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Cauchy problem for the linearized KdV equation on general metric star graphs

Z. A. Sobirov[1,2], M. I. Akhmedov[2], H. Uecker[3]

[1]Faculty of Mathematics, National University of Uzbekistan,
Vuzgorodok, 100047 Tashkent, Uzbekistan
[2]Applied Mathematics Department of Tashkent Financial Institute,
100000 Tashkent, Uzbekistan
[3]Institut für Mathematik, Universität Oldenburg, D26111 Oldenburg, Germany

hannes.uecker@uni-oldenburg.de

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1. Introduction

The Korteweg-de Vries (KdV) equation has attracted much attention in the literature, both in the context of physics and mathematics. This equation was found to permit soliton solutions and allow the modeling of solitary wave propagation on a water surface, a phenomena first discovered by Scott Russell in 1834. The KdV equation is also used, e.g., to model the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems such as ion-acoustic waves in a collisionless plasma, and magnetosonic waves in a magnetized plasma etc [11]. The linearized KdV provides an asymptotic description of linear, undirectional, weakly dispersive long waves, for example, shallow water waves. Earlier, it was proven that via normal form transforms, the solution of the KdV equation can be reduced to the solution for the linear KdVequation [12]. Belashov and Vladimirov [12] numerically investigated the evolution of a single disturbance $u(0,x) = u_0 \exp(-x^2/l^2)$ and showed that in the limit $l \to 0$, $u_0 l^2 = \mathrm{const}$, the solution of the KdV equation is qualitatively similar to the solution of the linearized KdV equation. Boundary value problems on half lines were considered in [2, 5, 7].

Here, summarizing and extending the results in [13], we investigate the linearized KdV equation on star graphs Γ with $m + k$ semi-infinite bonds connected at one point, called the vertex. The bonds are denoted by B_j, $j = 1, 2, ..., k + m$, the coordinate x_j on B_j is defined from $-\infty$ to 0 for $j = 1, 2, ..., k$, and from 0 to $+\infty$ for $j = k + 1, ..., k + m$ such that on each bond, the vertex corresponds to 0. On each bond we consider the linear equation:

$$\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x_j^3}\right) u_j(x_j, t) = f_j(x, t), \quad t > 0,\ x_j \in B_j. \tag{1}$$

Below, we will also use x instead of x_j $(j = 1, 2, ..., k + m)$. We investigate an initial value problem, and using the method of potentials, construct solution formulas.

2. Formulation of the problems

To solve linear KdV equation on an interval, one needs to impose three boundary conditions (BC): two on the left end of the x-interval and one on the right end, (see, e.g., [5,6] and references therein). For the star graph with $m+k$ semi-infinite bonds, we need to impose $k+2m$ BCs, which should also provide connection between the bonds. In detail, we require:

$$u_1(0;t) = a_j u_j(0;t), \quad j = \overline{2, k+m}, \tag{2}$$

$$u_x^+(+0;t) = B u_x^-(-0;t), \tag{3}$$

$$\sum_{i=1}^{k} a_i^{-1} u_{ixx}(-0;t) = \sum_{i=k+1}^{k+m} a_i^{-1} u_{ixx}(+0;t), \tag{4}$$

for $t>0$, where $u^-(x;t) = (u_1(x,t), \dots, u_k(x,t))^T$, $u^+(x;t) = (u_{k+1}(x,t), \dots, u_{k+m}(x,t))^T$, subscripts x and double x mean the first and the second order partial derivatives with respect to x, a_k are non-zero constants, B is a $m \times k$ matrix.

Furthermore, we assume that the $f_j(x,t)$ and the initial conditions:

$$u_j(x,0) = u_{0j}(x), \ \ x \in \overline{B_j}, \ (j = 1, 2, ..., k+m), \tag{5}$$

are sufficiently smooth enough and bounded, and that $u_{0,j}$ satisfies the vertex conditions (2) – (4).

It should be noted that the above vertex conditions are not the only possible ones. The main motivation for our choice is caused by the fact that they guarantee uniqueness of the solution and, if the solutions decay (to zero) at infinity,the norm (energy) conservation.

Here, we introduce some notation that will be useful in the following. For any vector, $\mathbf{v} = (v_1, v_2, \dots, v_{k+m})^T$ we put $\tilde{\mathbf{v}} = (v_2, \dots, v_{k+m})^T$, $\mathbf{v}^- = (v_1, \dots, v_k)^T$, $\tilde{\mathbf{v}}^- = (v_2, \dots, v_k)^T$, $\mathbf{v}^+ = (v_{k+1}, \dots, v_{k+m})^T$.

3. Existence and uniqueness of solutions

Lemma 1. *Let $I_k - B^T B$ be negatively defined matrix. Then the problem has at most one solution in $H^3(\Gamma)$.*

Proof of Lemma 1. Using the equation (1) one can easily get:

$$\frac{\partial}{\partial t} \int_a^b u_j^2(x,t)dx = \left(2u_j u_{jxx} - u_{jx}^2\right) \Big|_{x=a}^{x=b} + 2 \int_a^b f_j(x,t) u_j(x,t) dx$$

for appropriate values of constants a and b. From this equality and vertex conditions (2) – (4) we have:

$$\frac{d}{dt} \|\mathbf{u}(\cdot,t)\|_\Gamma^2 \le \left(\partial \mathbf{u}^-(0,t)\right)^T \left(I_k - B^T B\right) \partial \mathbf{u}^-(0,t) + 2 \|\mathbf{u}(\cdot,t)\|_\Gamma \|\mathbf{f}(\cdot,t)\|_\Gamma .$$

According to condition of Lemma 1, we get:

$$\|\mathbf{u}(\cdot,t)\|_\Gamma \le \|\mathbf{u}(\cdot,0)\|_\Gamma + \int_0^t \|\mathbf{f}(\cdot,\tau)\|_\Gamma d\tau. \tag{6}$$

The inequality (6) proves the lemma.

Notice that equality in (6) (i.e. energy conservation) holds iff $B^T B = I_m$.

We shall construct solutions and prove existence theorems for data from the Schwartz class of smooth decreasing functions, and for data in Sobolev classes.

Let $S(B_k)$ be the Schwartz space of rapidly decaying functions on the closure of B_k, $k = 1, 2, 3$. We say $v(x,t) \in C^1([0,T];\, S(B_k))$ $(T > 0)$ if v and $\dfrac{\partial v}{\partial t}$ in $C([0,T];\, S(B_k))$.

Theorem 1. *Assume that $I_k - B^T B$ is negatively defined matrix, $u_{0k}(x) \in S(B_k)$, $f_k(x,t) \in C^1([0,T];\, S(B_k))$ for some $T > 0$ and that $u_{0k}^{(p)} \equiv \dfrac{\partial^{3p}}{\partial x^{3p}} u_{0k}(x)$ and $f_k^{(p)} = \dfrac{\partial^{3p}}{\partial x^{3p}} f_k(x,t)$ satisfy vertex conditions (1) – (5) for any nonnegative integer p. Then (1) – (5) has a solution in $C^1([0,T];\, S(B_k))$.*

To treat the case of Sobolev data consider function $\mathbf{v} = (v_1(x_1), v_2(x_2), \ldots, v_{k+m}(x_{k+m}))$ defined on the graph. We suppose that $v_k \in S(B_k)$ and the functions $v_k^{(p)} \equiv \dfrac{\partial^{3p}}{\partial x^{3p}} v_k(x)$ satisfy vertex conditions (2) –(4) for any non-negative integer p. We denote the set of all such functions v by $S^-(\Gamma)$ $(S^+(\Gamma))$, and define $W^-(\Gamma)$ (or $W^+(\Gamma)$) as the closure of the set $S^-(\Gamma)$ (or $S^+(\Gamma)$) with respect to the norm $\|v\|_{3,\Gamma} = \sum_{k=1}^{3} \|v_k\|_{H^3(B_k)}$.

Theorem 2. *Let $I_k - B^T B$ be negatively defined matrix, $\mathbf{u}_0 \equiv (u_{01}(x_1), u_{02}(x_2), \ldots, u_{0,k+m}(x_{k+m})) \in W^{\pm}(\Gamma)$. Then (1) – (5) has a unique solution in $L_\infty(0,T,W^{\pm}(\Gamma))$.*

First, we construct exact solutions, using the results from the theory of potentials for the linearized KdV equation. For that purpose, we give some preliminaries from [1, 3, 5].

4. Some preliminaries from potentials theory

The following functions are called fundamental solutions of the equation $u_t - u_{xxx} = 0$ (see [1, 3, 5, 12]):

$$U(x,t;\xi,\eta) = \begin{cases} \dfrac{1}{(t-\eta)^{1/3}} f\left(\dfrac{x-\xi}{(t-\eta)^{1/3}}\right), & \text{if } t > \eta; \\ 0, & \text{if } t \le \eta, \end{cases}$$

$$V(x,t;\xi,\eta) = \begin{cases} \dfrac{1}{(t-\eta)^{1/3}} \phi\left(\dfrac{x-\xi}{(t-\eta)^{1/3}}\right), & \text{if } t > \eta; \\ 0, & \text{if } t \le \eta, \end{cases}$$

where $f(x) = \dfrac{\pi}{3^{1/3}} Ai\left(-\dfrac{x}{3^{1/3}}\right)$, $\phi(x) = \dfrac{\pi}{3^{1/3}} Bi\left(-\dfrac{x}{3^{1/3}}\right)$ for $x \ge 0$, $\phi(x) = 0$ for $x < 0$ and $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions. The functions $f(x)$ and $\phi(x)$ are integrable and $\int_{-\infty}^{0} f(x)dx = \dfrac{\pi}{3}$, $\int_0^{+\infty} f(x)dx = \dfrac{2\pi}{3}$, $\int_0^{+\infty} \phi(x)dx = 0$. We summarize some properties of potentials for (1) from [3, 5]. For given ω, f and ϕ let:

$$u(x,t) = \int\limits_a^b U(x,t;\xi,0)\omega(\xi)d\xi, \quad v(x,t) = \int\limits_0^t \int\limits_a^b U(x,t;\xi,\tau) f(\xi,\tau) d\xi d\tau,$$

$$w^{(1)}(x,t) = \int\limits_\eta^t U_{x\xi}(x,\eta;a,t)\phi(\eta)d\eta, \quad w^{(2)}(x,t) = \int\limits_\eta^t V_{x\xi}(x,\eta;a,t)\phi(\eta)d\eta.$$

Lemma 2. *a) Let $\omega \in BV([a,b])$. Then $u(x,t)$ satisfies $u_t - u_{xxx} = 0$ for $t > 0$ and:*

$$\lim_{(x,y)\to(x_0,0)} u(x,t) = \begin{cases} \pi\omega(x_0), & \text{if } x_0 \in (a,b); \\ 0, & \text{if } x_0 \notin (a,b). \end{cases}$$

b) Let $f \in L^2((a,b)\times(0,T)$. Then, $v(x,t)$ satisfies $u_t - u_{xxx} = \pi f(x,t)$ in $(a,b)\times(0,T]$, $T > 0$ and initial condition $u(x,0) = 0$, $x \in (a,b)$.

c) If $\phi \in H^1(0,T)$, then :

$$\lim_{x\to a+0} w^{(1)}(x,t) = \frac{2\pi}{3}\phi(y), \quad \lim_{x\to a-0} w^{(1)}(x,t) = -\frac{\pi}{3}\phi(y), \quad \lim_{x\to a+0} w^{(2)}(x,t) = 0.$$

Now, we are ready to construct exact solutions for the considered problems. We assume that initial data and source terms in each bond are sufficiently smooth and bounded functions.

5. Integral formula for exact solution

We look for solution in the form:

$$\mathbf{u}(x,t) = \int_0^t U(x,t;0,\eta)\boldsymbol{\phi}(\eta)d\eta + \int_0^t V(x,t;0,\eta)\boldsymbol{\psi}(\eta)d\eta + \mathbf{F}(x,t), \tag{7}$$

where

$$F_k(x,t) = \frac{1}{\pi}\int_{B_k} U(x,t;\xi,0)u_k(\xi,0)d\xi + \frac{1}{\pi}\int_0^t\int_{B_k} U(x,t;\xi,\eta)f_k(\xi,\eta)d\xi d\eta.$$

$\boldsymbol{\phi}(t) = (\phi_1(t),\ldots,\phi_{k+m}(t))^T$, $\boldsymbol{\psi}(t) = (0,\ldots,0,\psi_{k+1}(t),\ldots,\psi_{k+m}(t))^T$ are unknown vector functions.

According to vertex conditions (2) – (4), we get:

$$\tilde{\mathbf{C}}\cdot\int_0^t \frac{f(0)}{(t-\tau)^{1/3}}\phi_1(\tau)d\tau = \int_0^t \frac{f(0)}{(t-\tau)^{1/3}}\tilde{\boldsymbol{\phi}}(\tau)d\tau + \int_0^t \frac{\varphi(0)}{(t-\tau)^{1/3}}\tilde{\boldsymbol{\psi}}(\tau)d\tau - \left(\tilde{\mathbf{C}}F_1 - \tilde{\mathbf{F}}\right)\Big|_{x=0}, \tag{8}$$

$$\begin{gathered} \int_0^t \frac{f'(0)}{(t-\tau)^{2/3}}\boldsymbol{\phi}^+(\tau)d\tau + \int_0^t \frac{\varphi'(0)}{(t-\tau)^{2/3}}\boldsymbol{\psi}^+(\tau)d\tau = \\ \mathbf{B}\int_0^t \frac{f'(0)}{(t-\tau)^{2/3}}\boldsymbol{\phi}^-(\tau)d\tau - \left(\partial\mathbf{F}^+(0,t) - \mathbf{B}\cdot\partial\mathbf{F}^-(0,t)\right), \end{gathered} \tag{9}$$

$$2(\mathbf{C}^-)^T f^-(t) + (\mathbf{C}^+)^T f^+(t) = \frac{3}{\pi}\left(-\mathbf{C}^-\middle|\mathbf{C}^+\right)\partial^2\mathbf{F}(0,t), \tag{10}$$

where $\mathbf{C} = \left(1, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots, \dfrac{1}{a_{k+m}}\right)$.

Abel's integral equations (8), (9) can be written in terms of fractional integrals [9]:

$$J^\alpha_{(0,t)}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad 0 < \alpha < 1,$$

and solved using the inverse operators, i.e. the Riemann-Liouville fractional derivatives [8, 9] defined by:

$$D^{\alpha}_{(0,t)} f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad 0 < \alpha < 1.$$

Using the relation $D^{\alpha}_{(0,t)} J^{\alpha}_{(0,t)} = I$ from (8) and (9), we obtain the linear algebraic equations:

$$f(0)\left(\tilde{\mathbf{C}}\big| - \mathbf{I}_{m+k-1}\right)\boldsymbol{\phi}(t) - \left(\mathbf{0}_{(m+k-1)\times 1}\big|\mathbf{I}_{m+k-1}\right)\varphi(0)\boldsymbol{\psi}(t) = - \frac{1}{\Gamma(1/3)}\left(\tilde{\mathbf{C}}\big| - \mathbf{I}_{m+k-1}\right) D^{2/3}_{(0,t)} \mathbf{F}(0,t), \quad (11)$$

$$f'(0)\left(\mathbf{B}\big| - \mathbf{I}_m\right)\boldsymbol{\phi}(t) + \varphi'(0)\left(\mathbf{0}_{m\times k}\big| - \mathbf{I}_m\right)\boldsymbol{\psi}(t) = -\frac{1}{\Gamma(2/3)} D^{1/3}_{(0,t)}\left(\mathbf{B}\big| - \mathbf{I}_m\right)\partial\mathbf{F}(0,t). \quad (12)$$

We rewrite the system of equations (10), (11) and (12) in the following matrix form:

$$M \cdot \begin{pmatrix} \boldsymbol{\phi}(t) \\ \boldsymbol{\psi}^+(t) \end{pmatrix} = \mathbf{G}(t), \quad \mathbf{G}(t) = \begin{pmatrix} -\frac{1}{\Gamma(1/3)}\left(\tilde{\mathbf{C}}\big| - \mathbf{I}_{m+k-1}\right) D^{2/3}_{(0,t)} \mathbf{F}(0,t) \\ -\frac{1}{\Gamma(2/3)} D^{1/3}_{(0,t)}\left(\mathbf{B}\big| - \mathbf{I}_m\right)\partial\mathbf{F}(0,t) \\ \frac{3}{\pi}(-\mathbf{C}^-\big|\mathbf{C}^+)\partial^2\mathbf{F}(0,t) \end{pmatrix},$$

where

$$M = \left(\begin{array}{c|c|c|c} f(0)\tilde{C}^- & -f(0)I_{k-1} & 0_{(k-1)\times m} & 0_{(k-1)\times m} \\ f(0)C^+ & 0_{m\times(k-1)} & -f(0)I_m & -\phi(0)I_m \\ \multicolumn{2}{c|}{f'(0)B} & -f'(0)I_m & -\phi'(0)I_m \\ \multicolumn{2}{c|}{2C^-} & C^+ & 0_{1\times m} \end{array}\right)$$

Now we must prove that $\det(M) \neq 0$.

Suppose that $\det(M) = 0$. Then, the homogenous equation $M\boldsymbol{\alpha} = 0$ has a non-trivial, time independent solution $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{k+m})$ (notice that M is constant matrix). Therefore, putting in (7) $\boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 = (\alpha_1, \ldots, \alpha_{k+m})^T = \text{const}$, $\boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 = (0, \ldots, 0, \alpha_{k+m+1}, \ldots, \alpha_{k+2m}) = \text{const}$, we obtain a solution for the problem with $\mathbf{u}_0(x) \equiv 0$, $\mathbf{f}(x,t) \equiv 0$. According to the uniqueness theorem, we have:

$$\boldsymbol{\phi}_0 \int_0^t \frac{1}{(t-\tau)^{1/3}} f\left(\frac{x}{(t-\tau)^{1/3}}\right) d\tau + \boldsymbol{\psi}_0 \int_0^t \frac{1}{(t-\tau)^{1/3}} \varphi\left(\frac{x}{(t-\tau)^{1/3}}\right) d\tau = 0,$$

or

$$\boldsymbol{\phi}_0 f\left(\frac{x}{t^{1/3}}\right) + \boldsymbol{\psi}_0 \varphi\left(\frac{x}{t^{1/3}}\right) = 0,$$

for any fixed t. The last equality contradicts the condition of linear independence for the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. This proves the statement $\det(M) \neq 0$.

Summarizing the above studies, we obtain:

$$\mathbf{u}(x,t) = \mathbf{F}(x,t) + \int_0^t \mathbf{U}(x,t-\tau)M^{-1}\mathbf{G}(\tau)d\tau$$

with

$$\mathbf{U}(x,t) = \left(\begin{array}{ccccc} U(x,t;0,0)\mathbf{I}_k & | & \mathbf{0}_{k\times m} & | & \mathbf{0}_{k\times m} \\ \mathbf{0}_{m\times k} & | & U(x,t;0,0)\mathbf{I}_m & | & V(x,t;0,0)\mathbf{I}_m \end{array} \right).$$

6. Proof of existence theorems

Proof of Theorem 1. According to the theory of potentials [3, 5], the solutions constructed in the previous sections and their x-derivatives, up to the second order, are continuous functions in the closure of each bondof the graph.

Now, we consider the functions $v_j(x_j,t)$ that are solutions of the considered problem with initial conditions $v_j(x,0) = \frac{d^3}{dx^3}u_{0j}(x)$, and with f_j replaced by $\frac{\partial^3}{\partial x^3}f_j$. According to the conditions of the theorem, one can easily obtain $\frac{\partial^3 u_j}{\partial x^3}(x,t) = v_j(x,t)$. From this, we conclude that the functions $u_j(x,t)$, $j = 1,2,3$ and their x-derivatives of any order are continuous functions in the closure of B_j.

Now, we consider the half lines corresponding to each bond separately. Notice that $u_j(x,t)$ is a solution of the linearized KdV equation on the half line B_j and satisfies compatibility conditions at the point $x = 0$, $t = 0$. Applying Theorem 1.1 from [7], we get that these solutions define a C^1 map from $[0,T]$ into $S(\bar{B}_j)$.

Proof of Theorem 2. Above, we proved the estimate:

$$\|\mathbf{u}(\cdot,t)\|_\Gamma \le \|\mathbf{u}(\cdot,0)\|_\Gamma + \int_0^t \|\mathbf{f}(\cdot,\tau)\|_\Gamma \, d\tau.$$

Note that for the function v, constructed above, the same estimate holds.

Summing up these two estimates, we have:

$$\|\mathbf{u}(\cdot,t)\|_{3,\Gamma} \le \|\mathbf{u}(\cdot,0)\|_{3,\Gamma} + \int_0^t \|\mathbf{f}(\cdot,\tau)\|_{3,\Gamma} \, d\tau. \tag{13}$$

By construction, $S^{\pm}(\Gamma)$ is dense in $W^{\pm}(\Gamma)$. This, together with the *a priori* estimate (13) proves the theorem. Thus, we have shown the existence and uniqueness of the solution for the linearized KdV equation on a metric star graph and derived its explicit solution. The above approach can also be extended to cases of graphs with different topologies.

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