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The motion of a charged particle in the field by a frequency-modulated electromagnetic wave

G. F. Kopytov, A. A. Martynov, N. S. Akintsov

Kuban State University, Krasnodar, Russia

g137@mail.ru, martynov159@yandex.ru, akintsov777@mail.ru

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In this article, the exact solutions of equations of motion for a charged particle in a frequency-modulated wave are presented. We performed an analysis of the results for the motion of a charged particle in the field of frequency-modulated electromagnetic waves. A point of interest was a solution for the equations of the motion for a charged particle in the field of a plane electromagnetic wave. We investigated the interaction of high intensity laser pulses with solid targets in relation to the practical development of multi-frequency lasers and laser modulation technology. This study was undertaken because of the wide practical application of high-temperature plasma formed on the surface of the target and the search for new modes of laser plasma interaction

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1. Introduction

The creation of petawatt level laser systems in recent years, has allowed the study of a new unique physical object – relativistic laser plasma, produced when gas, clusters or solid targets are exposed to intense laser radiation [1]. Developments in different fields of physics and engineering; e.g. plasma physics, astrophysics, powerful relativistic high-frequency electronics and appliances. have increased the interest in studying the interaction between charged particles and electromagnetic waves. A special role in such interactions is assigned to relativistic charged particles in strong electromagnetic waves. The energy characteristics of a charged particle in the field of a frequency-modulated electromagnetic wave are of interest as a result of the practical development of multifrequency lasers and laser modulation techniques [2–4]. In this paper, we consider the dynamics of an electron in an intense frequency-modulated electromagnetic field of linear and circular polarization. The interaction of charged particles with ultrashort femtosecond laser pulses with radiation intensities of up to 1022 W/cm^2 is one of the main areas of laser physics at the moment. Previous literature [5] discussed the consistent derivation of a particle's average kinetic energy in an intense electromagnetic field by a frequency-modulated electromagnetic wave, but was not found by averaging coordinate, momentum, and energy values for the particle over the period of the particle's oscillation in the field plane of a monochromatic frequency-modulated electromagnetic wave.

The problem of the motion of a charged particle in the field of a plane monochromatic frequency-modulated electromagnetic wave was formulated and solved for linear and circular polarization of the wave [6], but the interest in this topic has appeared presently in connection with the development of high-power lasers. The peculiarity of this work lies in the fact that there are considered highly-fields for review before the end of the simple modes of interaction of charged particles with a frequency-modulated electromagnetic wave.

The aim of this work is to analyze the motion of a particle in the external field of frequency-modulated electromagnetic wave of high intensity and to derive the average kinetic energy of a particle over the oscillation period of the field.

2. Problem Statement

The equation of motion of a particle of mass m and charge q placed in an external field of a plane monochromatic wave has a known form (see, for example [7], paragraph 17). The equation of motion for a charged particle being acted upon by a high-Lorentz force is given by:

$$\frac{d\boldsymbol{p}}{dt} = q\boldsymbol{E} + \frac{q}{c}\left[\boldsymbol{V} \times \boldsymbol{H}\right], \tag{1}$$

where $\boldsymbol{p}$ – momentum of charged particle; $\boldsymbol{E}$ and $\boldsymbol{H}$ – electric and magnetic intensity of the laser field; $q > 0$ the absolute value of the electron charge. Equation (1) is supplemented by the initial conditions for the velocity and position of the electron:

$$\boldsymbol{V}(0) = \boldsymbol{V}_0, \quad \boldsymbol{r}(0) = \boldsymbol{r}_0.$$

The particle momentum $\boldsymbol{p}$ and velocity $\boldsymbol{V}$ are related by equality ([7], paragraph 9):

$$\boldsymbol{p} = \frac{m\boldsymbol{V}}{\sqrt{1 - \frac{V^2}{c^2}}}. \tag{2}$$

The change in the particle energy:

$$\varepsilon = \frac{mc^2}{\sqrt{1 - \frac{V^2}{c^2}}} = \sqrt{m^2c^4 + p^2c^2}, \tag{3}$$

is determined by the equation:

$$\frac{d\varepsilon}{dt} = q\boldsymbol{E}\boldsymbol{V}. \tag{4}$$

It follows from (2) and (3) that the energy ε, momentum $\boldsymbol{p}$, and velocity $\boldsymbol{V}$ of the particle are related by equations:

$$\boldsymbol{p} = \frac{\varepsilon\boldsymbol{V}}{c^2}, \quad \boldsymbol{V} = \frac{c^2\boldsymbol{p}}{\varepsilon}. \tag{5}$$

In this paper, it is assumed that the frequency of the electromagnetic wave is modulated harmonically $\phi = \mu\sin(\omega'\xi + \psi)$, where $\mu = \Delta\omega_m/\omega'$ – modulation index equal to the ratio of frequency deviation to the frequency of the modulating wave; ω' – frequency modulation; $\xi = t - z/c$; ψ – constant phase. We assign the plane wave as propagating along the axis$z$, then the vector components of electric and magnetic fields of plane monochromatic wave are given by:

$$\begin{cases} E_x = H_y = b_x \exp\big(-i\left(\omega\xi + \alpha + \mu\sin\left(\omega'\xi + \psi\right)\right)\big); \\ E_y = -H_x = f b_y \exp\big(-i\left(\omega\xi + \alpha + \mu\sin\left(\omega'\xi + \psi\right)\right)\big); \\ E_z = H_z = 0, \end{cases} \tag{6}$$

where ω is carrier frequency of the wave; α – constant phase; the x and y axes coincide with the b_x and b_y axes of the polarization ellipse of the wave and $b_x \geq b_y \geq 0$; $f = \pm 1$ is a polarization parameter (the upper and lower signs in the expression for E_y correspond to right and left polarization [5]).

We take the real part form (6) and apply the Jacobi-Anger expansion then obtain:

$$\begin{cases} E_x = H_y = b_x \sum\limits_{n=-\infty}^{+\infty} J_n(\mu) \cos \Phi_n, \\ E_y = -H_x = f b_y \sum\limits_{n=-\infty}^{+\infty} J_n(\mu) \cos \Phi_n, \\ E_z = H_z = 0, \end{cases} \tag{7}$$

where $J_n(\mu)$ is the n-th Bessel function; $\Phi_n = (\omega + n\omega')\,\xi + \alpha + n\psi$.

As can be seen from (7), the frequency spectrum of the modulated wave is symmetrical and is not theoretically limited, but when $n \gg \mu$ Bessel function becomes negligible and the width of the spectrum can be limited. Practical spectral width is determined by the expression $\Delta\omega = 2(\mu + 1)\,\omega'$. In (7), index n can vary from $-N$ to N, where the number $N \approx \mu + 1$. Thus, if $\mu \ll 1$ and $N = 1$, then spectrum width $\Delta\omega = 2\omega'$ coincides with the width of the spectrum of a harmonic amplitude-modulated wave [5]. When $\mu \gg 1$ and $N = \mu$, spectral width is equal to twice the frequency deviation $\Delta\omega = 2\Delta\omega_m$.

3. Solution of the equation of the charge motion

The solution of equations (1) and (4) with $\boldsymbol{E}$ and $\boldsymbol{H}$ from (7) has the form:

$$\begin{gathered} p_x = \frac{q b_x}{\omega} \sum_{n=-N}^{N} \frac{J_n(\mu) \sin \Phi_n}{(1 + n\eta)} + \chi_x, \\ p_y = \frac{f q b_y}{\omega} \sum_{n=-N}^{N} \frac{J_n(\mu) \sin \Phi_n}{(1 + n\eta)} + \chi_y, \\ p_z = \gamma g; \\ \varepsilon = c\gamma\,(1 + g)\,, \end{gathered} \tag{8}$$

where χ_x, χ_y and γ are constants ($\gamma \geq 0$ because $\varepsilon \geq mc^2$):

$$\begin{gathered} g = h + \frac{q}{\gamma^2 \omega}\,(b_x \chi_x + f b_y \chi_y) \sum_{n=-N}^{N} \frac{J_n(\mu) \sin \Phi_n}{(1 + n\eta)} + \\ \frac{q^2 \left(b_x^2 + b_y^2\right)}{2\gamma^2 \omega^2} \sum_{\substack{n,k=-N \\ n \neq k}}^{N} \frac{J_n(\mu)\, J_k(\mu) \sin \Phi_n \sin \Phi_k}{(1 + n\eta)\,(1 + k\eta)} - \frac{q^2 \left(b_x^2 + b_y^2\right)}{4\gamma^2 \omega^2} \sum_{n=-N}^{N} \frac{J_n^2(\mu) \cos(2\Phi_n)}{(1 + n\eta)^2}; \end{gathered} \tag{9}$$

$$h = \frac{1}{2} \left\{ \frac{m^2 c^2 + \chi_x^2 + \chi_y^2}{\gamma^2} - 1 + \frac{q^2 \left(b_x^2 + b_y^2\right)}{2\gamma^2 \omega^2} \sum_{n=-N}^{N} \frac{J_n^2(\mu)}{(1 + n\eta)^2} \right\}. \tag{10}$$

$\Phi_k = (\omega + k\omega')\,\xi + \alpha + k\psi$, k is index of the Bessel function.

From (8) and (5), we obtain the parametric representation (the parameter ξ) of the particle velocity:

$$\begin{gathered}
V_x = \frac{dx}{dt} = \frac{c}{\gamma}\left(1 - \frac{V_z}{c}\right)\left(\frac{qb_x}{\omega}\sum_{n=-N}^{N}\frac{J_n(\mu)\sin\Phi_n}{(1+n\eta)} + \chi_x\right) = \\
\frac{c}{(1+g)\gamma}\left(\frac{qb_x}{\omega}\sum_{n=-N}^{N}\frac{J_n(\mu)\sin\Phi_n}{(1+n\eta)} + \chi_x\right), \\
V_y = \frac{dy}{dt} = \frac{c}{\gamma}\left(1 - \frac{V_z}{c}\right)\left(\frac{fqb_y}{\omega}\sum_{n=-N}^{N}\frac{J_n(\mu)\sin\Phi_n}{(1+n\eta)} + \chi_y\right) = \\
\frac{c}{(1+g)\gamma}\left(\frac{fqb_y}{\omega}\sum_{n=-N}^{N}\frac{J_n(\mu)\sin\Phi_n}{(1+n\eta)} + \chi_y\right), \\
V_z = \frac{dz}{dt} = \frac{cg}{1+g}.
\end{gathered} \tag{11}$$

Through the constants χ_x, χ_y and γ, determined by the initial phase of the wave:

$$\Phi_{n0} = -(\omega + n\omega')\frac{z}{c} + \alpha + n\psi, \tag{12}$$

and the initial velocity of the particle V_0; from (3), (8) and (11) we find:

$$\begin{gathered}
\chi_x = \frac{mV_{x0}}{\sqrt{1 - V_0^2/c^2}} - \frac{qb_x}{\omega}\sum_{n=-N}^{N}\frac{J_n(\mu)\sin\Phi_{n0}}{(1+n\eta)}; \\
\chi_y = \frac{mV_{y0}}{\sqrt{1 - V_0^2/c^2}} - \frac{fqb_y}{\omega}\sum_{n=-N}^{N}\frac{J_n(\mu)\sin\Phi_{n0}}{(1+n\eta)}; \\
\gamma = \frac{mc(1 - v_{z0}/c)}{\sqrt{1 - v_0^2/c^2}}.
\end{gathered} \tag{13}$$

From (11), we obtain the following solutions for coordinates of the particles as functions of the parameter ξ:

$$\begin{gathered}
x = x_0 + \sum_{n=-N}^{N}\frac{\chi_x(\Phi_n - \Phi_{n0})}{\gamma k(1+n\eta)} - \frac{qb_x}{\gamma\omega k}\sum_{n=-N}^{N}\frac{J_n(\mu)}{(1+n\eta)^2}(\cos\Phi_n - \cos\Phi_{n0}), \\
y = y_0 + \sum_{n=-N}^{N}\frac{\chi_y(\Phi_n - \Phi_{n0})}{\gamma k(1+n\eta)} - \frac{fqb_y}{\gamma\omega k}\sum_{n=-N}^{N}\frac{J_n(\mu)}{(1+n\eta)^2}(\cos\Phi_n - \cos\Phi_{n0}), \\
z = z_0 + \sum_{n=-N}^{N}\frac{h(\Phi_n - \Phi_{n0})}{k(1+n\eta)} - \frac{2q}{\gamma^2 k\omega}(b_x\chi_x + fb_y\chi_y)\sum_{n=-N}^{N}\frac{J_n(\mu)(\cos\Phi_n - \cos\Phi_{n0})}{(1+n\eta)^2} - \\
\frac{q^2\left(b_x^2 + b_y^2\right)}{2\gamma^2 k\omega}\sum_{n=-N}^{N}\frac{J_n^2(\mu)(\sin(2\Phi_n) - \sin(2\Phi_{n0}))}{(1+n\eta)^3} + \\
\frac{q^2\left(b_x^2 + b_y^2\right)}{2\gamma^2\omega k}\sum_{\substack{n,k=-N \\ n\neq k}}^{N}\frac{J_n(\mu)J_k(\mu)}{(1+n\eta)(1+k\eta)}\times \\
\left(\frac{\sin\Phi_{(n-k)}}{n-k} - \frac{\sin\Phi_{(n-k)0}}{n-k} - \frac{\sin\Phi_{(n+k)}}{n+k} + \frac{\sin\Phi_{(n+k)0}}{n+k}\right),
\end{gathered} \tag{14}$$

where $\Phi_{(n-k)} = \Phi_n - \Phi_k$, $\Phi_{(n+k)} = \Phi_n + \Phi_k$; $\Phi_{(n-k)0} = \Phi_{n0} - \Phi_{k0}$, $\Phi_{(n+k)0} = \Phi_{n0} + \Phi_{k0}$.

From (11) and (14), we determine that the motion of a particle in the external field of the plane monochromatic electromagnetic wave is the imposition of movement with the constant velocity $\boldsymbol{V}$ and vibrational motion with the frequency $\tilde{\omega} = 2\pi/\tilde{T}$ different from the frequency of the field ω and the frequency modulation ω':

$$x(t) = \tilde{x} + \tilde{V}_x t + \xi(t), \quad y(t) = \tilde{y} + \tilde{V}_y t + \eta(t), \quad z(t) = \tilde{z} + \tilde{V}_z t + \zeta(t), \tag{15}$$

where $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are constants and

$$\xi\left(t + \tilde{T}\right) = \xi(t), \quad \eta\left(t + \tilde{T}\right) = \eta(t), \quad \zeta\left(t + \tilde{T}\right) = \zeta(t) \tag{16}$$

are periodic function with same period.

We seek the solution of the equation for the coordinate z in (14) from (15). By substituting $z(t)$ from (15) into (14) and selecting constants $\tilde{z}$ and $\tilde{V}_z$ in the form:

$$\tilde{z} = z_0 + \left[\frac{2q}{\gamma^2 k\omega}\left(b_x\chi_x + f b_y\chi_y\right)\sum_{n=-N}^{N}\frac{J_n(\mu)\cos\Phi_{n0}}{(1+n\eta)^2} - \frac{q^2\left(b_x^2+b_y^2\right)}{2\gamma^2 k\omega}\sum_{n=-N}^{N}\frac{J_n^2(\mu)\sin(2\Phi_{n0})}{(1+n\eta)^3} - \right.$$
$$\left.\frac{q^2\left(b_x^2+b_y^2\right)}{2\gamma^2\omega k}\sum_{\substack{n,k=-N\\ n\neq k}}^{N}\frac{J_n(\mu)J_k(\mu)}{(1+n\eta)(1+k\eta)}\left(\frac{\sin\Phi_{(n-k)0}}{n-k} - \frac{\sin\Phi_{(n+k)0}}{n+k}\right)\right]\frac{1}{1+h}; \tag{17}$$

$$\tilde{V}_z = \frac{ch}{1+h}, \tag{18}$$

we obtain the equation for $\zeta(t)$:

$$(1+h)\,\zeta(t) =$$
$$-\frac{q}{\gamma^2\omega}\left[\frac{2}{k}\left(b_x\chi_x + f b_y\chi_y\right)\sum_{n=-N}^{N}\frac{J_n(\mu)\cos\Phi_n}{(1+n\eta)^2} + \frac{q\left(b_x^2+b_y^2\right)}{2k}\sum_{n=-N}^{N}\frac{J_n^2(\mu)\sin(2\Phi_n)}{(1+n\eta)^3} + \right.$$
$$\left.\frac{q\left(b_x^2+b_y^2\right)}{2k}\sum_{\substack{n,k=-N\\ n\neq k}}^{N}\frac{J_n(\mu)J_k(\mu)}{(1+n\eta)(1+k\eta)}\left(\frac{\sin\Phi_{(n-k)}}{n-k} - \frac{\sin\Phi_{(n+k)}}{n+k}\right)\right]. \tag{19}$$

It follows from (19) that $\zeta(t)$ is a periodic function defined by the period $\tilde{T}$. Let us find its period. The period $\tilde{T}$ of particle oscillation in the field of a frequency-modulated electromagnetic wave is determined from the formulae $\Phi\left(t + \tilde{T}\right) = \Phi(t) + 2\pi$, from which, taking into account (7), (15) and (16), it follows that:

$$\tilde{T}_n = \frac{2\pi}{\omega}\frac{(1+h)}{(1+n\eta)} = T\frac{(1+h)}{(1+n\eta)}. \tag{20}$$

One can see that the oscillation period of the particle differs from that of the field of the frequency-modulated electromagnetic wave.

We will seek the solution of the first equation in (14) in the form $x(t)$ from (14). By representing constants $\tilde{x}$ and $\tilde{V}_x$ in the form:

$$\tilde{x} = x_0 + \sum_{n=-N}^{N} \frac{\chi_x (z_0 - \tilde{z})}{\gamma k (1+n\eta)} + \frac{qb_x}{\gamma\omega k} \sum_{n=-N}^{N} \frac{J_n(\mu)}{(1+n\eta)^2} \cos\Phi_{n0},$$
$$\tilde{V}_x = \chi_x \frac{c}{\gamma}\left(1 - \frac{\tilde{V}_z}{c}\right) = \frac{\chi_x}{\gamma}\frac{c}{(1+h)}, \tag{21}$$

we find that:

$$\xi(t) = -\sum_{n=-N}^{N} \frac{\chi_x}{\gamma(1+n\eta)}\zeta(t) - \frac{qb_x}{\gamma\omega k}\sum_{n=-N}^{N} \frac{J_n(\mu)}{(1+n\eta)^2}\cos\Phi_n. \tag{22}$$

Similarly, we obtain for $y(t)$ in (15):

$$\tilde{y} = y_0 + \sum_{n=-N}^{N} \frac{\chi_y (z_0 - \tilde{z})}{\gamma k (1+n\eta)} + \frac{fqb_y}{\gamma\omega k} \sum_{n=-N}^{N} \frac{J_n(\mu)}{(1+n\eta)^2} \cos\Phi_{n0},$$
$$V_y = \chi_y \frac{c}{\gamma}\left(1 - \frac{V_z}{c}\right) = \frac{\chi_y}{\gamma}\frac{c}{(1+h)\gamma}, \tag{23}$$
$$\eta(t) = -\sum_{n=-N}^{N} \frac{\chi_y}{\gamma k(1+n\eta)}\zeta(t) - \frac{fqb_y}{\gamma\omega k}\sum_{n=-N}^{N} \frac{J_n(\mu)}{(1+n\eta)^2}\cos\Phi_n.$$

4. The motion of a particle averaged over an oscillation period

In this section, we will perform the averaging of the coordinate $\boldsymbol{r}(t)$, velocity $\boldsymbol{V}(t)$, momentum $\boldsymbol{p}(t)$, and energy $\varepsilon(t)$ of the particles over its oscillation period (20) with (8), (11) and (14) in the field of a frequency-modulated electromagnetic wave.

Consider a new variable of the integration $\xi' = \xi(t')$, then:

$$\Phi'_n = \Phi_n(t');$$
$$dt' = \frac{d\Phi'_n}{\omega(1+n\eta)}\frac{1}{1 - V_z/c} = \frac{1+g}{\omega(1+n\eta)} d\Phi'_n. \tag{24}$$

Since the motion of particle is a superposition of two kinds of periodic motion with frequencies ω and ω', averaging will be carried out according to the formula:

$$\bar{f}(t) = \frac{1}{\tilde{T}_n}\int_{\Phi(t)}^{\Phi(\tilde{t})} f(t')\frac{1+g}{\omega(1+n\eta)} d\Phi'_n, \tag{25}$$

where $f(t')$ is an arbitrary function taking into account (7), (15) and (20).

For the coordinate x in (14), we have:

$$\bar{x}(t) = \frac{1}{\tilde{T}}\int_t^{\tilde{t}} x(t')\,dt' = \left(x_0 - \sum_{n=-N}^{N} \frac{\chi_x \Phi_{n0}}{\gamma k(1+n\eta)} + \frac{qb_x}{\gamma\omega k}\sum_{n=-N}^{N}\frac{J_n(\mu)}{(1+n\eta)^2}\cos\Phi_{n0}\right) +$$
$$\frac{\chi_x}{\gamma k}\sum_{n=-N}^{N}\frac{1}{\tilde{T}_n}\int_t^{\tilde{t}}\frac{\Phi_n(t')}{(1+n\eta)}dt' - \frac{qb_x}{\gamma\omega k}\sum_{n=-N}^{N}\frac{1}{\tilde{T}_n}\int_t^{\tilde{t}}\frac{J_n(\mu)}{(1+n\eta)^2}\cos\Phi_n(t')\,dt', \tag{26}$$

where

$$\tilde{t} = t + \tilde{T}_n \tag{27}$$

see (7), (15) and (28).

$$\begin{aligned}\Phi_n(t) = (1+n\eta)\left[\omega t - k\left(\tilde{z} + \tilde{V}_z t + \zeta(t)\right)\right] + \alpha + n\psi = \\ (1+n\eta)\left[\tilde{\omega} t - k\tilde{z} - k\zeta(t)\right] + \alpha + n\psi.\end{aligned} \tag{28}$$

By using (27) and (28), we obtain the expression:

$$\int_t^{\tilde{t}} \Phi_n(t')\,dt' = (\alpha + n\psi + (1+n\eta)[\tilde{\omega} t - k\tilde{z}])\,\tilde{T} - (1+n\eta)\,k\int_t^{\tilde{t}} \zeta(t)\,dt' \tag{29}$$

for the first integral in the right-hand side of (26). The integral in the right-hand side of (29) is independent of t, because $\zeta(t)$ is periodic function with a period $\tilde{T}_n$. This integral is zero. The Fourier component of the function $\zeta(t)$ multiplied by $\tilde{T}_n$.

Expression (29) can then be transformed to:

$$\int_t^{\tilde{t}} \Phi_n(t')\,dt' = \left(\alpha + n\psi - (1+n\eta)\,k\left(\tilde{z} + \bar{\zeta}\right)\right)\tilde{T} + 2\pi t, \tag{30}$$

where $\bar{\zeta}$ is the average value of the function $\zeta(t)$ in the time interval equal the period $\tilde{T}_n$.

By substituting into (26) the values of integrals from (30) with $\bar{\zeta} = 0$, we finally obtain:

$$\bar{x}(t) = \tilde{x} + \tilde{V}_x\left(t + \tilde{T}_n/2\right), \tag{31}$$

where $\tilde{x}$ and $\tilde{V}_x$ are defined by expressions (21).

In the same way, we find:

$$\bar{y}(t) = \tilde{y} + \tilde{V}_y\left(t + \tilde{T}/2\right), \tag{32}$$

from (32) $\tilde{y}$ and $\tilde{V}_y$ are defined by expressions in (23).

Finally, taking into account that $\bar{\zeta} = 0$, the expression for:

$$\bar{z}(t) = \tilde{z} + \tilde{V}_z\left(t + \tilde{T}/2\right), \tag{33}$$

where $\tilde{z}$ and $\tilde{V}_z$ are defined by expressions (17) and (18).

Averaging the components (11) of the particle velocity, we obtain:

$$\bar{V}_x = \tilde{V}_x, \quad \bar{V}_y = \tilde{V}_y, \quad \bar{V}_z = \tilde{V}_z. \tag{34}$$

As might be expected, the speed of the particle $\bar{\boldsymbol{V}}$ in (26) corresponds $\tilde{\boldsymbol{V}}$ with (18), (21) and (23).

From the average value of the longitudinal component of the particle momentum, we obtain the expression:

$$\bar{p}_x = \chi_x + \frac{q^2 b_x \left(b_x\chi_x + f b_y\chi_y\right)}{\gamma^2\omega^2\left(1+h\right)}\sum_{n=-N}^{N}\frac{J_n^2\left(\mu\right)}{\left(1+n\eta\right)^2};$$
$$\bar{p}_y = \chi_y + \frac{q^2 b_y \left(b_x\chi_x + f b_y\chi_y\right)}{\gamma^2\omega^2\left(1+h\right)}\sum_{n=-N}^{N}\frac{J_n^2\left(\mu\right)}{\left(1+n\eta\right)^2};$$
$$\bar{p}_z = \frac{\gamma}{1+h}\left\{h + h^2 + \frac{q^2}{2\gamma^4\omega^2}\left(b_x\chi_x + f b_y\chi_y\right)^2\sum_{n=-N}^{N}\frac{J_n^2\left(\mu\right)}{\left(1+n\eta\right)^2} + \right. \tag{35}$$
$$\left. \frac{q^4\left(b_x^2+b_y^2\right)^2}{16\gamma^4\omega^4}\sum_{\substack{n,k=-N\\ n\neq k}}^{N}\frac{J_n^2\left(\mu\right)J_k^2\left(\mu\right)}{\left(1+n\eta\right)^2\left(1+k\eta\right)^2} + \frac{q^4\left(b_x^2+b_y^2\right)^2}{32\gamma^4\omega^4}\sum_{n=-N}^{N}\frac{J_n^4\left(\mu\right)}{\left(1+n\eta\right)^4}\right\}.$$

The average energy $\bar{\varepsilon}$ of the particles is determined by formula:

$$\bar{\varepsilon} = \frac{c\gamma}{1+h}\left\{\left(1+h\right)^2 + \frac{q^2}{2\gamma^4\omega^2}\left(b_x\chi_x + f b_y\chi_y\right)^2\sum_{n=-N}^{N}\frac{J_n^2\left(\mu\right)}{\left(1+n\eta\right)^2} + \right.$$
$$\left. \frac{q^4\left(b_x^2+b_y^2\right)^2}{16\gamma^4\omega^4}\sum_{\substack{n,k=-N\\ n\neq k}}^{N}\frac{J_n^2\left(\mu\right)J_k^2\left(\mu\right)}{\left(1+n\eta\right)^2\left(1+k\eta\right)^2} + \frac{q^4\left(b_x^2+b_y^2\right)^2}{32\gamma^4\omega^4}\sum_{n=-N}^{N}\frac{J_n^4\left(\mu\right)}{\left(1+n\eta\right)^4}\right\}. \tag{36}$$

5. The case of an arbitrary polarization for a particle being initially at rest

Consider the case when the particle is initially at rest ($V_0 = 0$) and the Bessel functions indices are equal to each other ($k = n$). Formula (13) expresses χ_x, χ_y, γ and taking into account that:

$$\Phi_n\left(0\right) = \Phi_{n0} = \left(\omega + n\omega'\right)\xi_0 + \alpha + n\psi; \quad \xi_0 = -z_0/c,$$

we obtain:

$$\chi_x = -\frac{qb_x}{\omega}\sum_{n=-N}^{N}\frac{J_n\left(\mu\right)\sin\Phi_{n0}}{\left(1+n\eta\right)};$$
$$\chi_y = -\frac{fqb_y}{\omega}\sum_{n=-N}^{N}\frac{J_n\left(\mu\right)\sin\Phi_{n0}}{\left(1+n\eta\right)}; \tag{37}$$
$$\gamma = mc.$$

For a wave with an arbitrary polarization [8]:

$$b_x^2 \pm b_y^2 = \rho^2 b^2, \tag{38}$$

where ρ is the ellipticity parameter ($\rho = \pm 1$ corresponds to the linear polarization and $\rho = \pm 1/\sqrt{2}$ does to the circular one).

In other cases, the value ρ corresponds to an elliptical polarization ($0 \leq \rho \leq 1$), in which:

$$\chi_x^2 + \chi_y^2 = \frac{q^2\rho^2 b^2}{\omega^2}\sum_{n=-N}^{N}\frac{J_n^2\left(\mu\right)\sin^2\Phi_{n0}}{\left(1+n\eta\right)^2}, \tag{39}$$

$$(b_x\chi_x + f b_y\chi_y)^2 = \frac{q^2\rho^4 b^4}{\omega^2}\sum_{n=-N}^{N}\frac{J_n^2(\mu)\sin^2\Phi_{n0}}{(1+n\eta)^2}. \tag{40}$$

From (10), we obtain the value of h at the initial time:

$$h = \frac{1}{4}\left\{\sigma\left(\sum_{n=-N}^{N}\frac{J_n^2(\mu)\left(1+2\sin^2\Phi_{n0}\right)}{(1+n\eta)^2}\right)\right\}, \tag{41}$$

and according to (41):

$$\sigma = \frac{q^2\rho^2 b^2}{m^2c^2\omega^2} = \frac{2q^2}{\pi m^2 c^5} I\lambda^2, \tag{42}$$

where $I = c\rho^2 b^2/4\pi$ is the intensity of the elliptically polarized electromagnetic wave, and $\lambda = 2\pi c/\omega$ is the wavelength.

The oscillation period of a particle is:

$$\tilde{T}_n = T\left(\frac{1}{(1+n\eta)} + \frac{\sigma}{4}\left(\sum_{n=-N}^{N}\frac{J_n^2(\mu)\left(1+2\sin^2\Phi_{n0}\right)}{(1+n\eta)^3}\right)\right). \tag{43}$$

By substituting (37) – (41) in (36), we obtain the average energy of a particle at rest in the initial wave of arbitrary polarization:

$$\begin{aligned}\bar{\varepsilon} - mc^2 = \frac{mc^2\sigma}{4}&\left\{\sum_{n=-N}^{N}\frac{J_n^2(\mu)\left(1+2\sin^2\Phi_{n0}\right)}{(1+n\eta)^2}+\right.\\ \sum_{n=-N}^{N}\frac{J_n^4(\mu)}{(1+n\eta)^4}&\left.\frac{\sigma\left(2\sin^2\Phi_{n0}+1/8\right)}{\left(1+\left(\sigma J_n^2(\mu)\left(1+2\sin^2\Phi_{n0}\right)\right)/\left(4(1+n\eta)^2\right)\right)}\right\}.\end{aligned} \tag{44}$$

The maximum average energy is obtained for the phase $\Phi_{n0} = \pi/2$ or $3\pi/2$, when the field at the point where a particle in located initially zero. In this case, we have:

$$\bar{\varepsilon} - mc^2 = \frac{3mc^2\sigma}{4}\left\{\sum_{n=-N}^{N}\frac{J_n^2(\mu)}{(1+n\eta)^2}+\sum_{n=-N}^{N}\frac{J_n^4(\mu)}{(1+n\eta)^4}\frac{17\sigma}{\left(24+18\sigma J_n^2(\mu)/\left((1+n\eta)^2\right)\right)}\right\}. \tag{45}$$

The minimum average energy corresponds to the phase $\Phi_{n0} = 0$ or π and is determined by the expression:

$$\bar{\varepsilon} - mc^2 = \frac{mc^2\sigma}{4}\left\{\sum_{n=-N}^{N}\frac{J_n^2(\mu)}{(1+n\eta)^2}+\sum_{n=-N}^{N}\frac{J_n^4(\mu)}{(1+n\eta)^4}\frac{\sigma}{\left(8+2\sigma J_n^2(\mu)/(1+n\eta)^2\right)}\right\}. \tag{46}$$

The energy $\langle\bar{\varepsilon}\rangle$ of the charged particle, being further averaged over the initial phase Φ_{n0}, in the plane monochromatic arbitrarily polarized wave is given by:

$$\begin{aligned}\langle\bar{\varepsilon}\rangle - mc^2 = \frac{mc^2\sigma}{4}&\left\{\sum_{n=-N}^{N}\frac{6J_n^2(\mu)}{(1+n\eta)^2}-\right.\\ \sum_{n=-N}^{N}&\left.\frac{7\sigma J_n^4(\mu)+32J_n^2(\mu)(1+n\eta)^2}{2(1+n\eta)^2\sqrt{3J_n^4(\mu)\sigma^2+16\sigma J_n^2(\mu)(1+n\eta)^2+16(1+n\eta)^4}}\right\}.\end{aligned} \tag{47}$$

For the private case of wave circular polarization when the difference between E_x and E_y corresponds to $\pi/2$ or $3\pi/2$ (see (6) and (7)). This means that waves are located in opposite phases and cancel each other out. We obtain the average energy $\bar{\varepsilon}$ of the particle:

$$\bar{\varepsilon} - mc^2 = \frac{\sigma mc^2}{2}\left\{\sum_{n=-N}^{N}\frac{J_n^2(\mu)}{(1+n\eta)^2} + \sigma\sum_{n=-N}^{N}\frac{J_n^4(\mu)}{(1+n\eta)^4\left(4+2\sigma\sum\limits_{n=-N}^{N}\frac{J_n^2(\mu)}{(1+n\eta)^2}\right)}\right\}. \quad (48)$$

The resulting formulas (44), (45), (46) (47) and (48) for the average kinetic energy of the particles comprise an explicit dependence on the initial particle velocity, amplitude of the electromagnetic wave, a frequency modulation index, the frequency of the carrier wave, frequency modulation, intensity and polarization, which allow one to make practical calculations. When $\mu \ll 1$, $N = 1$, the formulas (44), (45), (46) (47) and (48) become the special case of linear and circular polarization form [6].

6. Conclusions

This article offers the exact solution for the equations of a charged particle's motion in the external field of a frequency-modulated electromagnetic wave. This solution indicates the dependence of the electron velocity on the intensity of the monochromatic frequency-modulated electromagnetic wave for the cases of elliptical polarization which are, therefore, the cases of different initial conditions of the charged particle motion and wave polarization. In the electromagnetic wave (7) of the field $\boldsymbol{E}$ and $\boldsymbol{H}$ is periodic with average electric and magnetic field values of zero. One would assume that such fields will have an alternating effect on charged particles and the average deviation caused by this influence is also zero. However, this assumption is incorrect. In particular, in the field of a plane frequency-modulated electromagnetic wave, the particle performs a systematic drift in the direction of the electromagnetic field, as well as the drift direction of wave propagation. The values of the momentum and energy of the particle, averaged over the period of vibration, were calculated. The oscillation period of the particle differs from that of the field. As the field intensity is increased, the frequency of the oscillatory motion of the particle tends to zero according to (20). The motion of the particle was shown to be the superposition of motion at a constant velocity and vibrational motion with the frequency of the electromagnetic field and the frequency modulation different from the field frequency. In the absence of the frequency modulation, all the formulae go to the appropriate formulae given in [6]. The solutions obtained are presented in the explicit dependence on the initial data, the amplitude of the electromagnetic wave, the wave intensity and its polarization parameter that allows practical application of the solutions. We have obtained the exact criterion for the applicability of relativistic equations of motion for a charged particle in a frequency-modulated electromagnetic field, depending on the intensity and duration of the pulse. This implies that the accuracy of the analytical calculation increases with its time duration and decreases with the intensity of the electromagnetic pulse. The practical significance of the research is that the results can be used to develop relativistic electronic devices. In addition, the results may be of interest for astrophysical research or studies involving plasma in an external electromagnetic field.

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