An introduction to the spectral asymptotics of a damped wave equation on metric graphs

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This paper summarizes the main results of [1] for the spectral asymptotics of the damped wave equation. We define the notion of a high frequency abscissa, a sequence of eigenvalues with imaginary parts going to plus or minus infinity and real parts going to some real number. We give theorems on the number of such high frequency abscissas for particular conditions on the graph. We illustrate this behavior in two particular examples.

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1. Introduction

The current text is a brief introduction to the spectral asymptotics of the damped wave equation on metric graphs. Our paper summarizes the main results of the paper [1] and gives ideas of their proofs. If the reader wants a detailed study of this problem or proofs of certain theorems, we refer to this paper. Its main results were obtained in collaboration with prof. Pedro Freitas during my stay in Lisbon.

Our aim is to study the damped wave equation

$$\partial_{tt}u(t,x) + 2a(x)\partial_t u(t,x) = \partial_{xx}u(t,x) + b(x)u(t,x) \tag{1}$$

on a metric graph. The problem of damped wave equation was studied in detail for a segment with Dirichlet conditions on both ends [2]. Paper [1], to the author's knowledge, is the first attempt to treat the problem for the graph. In the case of a segment, there exists a sequence of eigenvalues with imaginary parts going to plus and minus infinity and real part approaching the negative average of the damping function on the segment. In paper [2], an asymptotic expansion of the eigenvalues was obtained.

We show that in the case of a metric graph, there are several sequences of eigenvalues which we call *high frequency abscissas*. Our main results are three theorems on the number of these high frequency abscissas. This paper is structured as follows: in the second section we describe the model, next we give theorems on the asymptotics of eigenvalues and eigenfunctions and locations of eigenvalues and high frequency abscissas; next, we introduce the method of pseudo orbit expansion; in section 5, we give three main theorems on the number of high frequency abscissas; and finally, we show two particular examples to illustrate their behavior.

2. Description of the model

Let us consider a metric graph Γ with $N < \infty$ finite edges $\{e_j\}_{j=1}^N$ of lengths $\{l_j\}_{j=1}^N$. On each edge we consider a damped wave equation:

$$\partial_{tt} w_j(t,x) + 2a_j(x)\partial_t w_j(t,x) = \partial_{xx} w_j(t,x) + b(x) w_j(t,x), \tag{2}$$

with damping functions $a_j(x)$ and potentials $b_j(x)$ real and bounded. The functions at the j-th vertex are connected by coupling conditions similar to the case of quantum graphs

$$(U_j - I)\Psi_j + i(U_j + I)\Psi_j' = 0\,,$$

where U_j is a unitary square matrix, I is a unit matrix, Ψ_j is the vector of limits of functional values in the vertex from all neighboring edges and, similarly, Ψ_j' is the vector of outgoing derivatives. The coupling on the whole graph can be described by a large $2N \times 2N$ unitary matrix U (for more details see [3, 4]), which describes not only the coupling, but also the topology of the graph. Then, the coupling conditions are:

$$(U - I)\Psi + i(U + I)\Psi' = 0\,. \tag{3}$$

The ansatz $w_j(t,x) = \mathrm{e}^{\lambda t} u_j(x)$ leads to the differential equation:

$$\partial_{xx} u_j(x) - (\lambda^2 + 2\lambda a_j(x) - b_j(x)) u_j(x) = 0\,. \tag{4}$$

Our aim is to solve this equation and find complex numbers λ. Its real parts give the time decay for the solutions to the damped wave equation.

There exists a second approach to the problem, which is equivalent to the previous approach. One finds the eigenvalues of a non-self-adjoint operator:

$$H = \begin{pmatrix} 0 & I \\ I\frac{\mathrm{d}^2}{\mathrm{d}x^2} + B & -2A \end{pmatrix},$$

where A and B are $N \times N$ diagonal matrices with $a_j(x)$ and $b_j(x)$ on the diagonal. The domain of this operator consists of functions $(\psi_1(x), \psi_2(x))^{\mathrm{T}}$ with components of both ψ_1 and ψ_2 in $W^{2,2}(e_j)$ for the corresponding edge and satisfying coupling conditions (3) at the vertices.

In the following text, we will sometimes use the term *standard conditions*. These conditions (sometimes referred to in the literature as Kirchhoff, Neumann or free coupling) imply that the function is continuous at the vertex and the sum of outgoing derivatives is equal to zero. The corresponding vertex coupling matrix is $U = 2/dJ - I$, where d is the degree of a given vertex and J has all entries equal to one.

3. Eigenfunction and eigenvalue asymptotic properties and the locations of high frequency abscissas

First, we present a theorem from [2] on the asymptotic behavior of eigenfunctions on a segment.

Theorem 3.1. *Let $a \in \mathcal{C}^{m+1}[0,1]$ and $b \in \mathcal{C}^m[0,1]$. Then there exist two linearly independent solutions $u_\pm(x,\lambda)$ of equation (4) satisfying the initial condition $u_\pm(0,\lambda) = 1$ having the asymptotics:*

$$u_\pm(x,\lambda) = \mathrm{e}^{\pm\lambda x \pm \int_0^x \phi_\pm(t,\lambda)\,\mathrm{d}t}, \tag{5}$$

in the $C^2[0,1]$ *norm as* $\operatorname{Im}\lambda \to \infty$ *with:*

$$\phi_\pm(x,\lambda) = \sum_{i=0}^{m} \frac{\phi_i^\pm(x)}{\lambda^i} + \mathcal{O}(\lambda^{-m-1}), \tag{6}$$

and

$$\phi_0^{(\pm)}(x) = a(x)\,, \quad \phi_1^{(\pm)}(x) = -\frac{1}{2}(\pm a'(x) + a^2(x) + b(x))\,,$$
$$\phi_i^{(\pm)}(x) = -\frac{1}{2}\left(\pm \phi'^{(\pm)}_{i-1} + \sum_{s=0}^{i-1} \phi_s^{(\pm)}\phi_{i-s-1}^{(\pm)}\right).$$

Now, we can formulate the theorem on the asymptotics of eigenvalues for a graph with all the edges of lengths equal to one.

Theorem 3.2. *Let us assume a graph with N finite edges of lengths 1 with the coupling between vertices given by matrix U. Let on each edge be damping $a_j \in \mathcal{C}^{N+1}([0,1])$ and potential $b_j \in \mathcal{C}^N([0,1])$. Then, there exists such a $K_0 \in \mathbb{R}_+$ that for $K > K_0$, if $\lambda = r + iK$ is an eigenvalue, then $\lambda + 2\pi i + \mathcal{O}(1/K)$ is also an eigenvalue. Similarly, if $\lambda = r - iK$ is an eigenvalue, then $\lambda - 2\pi i + \mathcal{O}(1/K)$ is also an eigenvalue. This means that there exist sequences of eigenvalues with the asymptotics $\lambda_{ns} = 2\pi i n + c_0^s + \mathcal{O}(1/n)$.*

Idea of the proof: Since two linearly independent solutions exist, according to the previous theorem, one can write the general solution as their linear combination. Substituting for the coupling conditions, one finds the secular equation in the form:

$$\begin{aligned} &P_0 \mathrm{e}^{\lambda+\langle a_1\rangle+\lambda+\langle a_2\rangle+\cdots+\lambda+\langle a_N\rangle+\mathcal{O}(1/\lambda)} + P_{11}\mathrm{e}^{-\lambda-\langle a_1\rangle+\lambda+\langle a_2\rangle+\cdots+\lambda+\langle a_N\rangle+\mathcal{O}(1/\lambda)} + \\ &\quad P_{12}\mathrm{e}^{\lambda+\langle a_1\rangle-\lambda-\langle a_2\rangle+\cdots+\lambda+\langle a_N\rangle+\mathcal{O}(1/\lambda)} + \cdots + P_{21}\mathrm{e}^{-\lambda-\langle a_1\rangle-\lambda-\langle a_2\rangle+\cdots+\lambda+\langle a_N\rangle+\mathcal{O}(1/\lambda)} + \\ &\qquad \cdots + P_{N1}\mathrm{e}^{-\lambda-\langle a_1\rangle-\lambda-\langle a_2\rangle-\cdots-\lambda-\langle a_N\rangle+\mathcal{O}(1/\lambda)} = 0\,, \end{aligned}$$

where P_{mn} is a polynomial in λ of degree $2N$ with m minuses before λ; n only distinguishes different polynomials. Since $1/\lambda = \mathcal{O}(1/K)$, one finds that the first term of the asymptotics is equal to zero for $\lambda_0 + 2\pi i$ if λ_0 is an eigenvalue. Hence, such $\lambda = \lambda_0 + 2\pi i + \mathcal{O}(1/\lambda_0)$ exists for which the secular equation is equal to zero. □

Now, we define the notion of a high frequency abscissa, which will be very important in subsequent sections.

Definition 3.3. We say that c_0 is a *high frequency abscissa* of the operator H if there exists a sequence of eigenvalues of H, say $\{\lambda_n\}_{n=1}^\infty$, such that:

$$\lim_{n\to\infty} \operatorname{Im}\lambda_n = \pm\infty \text{ and } \lim_{n\to\infty} \operatorname{Re}\lambda_n = c_0.$$

The next theorem says that only the average of the damping function on each edge is important for the location of high frequency abscissas.

Theorem 3.4. *Let Γ be a graph with N commensurate edges of lengths $l_j = m_j l_0$, $m_j \in \mathbb{N}$, $j = 1,\dots,N$, with the coupling conditions (3). Let the damping functions $a_j(x)$ and $b_j(x)$ be bounded and continuous on each edge. Let λ_n be eigenvalues of the corresponding problem (4) and μ_n eigenvalues for a_j and b_j replaced by their averages. Then, the constant terms c_0 in the asymptotic expansion of λ_n coincide with the corresponding constant terms in the asymptotic expansion of μ_n.*

Now, we write a theorem on the location of nonreal eigenvalues, which has a nice corollary. It shows that the high frequency abscissas are located between the negative maximum of the averages of the damping functions on each edge and the negative minimum of these damping functions.

Theorem 3.5. *Let us consider a damped wave equation on a graph with N edges of lengths l_j, bounded damping coefficients $a_j(x)$ and potentials $b_j(x)$, and the coupling conditions given by (3). If λ is an eigenvalue of H with nontrivial imaginary part $\Im(\lambda) \neq 0$, then its real part satisfies:*

$$\Re(\lambda) = -\frac{\sum_{j=1}^{N} \int_0^{l_j} a_j(x)|u_j(x)|^2\,\mathrm{d}x}{\sum_{j=1}^{N} \|u_j(x)\|_2^2},$$

where $u_j(x)$ denotes the corresponding wavefunction components.

Idea of the proof: The main idea of the proof is to take the equation (4), multiply it on the left by $\bar{u_j}(x)$, integrate over each edge and sum over all the edges. The imaginary part of the result is:

$$0 = 2\mathrm{i}\,\Im(\lambda) \sum_{j=1}^{N} \int_{e_j} (a_j(x) + \Re(\lambda))\,|u_j(x)|^2\,\mathrm{d}x,$$

from which the conclusion follows. □

Corollary 3.6. *Let us consider a damped wave equation on graph Γ with damping functions on the edges $a_j(x)$ and potentials $b_j(x)$. We denote the average of the damping function on each edge by $\bar{a}_j$. Then, the real part of nonreal eigenvalues of H (and therefore also all high frequency abscissas) lie in the interval $[-\max_j \bar{a}_j, -\min_j \bar{a}_j]$.*

4. Pseudo orbit expansion

There is a different approach to the secular equation than the one shown in the previous sections. The secular equation can be constructed by the method of pseudo orbit expansion, which has been developed for quantum graphs [5–7]. This theory was adapted for the damped wave equation in [1], and now, we summarize its main ideas.

First, the metric graph Γ is replaced by a directed graph Γ_2, each edge is replaced by two edges e_j and $\hat{e}_j$ in both directions. The functional values on both corresponding directed edges must be the same, hence if we use the ansatz:

$$f_{e_j}(x) = \alpha_{e_j}^{\mathrm{in}} \mathrm{e}^{\tilde{\lambda}_j x} + \alpha_{e_j}^{\mathrm{out}} \mathrm{e}^{-\tilde{\lambda}_j x}\,,$$
$$f_{\hat{e}_j}(x) = \alpha_{\hat{e}_j}^{\mathrm{in}} \mathrm{e}^{\tilde{\lambda}_j x} + \alpha_{\hat{e}_j}^{\mathrm{out}} \mathrm{e}^{-\tilde{\lambda}_j x}\,,$$

we have from $f_{e_j}(x) = f_{\hat{e}_j}(l_j - x)$ the relation between the coefficients of this ansatz:

$$\alpha_{\hat{e}_j}^{\mathrm{out}} = \mathrm{e}^{\tilde{\lambda}_j l_j} \alpha_{e_j}^{\mathrm{in}}\,, \qquad \alpha_{e_j}^{\mathrm{out}} = \mathrm{e}^{\tilde{\lambda}_j l_j} \alpha_{\hat{e}_j}^{\mathrm{in}}\,, \tag{7}$$

where $\tilde{\lambda}_j = \sqrt{\lambda^2 + 2\lambda a_j - b_j}$. Furthermore, we will now define several variables. The vertex scattering matrix maps the vector $\vec{\alpha}_v^{\mathrm{in}}$ into $\vec{\alpha}_v^{\mathrm{out}}$ by the relation $\vec{\alpha}_v^{\mathrm{out}} = \sigma_v(\lambda)\vec{\alpha}_v^{\mathrm{in}}$. Here, $\vec{\alpha}_v^{\mathrm{in,out}} = (\alpha_{e_{v1}}^{\mathrm{in,out}}, \dots, \alpha_{e_{vd}}^{\mathrm{in,out}})^{\mathrm{T}}$ and v denotes the vertex. The matrix $\Sigma(\lambda)$ is block-diagonalizable and it is written in the basis corresponding to:

$$\vec{\alpha} = (\alpha_{e_1}, \dots, \alpha_{e_N}, \alpha_{\hat{e}_1}, \dots, \alpha_{\hat{e}_N})^{\mathrm{T}}\,.$$

This is block diagonal with blocks $\sigma_v(\lambda)$ if written in the following basis:

$$(\alpha_{e_{v_1 1}}^{\mathrm{in}}, \dots, \alpha_{e_{v_1 d_1}}^{\mathrm{in}}, \alpha_{e_{v_2 1}}^{\mathrm{in}}, \dots, \alpha_{e_{v_2 d_2}}^{\mathrm{in}}, \dots,)^{\mathrm{T}}.$$

Furthermore, we define

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad L = \exp\left(\mathrm{diag}\,(-\tilde\lambda_1 l_1, \dots, -\tilde\lambda_N l_N, -\tilde\lambda_1 l_1, \dots, -\tilde\lambda_N l_N)\right)\,,$$

which then allows us to write:

$$\begin{pmatrix} \vec\alpha_e^{\mathrm{in}} \\ \vec\alpha_{\hat e}^{\mathrm{in}} \end{pmatrix} = L \begin{pmatrix} \vec\alpha_{\hat e}^{\mathrm{out}} \\ \vec\alpha_e^{\mathrm{out}} \end{pmatrix} = LJ \begin{pmatrix} \vec\alpha_e^{\mathrm{out}} \\ \vec\alpha_{\hat e}^{\mathrm{out}} \end{pmatrix} = LJ\Sigma(\lambda) \begin{pmatrix} \vec\alpha_e^{\mathrm{in}} \\ \vec\alpha_{\hat e}^{\mathrm{in}} \end{pmatrix}\,,$$

where we have used the definition of the matrix L and relations (7), then the definition of the matrix J and finally the definition of the matrix Σ. Since the vectors on the left and the right side are the same, we obtain the secular equation:

$$\det\,(LJ\Sigma(\lambda) - I_{2N\times 2N}) = 0\,. \tag{8}$$

Next, following the terminology of [7], we define the following notions.

Definition 4.1. *A* periodic orbit *is a closed trajectory on the graph* Γ_2. *An* irreducible pseudo orbit $\bar\gamma$ *is a collection of periodic orbits where none of the directed bonds is contained more than once. Let* $m_{\bar\gamma}$ *denote the number of periodic orbits in* $\bar\gamma$, $L_{\bar\gamma} = \sum_{e\in\bar\gamma} \tilde\lambda_e l_e$ *where the sum is over all directed bonds in* $\bar\gamma$ *and* $\tilde\lambda_e = \sqrt{\lambda^2 + 2a_e\lambda - b_e}$. *The coefficients* $A_{\bar\gamma} = \Pi_{\gamma_j\in\bar\gamma} A_{\gamma_j}$ *with* A_{γ_j} *given as multiplication of entries of* $S(\lambda) = J\Sigma(\lambda)$ *along the trajectory* γ_j.

We give without the proof a theorem which gives the secular equation (8) in the terms of pseudo orbit expansion.

Theorem 4.2. *The secular equation for the damped wave equation on a metric graph is given by:*

$$\sum_{\bar\gamma} (-1)^{m_{\bar\gamma}}\, A_{\bar\gamma}(\lambda) \exp(-L_{\bar\gamma}(\lambda)) = 0$$

with $L_{\bar\gamma}$ *being the sum of the lengths of all directed edges along a particular irreducible pseudo orbit* $\bar\gamma$.

5. Number of distinct high frequency abscissas

In this section, we state the three main theorems of this paper. These theorems give upper and lower bounds on the number of distinct high frequency abscissas for a graph which has all edges of lengths 1. The first theorem gives an upper bound for a graph with general coupling conditions.

Theorem 5.1. *Let* Γ *be an equilateral graph with* N *edges of the length* 1. *Let us assume a damped wave equation on* Γ *with damping and constant potential functions constant on each edge* $a_j(x) \equiv a_j$, $b_j(x) \equiv b_j$ *and with general coupling given by (3) for a given unitary matrix* U. *Then there are at most* $2N$ *high frequency abscissas.*

Idea of the proof: We perform an expansion according to the theorem 3.2. In the first term of the n-asymptotics of the secular equation (written by the pseudo orbit expansion) is a polynomial equation in $y = \mathrm{e}^{c_0}$ of order $2N$. This polynomial equation has $2N$ complex solutions, therefore, there are at most $2N$ different numbers c_0 and $2N$ distinct high frequency abscissas. □

For a special type of graphs, the bound can be improved. In the second theorem, we consider a bipartite graph, the graph which can be colored by only two colors, with the neighboring vertices having different colors. Another definition is that there is not a loop of edges of odd length. In this case, there are at most N distinct high frequency abscissas.

Theorem 5.2. *Let Γ be a graph with N edges all of which have lengths equal to 1, (general) Robin coupling at the boundary and standard coupling otherwise. Let us suppose that the graph is bipartite. Then, for any damping functions bounded and $\mathcal{C}^2$ at each edge, there are at most N high frequency abscissas.*

Idea of the proof: Similarly to the previous theorem, we can construct the leading term of the n asymptotics of the secular equation. In the pseudo orbit expansion, we obtain only pseudo orbits, which have even length. Due to this fact there are only terms with e^{2c_0} in the secular equation. The first term of the n-expansion is a polynomial equation in e^{2c_0} of order N. Hence, there are at most N high frequency abscissas. □

The third theorem gives a lower bound on the number of high frequency abscissas. A tree graph with vertices of odd degree is considered.

Theorem 5.3. *Let Γ be a tree graph with N edges all with unit length, Robin coupling at the boundary and standard coupling otherwise. Let us suppose that all vertices have odd degree. Then, there always exists such a damping, for which the number of high frequency abscissas is greater than or equal to N.*

Idea of the proof: The main idea is that the contribution of the pseudo orbits to the coefficient in the secular equation cancels if and only if there is a vertex of Γ with a degree $2v$ and the pseudo orbit contains exactly v edges which emanate from this vertex. This can be proven using rather technical lemma 6.3 from the paper [1]. Hence, if a tree graph has all vertices of odd degree, then there is no cancellation and all the coefficients in the secular equation are nonzero. Now, we construct the damping function. We choose constant damping on each edge with $0 \ll a_N \ll a_{N-1} \ll \cdots \ll a_1$. Now we can rewrite the first term of the secular equation as:

$$C_N \mathrm{e}^{2a_1+2a_2+\cdots+2a_N} y^N + C_{N-1}\mathrm{e}^{2a_1+2a_2+\cdots+2a_{N-1}} \left[1+\mathcal{O}\left(\mathrm{e}^{-2(a_{N-1}-a_N)}\right)\right] y^{N-1} + \\ + \cdots + C_2 \mathrm{e}^{2a_1+2a_2}\left[1+\mathcal{O}\left(\mathrm{e}^{-2(a_2-a_3)}\right)\right] y^2 + C_1 \mathrm{e}^{2a_1}\left[1+\mathcal{O}\left(\mathrm{e}^{-2(a_1-a_2)}\right)\right] y + C_0 = 0\,,$$

with $y = \mathrm{e}^{2c_0}$. We recall that none of the coefficients C_i are equal to zero. Now, if y is close to e^{-2a_1}, the last two terms are dominant, for y close to e^{-2a_2} the terms with C_2 and C_1 are dominant, etc. Hence we have

$$y_j = -\frac{C_{j-1}}{C_j}\,\mathrm{e}^{-2a_j}\left[1+\mathcal{O}\left(\mathrm{e}^{-2(a_j-a_{j+1})}\right)\right]\,.$$

We obtain N distinct numbers y_j and hence N distinct numbers c_0 and N distinct high frequency abscissas. □

6. Examples

Now, we present two particular examples, which illustrate the behavior of the eigenvalues.

6.1. Two loops with different damping coefficients

The first example of a graph consists of two loops, each loop having three edges of lengths 1 (see figure 1). Let us assume that there is constant damping a_1 on the first loop and a_2 on the second loop. Therefore, one can use the ansatz $f_j(x) = \alpha_j \sinh(\tilde{\lambda}_j(\lambda)x) +$

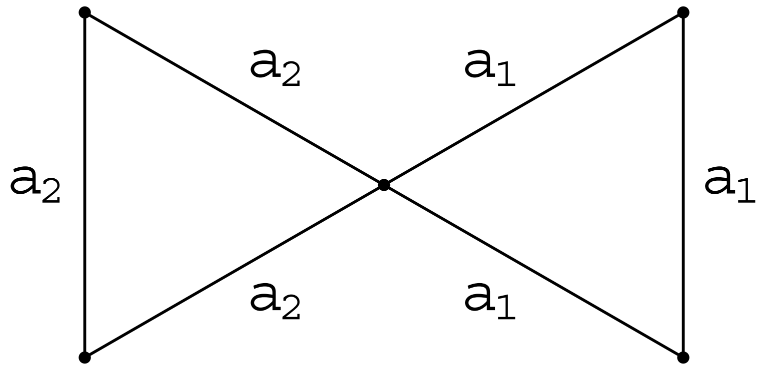

FIG. 1. Graph with two loops

$\beta_j \cosh(\tilde{\lambda}_j(\lambda)x)$, where j distinguishes the loop. We choose $x = 0$ at the middle of each loop. From the continuity at the central vertex, we have:

$$\alpha_j \sinh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda)\right) + \beta_j \cosh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda)\right) = -\alpha_j \sinh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda)\right) + \beta_j \cosh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda)\right).$$

Therefore, we either have $\alpha_1 = \alpha_2 = 0$ or $\sinh\left(\frac{3}{2}\tilde{\lambda}_1(\lambda)\right) = 0$ or $\sinh\left(\frac{3}{2}\tilde{\lambda}_2(\lambda)\right) = 0$.

First, we will assume $\alpha_1 = \alpha_2 = 0$. From the standard conditions at the central vertex we have:

$$\beta_1 \cosh\frac{3\tilde{\lambda}_1(\lambda)}{2} = \beta_2 \cosh\frac{3\tilde{\lambda}_2(\lambda)}{2},$$
$$\beta_1\tilde{\lambda}_1(\lambda) \sinh\frac{3\tilde{\lambda}_1(\lambda)}{2} + \beta_2\tilde{\lambda}_2(\lambda) \sinh\frac{3\tilde{\lambda}_2(\lambda)}{2} = 0.$$

where

$$\tilde{\lambda}_j \equiv \tilde{\lambda}_j(\lambda) = \sqrt{\lambda^2 + 2a_j\lambda - b_j}.$$

This set of equations is solvable under the condition:

$$\tilde{\lambda}_2 \sinh\frac{3\tilde{\lambda}_2}{2} \cosh\frac{3\tilde{\lambda}_1}{2} + \tilde{\lambda}_1 \sinh\frac{3\tilde{\lambda}_1}{2} \cosh\frac{3\tilde{\lambda}_2}{2} = 0.$$

or, equivalently by:

$$(\tilde{\lambda}_1 + \tilde{\lambda}_2) \sinh\frac{3(\tilde{\lambda}_1 + \tilde{\lambda}_2)}{2} + (\tilde{\lambda}_1 - \tilde{\lambda}_2) \sinh\frac{3(\tilde{\lambda}_1 - \tilde{\lambda}_2)}{2} = 0.$$

Using the asymptotic expansion $\lambda_n = 2\pi i n + c_0 + \mathcal{O}\left(\frac{1}{n}\right)$ one obtains

$$4\pi i n\left(\mathrm{e}^{6\pi i n + \frac{3}{2}(a_1+a_2+2c_0)} - \mathrm{e}^{-6\pi i n - \frac{3}{2}(a_1+a_2+2c_0)}\right) + \mathcal{O}(1) = 0,$$

and therefore:

$$3(a_1 + a_2 + 2c_0) + 2\pi i n = 0, \tag{9}$$
$$c_0^{(s)} = -\frac{a_1 + a_2}{2} + \frac{s\pi i}{6}, \quad s \in \{0, \ldots, 5\}$$

Now, let us return to the condition $\sinh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda_n)\right) = 0$. This leads to:

$$3a_j + 3c_0^{(s)} + \mathcal{O}\left(\frac{1}{n}\right) = 2\pi i s \quad \Rightarrow \quad c_0^s = -a_j + \frac{2\pi i s}{3}, \quad s \in \{0, 1, 2\}$$

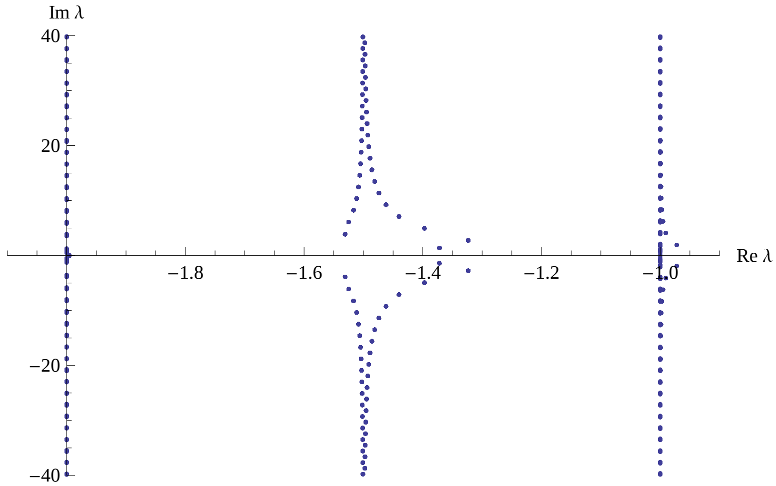

FIG. 2. Spectrum of a graph in figure 1, $a_1 = 2$, $a_2 = 1$, $b_1 = 0$, $b_2 = 0$

Hence, we have three high frequency abscissas at $-a_1$, $-a_2$ and $-\frac{a_1+a_2}{2}$. The eigenfunctions for the first two abscissas are supported on the first loop or the second loop, respectively. The third one has eigenfunction supported on both loops. Eigenvalues for particular choice $a_1 = 2$ and $a_2 = 1$ are shown in figure 2.

6.2. Star graph with different lengths of the edges

The second example illustrates eigenvalue behavior in the case when the lengths of the edges are not equal to one. Let us consider a star graph consisting of three edges of lengths l_1, l_2 and l_3. We assume Dirichlet coupling at the free ends and standard coupling in the central vertex.

If we use the ansatz $f_j(x) = \alpha_j \sinh \tilde{\lambda}_j x$ on each edge with $x = 0$ at the free end, we obtain the secular equation:

$$\sum_{j=1}^{3} \tilde{\lambda}_j \cosh \tilde{\lambda}_j l_j \prod_{\substack{i=1 \\ i \neq j}}^{3} \sinh \tilde{\lambda}_i l_i = 0 \,.$$

In figure 3, we show the eigenvalues for particular choice of the damping $a_1 = 3$, $a_2 = 4$, $a_3 = 5$ and the lengths of the edges $l_1 = 1$, $l_2 = 1$, $l_3 = 1.03$. If we wanted to apply the theorems from the previous sections, we would have 303 edges of lengths 0.01, which means that the bound on the number of the high frequency abscissas would be 606. In figures 4 and 5, the behavior is shown for other combinations of edge lengths.

7. Conclusion

We have summarized the main results of a paper [1]. The main results are three theorems in section 5 on graphs with edges of unit lengths. If we have a graph with general coupling, the number of high frequency abscissas is bounded from above by $2N$. For a bipartite graph with standard coupling, the bound can be improved to N. And finally, for a tree graph with vertices of odd degree, one can find such a damping for which there is at least N high frequency abscissas.

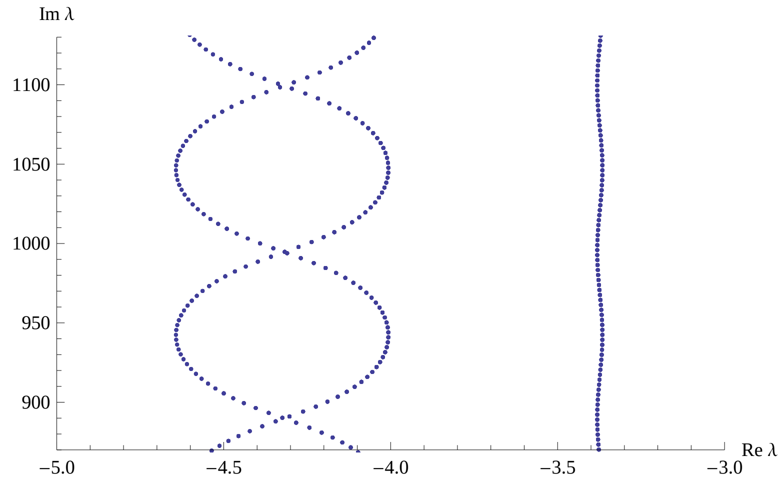

FIG. 3. Spectrum of a star graph with different edges lengths, $l_1 = 1$, $l_2 = 1$, $l_3 = 1.03$

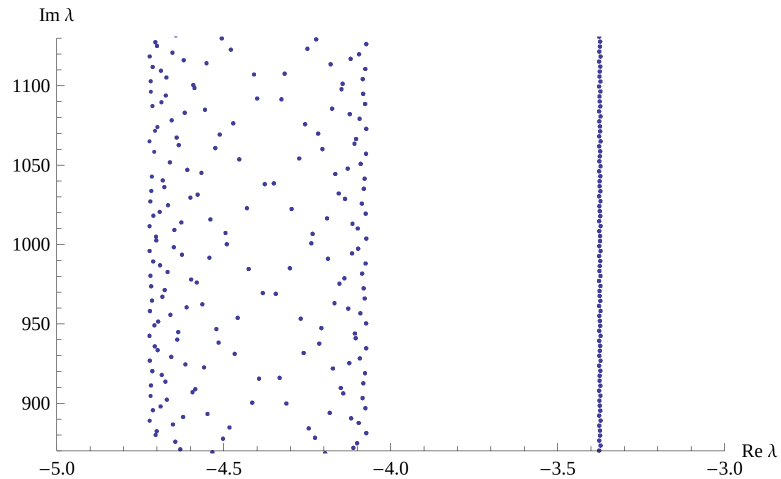

FIG. 4. Spectrum of a star graph with different edges lengths, $l_1 = 1$, $l_2 = 1$, $l_3 = 1.41$

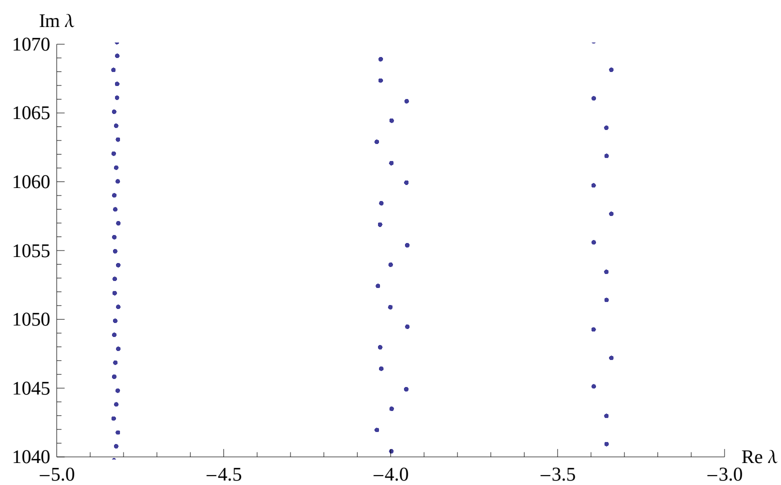

FIG. 5. Spectrum of a star graph with different edges lengths, $l_1 = 1.5$, $l_2 = 2.1$, $l_3 = 3.1$

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