 operator matrices $H(K)$, $K \in \mathbb{T}^3$, associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. We find a finite set

$\Lambda \subset \mathbb{T}^3$ and under some smoothness assumptions on the parameters of a family of Friedrichs models $h(k)$, $k \in \mathbb{T}^3$, we obtain the following results: if $h(0)$ has a zero energy resonance, then for the number $N(K,z)$ of eigenvalues of $H(K)$ lying on the left of z, $z<0$, we establish the asymptotics $N(K,z) \sim \mathcal{U}_0 |\log|z||$ with $0 < \mathcal{U}_0 < \infty$ for all $K \in \Lambda$.

We show the finiteness of negative eigenvalues of $H(K)$ for $K \in \Lambda$, if the operator $h(0)$ has a zero eigenvalue or a zero is the regular type point for $h(0)$ with $h(0) \geq 0$.

We point out that the operator $H(K)$ has been considered before in [3,4,8] for $K=0$ and $n=1$, where the existence of Efimov's effect has been proven. Moreover, similar asymptotics for the number of eigenvalues was obtained in [3].

2. Family of 3×3 operator matrices and main results

We denote by $\mathbb{T}^3$ the three-dimensional torus, the cube $(-\pi,\pi]^3$ with appropriately identified sides equipped with its Haar measure. Let $\mathcal{H}_0 := \mathbb{C}$ be the field of complex numbers, $\mathcal{H}_1 := L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $\mathcal{H}_2 := L_2^{\rm s}((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. The spaces $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are called zero-, one- and two-particle subspaces of a bosonic Fock space $\mathcal{F}_{\rm s}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively.

Let us consider the following family of 3×3 operator matrices $H(K)$, $K \in \mathbb{T}^3$ acting in the Hilbert space $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ as :

$$H(K) := \begin{pmatrix} H_{00}(K) & H_{01} & 0 \\ H_{01}^* & H_{11}(K) & H_{12} \\ 0 & H_{12}^* & H_{22}(K) \end{pmatrix},$$

with the entries:

$$H_{00}(K)f_0 = w_0(K)f_0, \quad H_{01}f_1 = \int_{\mathbb{T}^3} v_0(s)f_1(s)ds, \quad (H_{11}(K)f_1)(p) = w_1(K;p)f_1(p),$$

$$(H_{12}f_2)(p) = \int_{\mathbb{T}^3} v_1(s)f_2(p,s)ds, \quad (H_{22}(K)f_2)(p,q) = w_2(K;p,q)f_2(p,q),$$

where $f_i \in \mathcal{H}_i$, $i=0,1,2$; $w_0(\cdot)$ and $v_i(\cdot)$, $i=0,1$ are real-valued bounded functions on $\mathbb{T}^3$, the functions $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ are defined by the equalities:

$$w_1(K;p) := l_1\varepsilon(p) + l_2\varepsilon(K-p) + 1, \quad w_2(K;p,q) := l_1\varepsilon(p) + l_1\varepsilon(q) + l_2\varepsilon(K-p-q),$$

respectively, with $l_1, l_2 > 0$ and

$$\varepsilon(q) := \sum_{i=1}^{3} (1 - \cos(nq^{(i)})), \quad q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{T}^3, \quad n \in \mathbb{N}.$$

Here, H_{ij}^* $(i<j)$ denotes the adjoint operator to H_{ij} and

$$(H_{01}^* f_0)(p) = v_0(p)f_0, \ (H_{12}^* f_1)(p,q) = \frac{1}{2}(v_1(p)f_1(q) + v_1(q)f_1(p)), \ f_i \in \mathcal{H}_i, \ i=0,1.$$

Under these assumptions, the operator $H(K)$ is bounded and self-adjoint.

We remark that the operators H_{01} and H_{12} resp. H_{01}^* and H_{12}^* are called annihilation resp. creation operators [6], respectively. In this note, we consider the case where the number of annihilations and creations of the particles of the considering system is equal to 1. It means that $H_{ij} \equiv 0$ for all $|i-j|>1$.

We denote by $\sigma_{\rm ess}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$, respectively, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator.

To study the spectral properties of the operator $H(K)$, we introduce a family of bounded self-adjoint operators (Friedrichs models) $h(k)$, $k \in \mathbb{T}^3$, which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as follows:

$$h(k) := \begin{pmatrix} h_{00}(k) & h_{01} \\ h_{01}^* & h_{11}(k) \end{pmatrix},$$

where

$$h_{00}(k)f_0 = (l_2\varepsilon(k)+1)f_0, \quad h_{01}f_1 = \frac{1}{\sqrt{2}}\int\limits_{\mathbb{T}^3} v_1(s)f_1(s)ds,$$

$$(h_{11}(k)f_1)(q) = E_k(q)f_1(q), \quad E_k(q) := l_1\varepsilon(q) + l_2\varepsilon(k-q).$$

It is easily to seen that $\sigma_{\rm ess}(h(0)) = [0; 6(l_1+l_2)]$.

The following theorem describes the location of the essential spectrum of operator $H(K)$ by the spectrum of the family $h(k)$ of Friedrichs models.

Theorem 2.1. *For the essential spectrum of $H(K)$, the equality*

$$\sigma_{\rm ess}(H(K)) = \bigcup_{p\in\mathbb{T}^3}\{\sigma_{\rm disc}(h(K-p)) + l_1\varepsilon(p)\} \cup [m_K; M_K]$$

holds, where the numbers m_K and M_K are defined by:

$$m_K := \min_{p,q\in\mathbb{T}^3} w_2(K;p,q) \quad \textit{and} \quad M_K := \max_{p,q\in\mathbb{T}^3} w_2(K;p,q).$$

Let us consider the following subset of $\mathbb{T}^3$:

$$\Lambda := \left\{(p^{(1)},p^{(2)},p^{(3)}) : p^{(i)} \in \left\{0, \pm\frac{2}{n}\pi; \pm\frac{4}{n}\pi; \ldots; \pm\frac{n'}{n}\pi\right\} \cup \Pi_n,\ i=1,2,3\right\},$$

where

$$n' := \begin{cases} n-2, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \Pi_n := \begin{cases} \{\pi\}, & \text{if } n \text{ is even} \\ \emptyset, & \text{if } n \text{ is odd} \end{cases}$$

Direct calculation shows that the cardinality of Λ is equal to n^3. It is easy to check that for any $K \in \Lambda$, the function $w_2(K;\cdot,\cdot)$ has non-degenerate zero minimum at the points of $\Lambda \times \Lambda$, that is, $m_K = 0$ for $K \in \Lambda$.

The following assumption we needed throughout the note: the function $v_1(\cdot)$ is either even or odd function on each variable and there exists all second order continuous partial derivatives of $v_1(\cdot)$ on $\mathbb{T}^3$.

Let us denote by $C(\mathbb{T}^3)$ and $L_1(\mathbb{T}^3)$ the Banach spaces of continuous and integrable functions on $\mathbb{T}^3$, respectively.

Definition 2.2. *The operator $h(0)$ is said to have a zero energy resonance if the number 1 is an eigenvalue of the integral operator given by:*

$$(G\psi)(q) = \frac{v_1(q)}{2(l_1+l_2)}\int\limits_{\mathbb{T}^3}\frac{v_1(s)\psi(s)}{\varepsilon(s)}ds, \quad \psi \in C(\mathbb{T}^3)$$

and at least one (up to a normalization constant) of the associated eigenfunction ψ satisfies the condition $\psi(p') \neq 0$ for some $p' \in \Lambda$. If the number 1 is not an eigenvalue of the operator G, then we say that $z = 0$ is a regular type point for the operator $h(0)$.

We note that in Definition 2.2, the requirement for the existence of an eigenvalue 1 of G corresponds to the existence of a solution of $h(0)f = 0$ and the condition $\psi(p') \neq 0$ for some $p' \in \Lambda$ implies that the solution f of this equation does not belong to $\mathcal{H}_0 \oplus \mathcal{H}_1$. More precisely, if the operator $h(0)$ has a zero energy resonance, then the solution $\psi(\cdot)$ of $G\psi = \psi$ is equal to $v_1(\cdot)$ (up to constant factor) and the vector $f = (f_0, f_1)$, where

$$f_0 = \text{const} \neq 0, \quad f_1(q) = -\frac{v_1(q) f_0}{\sqrt{2}(l_1 + l_2)\varepsilon(q)}, \tag{2.1}$$

obeys the equation $h(0)f = 0$ with $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$. If the operator $h(0)$ has a zero eigenvalue, then the vector $f = (f_0, f_1)$, where f_0 and f_1 are defined by (2.1), again obeys the equation $h(0)f = 0$ and $f_1 \in L_2(\mathbb{T}^3)$.

Denote by $\tau_{\text{ess}}(K)$ the bottom of the essential spectrum of $H(K)$ and by $N(K, z)$ the number of eigenvalues of $H(K)$ on the left of z, $z \leq \tau_{\text{ess}}(K)$.

Note that if the operator $h(0)$ has either a zero energy resonance or a zero eigenvalue, then for any $K \in \Lambda$ and $p \in \mathbb{T}^3$ the operator $h(K - p) + l_1\varepsilon(p)I$ is non-negative, where I is the identity operator in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Hence Theorem 2.1 and equality $m_K = 0$, $K \in \Lambda$ imply that $\tau_{\text{ess}}(K) = 0$ for all $K \in \Lambda$.

The main results of the present note are as follows.

Theorem 2.3. *Let $K \in \Lambda$ and one of the following assumptions hold:*
(i) *the operator $h(0)$ has a zero eigenvalue;*
(ii) *$h(0) \geq 0$ and a zero is the regular type point for $h(0)$.*
Then the operator $H(K)$ has a finite number of negative eigenvalues.

Theorem 2.4. *Let $K \in \Lambda$. If the operator $h(0)$ has a zero energy resonance, then the operator $H(K)$ has infinitely many negative eigenvalues accumulating at zero and the function $N(K, \cdot)$ obeys the relation:*

$$\lim_{z \to -0} \frac{N(K, z)}{|\log |z||} = \mathcal{U}_0, \quad 0 < \mathcal{U}_0 < \infty. \tag{2.2}$$

Remark 2.5. *The constant $\mathcal{U}_0$ does not depend on the functions $v_0(\cdot)$, $v_1(\cdot)$ and the cardinality of the set Λ. It is positive and depends only on the ratio l_2/l_1.*

Remark 2.6. *Clearly, by equality (2.2), the infinite cardinality of the negative discrete spectrum of $H(K)$ follows automatically from the positivity of $\mathcal{U}_0$.*

3. Sketch of proof of the main results

For any $k \in \mathbb{T}^3$, we define an analytic function $\Delta(k\,; \cdot)$ (the Fredholm determinant associated with the operator $h(k)$) in $\mathbb{C} \setminus [E_{\min}(k); E_{\max}(k)]$ by:

$$\Delta(k\,; z) := l_2\varepsilon(k) + 1 - z - \frac{1}{2} \int\limits_{\mathbb{T}^3} \frac{v_1^2(s)ds}{E_k(s) - z},$$

where the numbers $E_{\min}(k)$ and $E_{\max}(k)$ are defined by

$$E_{\min}(k) := \min_{q \in \mathbb{T}^3} E_k(q) \quad \text{and} \quad E_{\max}(k) := \max_{q \in \mathbb{T}^3} E_k(q).$$

Set

$$\Sigma(K) := \bigcup_{p \in \mathbb{T}^3} \{\sigma_{\text{disc}}(h(K - p)) + l_1\varepsilon(p)\} \cup [m_K; M_K].$$

Let us consider 2×2 block operator matrix $\widehat{T}(K, z)$, $z \in \mathbb{C} \setminus \Sigma(K)$ acting on $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\widehat{T}(K, z) := \begin{pmatrix} \widehat{T}_{00}(K, z) & \widehat{T}_{01}(K, z) \\ \widehat{T}_{10}(K, z) & \widehat{T}_{11}(K, z) \end{pmatrix},$$

with the entries $\widehat{T}_{ij}(K, z) : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0, 1$:

$$\widehat{T}_{00}(K, z)g_0 = (1 + z - w_0(K))g_0, \quad \widehat{T}_{01}(K, z) = -H_{01};$$

$$(\widehat{T}_{10}(K, z)g_0)(p) = -\frac{v_0(p)g_0}{\Delta(K - p\,; z - l_1\varepsilon(p))};$$

$$(\widehat{T}_{11}(K, z)g_1)(p) = \frac{v_1(p)}{2\Delta(K - p\,; z - l_1\varepsilon(p))} \int\limits_{\mathbb{T}^3} \frac{v_1(s)g_1(s)ds}{w_2(K; p, s) - z},$$

$g_i \in \mathcal{H}_i$, $i = 0, 1$.

The following lemma is an analog of the well-known Faddeev's result for the operator $H(K)$ and establishes a connection between eigenvalues of $H(K)$ and $\widehat{T}(K, z)$.

Lemma 3.1. *The number $z \in \mathbb{C} \setminus \Sigma(K)$ is an eigenvalue of the operator $H(K)$ if and only if the number $\lambda = 1$ is an eigenvalue of the operator $\widehat{T}(K, z)$. Moreover, the eigenvalues z and 1 have the same multiplicities.*

The inclusion $\Sigma(K) \subset \sigma_{\text{ess}}(H(K))$ in the proof of Theorem 2.1 is established with the use of a well-known Weyl criterion. An application of Lemma 3.1 and analytic Fredholm theorem (see, e.g., Theorem VI.14 in [11]) proves inclusion $\sigma_{\text{ess}}(H(K)) \subset \Sigma(K)$.

To find conditions which guarantee for the finiteness or infiniteness of the number of eigenvalues of $H(K)$, $K \in \Lambda$, we establish in which cases the bottom of the essential spectrum of $h(0)$ is a threshold energy resonance or eigenvalue.

Lemma 3.2. (i) *The operator $h(0)$ has a zero eigenvalue if and only if $\Delta(0\,; 0) = 0$ and $v_1(q') = 0$ for all $q' \in \Lambda$;*

(ii) *The operator $h(0)$ has a zero energy resonance if and only if $\Delta(0\,; 0) = 0$ and $v_1(q') \neq 0$ for some $q' \in \Lambda$.*

Since $\Delta(K - p\,; z - l_1\varepsilon(p)) > 0$ for any $K, p \in \mathbb{T}^3$ and $z < \tau_{\text{ess}}(K)$, one can define a symmetric version of the operator $\widehat{T}(K, z)$ for $z < \tau_{\text{ess}}(K)$, which is important in our analysis of the discrete spectrum of $H(K)$, $K \in \mathbb{T}^3$. So, we consider the self-adjoint compact 2×2 block operator matrix $T(K, z)$, $z < \tau_{\text{ess}}(K)$ acting on $\mathcal{H}_0 \oplus \mathcal{H}_1$ as follows:

$$T(K, z) := \begin{pmatrix} T_{00}(K, z) & T_{01}(K, z) \\ T_{01}^*(K, z) & T_{11}(K, z) \end{pmatrix},$$

with the entries $T_{ij}(K, z) : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0, 1$:

$$T_{00}(K, z)g_0 = (1 + z - w_0(K))g_0, \quad T_{01}(K, z)g_1 = -\int\limits_{\mathbb{T}^3} \frac{v_0(s)g_1(s)ds}{\sqrt{\Delta(K - s\,; z - l_1\varepsilon(s))}};$$

$$(T_{11}(K, z)g_1)(p) = \frac{v_1(p)}{2\sqrt{\Delta(K - p\,; z - l_1\varepsilon(p))}} \int\limits_{\mathbb{T}^3} \frac{v_1(s)g_1(s)ds}{\sqrt{\Delta(K - s\,; z - l_1\varepsilon(s))}(w_2(K; p, s) - z)},$$

$g_i \in \mathcal{H}_i$, $i = 0, 1$.

To prove Theorem 2.3, first we show $N(K, z) = n(1, T(K, z))$ (so-called Birman-Schwinger principle for the operator $H(K)$), where $n(1, A)$ is the number of the eigenvalues (counted multiplicities) of the compact operator A bigger than 1. Then, under the conditions of Theorem 2.3, we prove that the operator $T(K,z)$ is continuous from the left up to $z=0$ and $T(K,0)$ is a compact operator. Using the Weyl inequality,

$$n(\lambda_1+\lambda_2, A_1+A_2) \le n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

for the sum of compact operators A_1 and A_2, and for any positive numbers λ_1 and λ_2, we have

$$n(1, T(K,z)) \le n(1/2, T(K,0)) + n(1/2, T(K,z) - T(K,0))$$

for all $z<0$. Hence, $\lim\limits_{z\to -0} N(K,z) = N(K,0) \le n(1/2, T(K,0)) < \infty$.

The study of the behavior of $T(K,z)$, $K\in\Lambda$, that is, proof of Theorem 2.4, is based on the analysis of the behavior of $\Delta(K-p\,; z-l_1\varepsilon(p))$ as $z\to -0$ and $|p-p'|\to 0$ for $K, p'\in\Lambda$.

Set

$$\Lambda_0 := \{q'\in\Lambda : v_1(q')\neq 0\}.$$

Lemma 3.3. *Let the operator $h(0)$ have a zero energy resonance and $K, p'\in\Lambda$. Then, the following decomposition:*

$$\Delta(K-p\,; z-l_1\varepsilon(p)) = \frac{2\pi^2}{n^2(l_1+l_2)^{3/2}}\Big(\sum_{q'\in\Lambda_0} v_1^2(q')\Big)\sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p'|^2 - \frac{2z}{n^2}} + O(|p-p'|^2) + O(|z|),$$

holds for $|p-p'|\to 0$ and $z\to -0$.

By applying Lemma 3.3, we single out the principal part of the operator $T(K,z)$ $K\in\Lambda$ as $z\to -0$, which is unitarily equivalent to the compact integral operator S_R, $R=1/2|\log|z||$ in $L_2((0,R), L_2(\mathbb{S}^2))$ with the kernel

$$S(y,t) := \frac{1}{4\pi^2}\frac{(l_1+l_2)^2}{\sqrt{l_1^2+2l_1l_2}}\frac{1}{(l_1+l_2)\cosh y + l_2 t},$$

where $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$; $y=x-x'$, $x,x'\in(0,R)$ and $t=\langle\xi,\eta\rangle$ is the inner product of the arguments $\xi,\eta\in\mathbb{S}^2$.

The eigenvalue asymptotics for the operator S_R have been studied in detail by Sobolev [12], by employing an argument used in the calculation of the canonical distribution of Toeplitz operators.

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