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Linearized KdV equation on a metric graph

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We address a linearized KdV equation on metric star graphs with one incoming finite bond and two outgoing semi-infinite bonds. Using the theory of potentials, we reduce the problem to systems of linear integral equations and show that they are uniquely solvable under conditions of the uniqueness theorem.

Keywords: KdV, IBVP, PDE on metric graphs, exact solution, third order differential equations.

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Introduction

The Korteweg – de Vries (KdV) equation is of importance for many problems of physics and related fields. In particular, soliton solutions of KdV equation have found applications in fluid mechanics [1-11]. A pioneering study of KdV equation dates back to Scott Russell, who was able to model the propagation of solitary wave on the water surface in 1834. The linearized KdV provides an asymptotic description of linear, weakly dispersive long waves, such as, e.g., shallow water waves. Earlier, it was proven that via the normal form transforms the solution of the KdV equation can be reduced to the solution of the linear KdV equation [12]. Namely, Belashov and Vladimirov [12] numerically investigated evolution of a single disturbance $u(0,x) = u_0 \exp(-x^2/l^2)$ and showed that in the limit $l \to 0$, $u_0 l^2 = const$, the solution of the KdV equation is qualitatively similar to that of the linearized KdV equation. The boundary value problems for KdV equation on half lines are considered in [2,5,7].

In this paper, we address the linearized KdV equation on a star graph Γ with one bounded bond and two semi-infinite bonds connected at one point, called the vertex. The bonds are denoted by B_j, $j = 1, 2, 3$, the coordinate x_1 on B_1 is defined from -1 to 0, and coordinates x_2 and x_3 on the bonds B_2 and B_3 are defined from 0 to such that on each bond the vertex corresponds to 0. On each bond we consider the linear equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x_j^3}\right) u_j(x_j, t) = f_j(x, t), \quad t > 0, x_j \in B_j, \quad j = 1, 2, 3. \tag{1}$$

Below, we will also use the notation x instead of x_j ($j = 1, 2, 3$). We treat a boundary value problem and using the method of potentials, reduce it to a system of integral equations. The solvability of the obtained system of integral equations is proven.

1. Formulation of the problems

To solve the linear KdV equation on an interval, one needs to impose three boundary conditions (BC): two on the left end of the$x$-interval and one on the right end, (see, e.g., [5-6] and references therein). For the above star graph, we need to impose 5 BCs at the vertex point, which should provide also connection between the bonds and 2 BCs at the left side of B_1. In detail, we require:

$$u_1(-1;t) = \phi_0(t), \quad u_{1x}(-1;t) = \phi_1(t), \tag{2}$$

$$u_1(0,t) = a_2 u_2(0,t) = a_3 u_3(0,t), \tag{3}$$

$$u_{1x}(0;t) = b_2 u_{2x}(0;t) = b_3 u_{3x}(0;t), \tag{4}$$

$$u_{1xx}(0;t) = a_2^{-1} u_{2xx}(0;t) + a_3^{-1} u_{3xx}(0;t), \tag{5}$$

for $0 < t < T,\ T = const.$

Furthermore, we assume that the functions $f_j(x,t), j = 1,2,3,$ are smooth enough and bounded. The initial conditions are given by:

$$u_j(x,0) = 0, \ \ x \in \overline{B_j}, \ (j = 1,2,3). \tag{6}$$

It should be noted that the above vertex conditions are not the only possible ones. The main motivation for our choice is caused by the fact that they guarantee uniqueness of the solution and, if the solutions decay (to zero) at infinity, the norm (energy) conservation.

2. Existence and uniqueness of solutions

Lemma 1. *Let $\frac{1}{b_2^2} + \frac{1}{b_3^2} \leqslant 1$. Then the problem (1)-(6) has at most one solution.*

Proof of Lemma 1. Using the equation (1) one can easily get:

$$\frac{d}{dt}\int\limits_a^b u_j^2(x,t)dx = \left(2u_j u_{jxx} - u_{jx}^2\right)\Big|_{x=a}^{x=b} + 2\int\limits_a^b f_j(x,t)u_j(x,t)dx,$$

for appropriate values of constants a and $b$on each bond. We put $\phi_0(t) \equiv 0$. Then, the above equalities and vertex conditions (2)-(5) yield:

$$\frac{d}{dt}\left(e^{-\varepsilon t}\|u\|^2\right) \leqslant e^{-\varepsilon t}\left(\frac{1}{\varepsilon^2}\|f\|^2 + \phi_1^2(t)\right),$$

$$\|u\|^2 \leqslant \int\limits_0^t e^{-\varepsilon(t-\tau)}\left(\frac{1}{\varepsilon^2}\|f(\cdot,\tau)\|^2 + \phi_1^2(\tau)\right)d\tau, \tag{7}$$

where $(u,v) = \int\limits_{-1}^{0} u_1 v_1 dx_1 + \int\limits_{0}^{+\infty} u_2 v_2 dx_2 + \int\limits_{0}^{+\infty} u_3 v_3 dx_3$, $||u|| = \sqrt{(u,u)}$ are L_2 scalar product and norm defined on graph, ε is an arbitrary positive number.

Uniqueness of the solution follows from (7).

Theorem 1. *Let $a_2^2 + a_3^2 + a_2^2 a_3^2 + \frac{a_2}{b_3} + \frac{a_3}{b_2} \neq 0, \frac{1}{b_2^2} + \frac{1}{b_3^2} \leqslant 1, \phi_0(t) \in C^2[0,T],\ \phi_1(t) \in C^1[0,T]$. Then the problem (1) – (6) has a unique solution in $C^1([0,T],\ C^3(\Gamma))$.*

Proof of Theorem 1.

To prove the theorem, we use the following functions are called fundamental solutions of the equation $u_t - u_{xxx} = 0$ (see [1, 3, 5, 12, 16]):

$$U(x,t;\xi,\eta) = \begin{cases} \frac{1}{(t-\eta)^{1/3}} f\left(\frac{x-\xi}{(t-\eta)^{1/3}}\right), & t > \eta, \\ 0 & t \leqslant \eta, \end{cases}$$

$$V(x,t;\xi,\eta)=\begin{cases}\frac{1}{(t-\eta)^{1/3}}\varphi\left(\frac{x-\xi}{(t-\eta)^{1/3}}\right), & t>\eta,\\ 0 & t\leqslant\eta,\end{cases}$$

where $f(x)=\frac{\pi}{3^{1/3}}Ai\left(-\frac{x}{3^{1/3}}\right)$, $\varphi(x)=\frac{\pi}{3^{1/3}}Bi\left(-\frac{x}{3^{1/3}}\right)$ for $x\geqslant 0$, $\varphi(x)=0$ for $x<0$ and $Ai(x)$ and $Bi(x)$ are the Airy functions. The functions $f(x)$ and $\varphi(x)$ are integrable and $\int\limits_{-\infty}^{0} f(x)dx=\frac{\pi}{3}$, $\int\limits_{0}^{+\infty} f(x)dx=\frac{2\pi}{3}$, $\int\limits_{0}^{+\infty}\varphi(x)dx=0$.

Below, we also use fractional integrals [9]:

$$J^{\alpha}_{(0,t)}f(t):=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\ \ 0<\alpha<1,$$

and the inverse of this operator, i.e. the Riemann-Liouville fractional derivatives [8, 9] defined by:

$$D^{\alpha}_{(0,t)}f(t):=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}f(\tau)d\tau,\ \ 0<\alpha<1.$$

We look for solution in the form:

$$u_1(x,t)=\int\limits_0^t U(x,t;0,\eta)\varphi_1(\eta)d\eta+\int\limits_0^t U(x,t;-1,\eta)\alpha(\eta)d\eta+\int\limits_0^t V(x,t;-1,\eta)\beta(\eta)d\eta+F_1(x,t),$$

$$u_2(x,t)=\int\limits_0^t U(x,t;0,\eta)\varphi_2(\eta)d\eta+\int\limits_0^t V(x,t;0,\eta)\psi_2(\eta)d\eta+F_2(x,t), \tag{8}$$

$$u_3(x,t)=\int\limits_0^t U(x,t;0,\eta)\varphi_3(\eta)d\eta+\int\limits_0^t V(x,t;0,\eta)\psi_3(\eta)d\eta+F_3(x,t), \tag{9}$$

where $F_k(x,t)=\frac{1}{\pi}\int\limits_0^t\int\limits_{B_k} U(x,t;\xi,\eta)f_k(\xi,\eta)d\xi d\eta,\ k=1,2,3.$

Satisfying the conditions (2) - (4), we have:

$$f(0)\alpha(t)+\varphi(0)\beta(t)+D^{2/3}_{(0,t)}\int\limits_0^t\varphi_1(\eta)f(-\frac{1}{(t-\eta)^{1/3}})dt=\frac{1}{\Gamma(\frac{1}{3})}D^{2/3}_{(0,t)}[\phi_0(t)-F_1(0,t)], \tag{10}$$

$$f'(0)\alpha(t)+\varphi'(0)\beta(t)+D^{1/3}_{(0,t)}\int\limits_0^t\varphi_1(\eta)f'(-\frac{1}{(t-\eta)^{1/3}})dt=\frac{1}{\Gamma(\frac{2}{3})}D^{1/3}_{(0,t)}[\phi_1(t)-F_{1x}(0,t)], \tag{11}$$

$$\begin{aligned}&f(0)\varphi_1(t)-a_2f(0)\varphi_2(t)-a_2\varphi(0)\psi_2(t)+\int\limits_0^t K_1\alpha(\eta)d\eta+\int\limits_0^t K_2\beta(\eta)d\eta=\\&=\tfrac{1}{\Gamma(\frac{1}{3})}D^{2/3}_{(0,t)}[F_2(0,t)-F_1(0,t)],\end{aligned} \tag{12}$$

$$\begin{aligned}&f(0)\varphi_1(t)-a_3f(0)\varphi_3(t)-a_3\varphi(0)\psi_3(t)+\int\limits_0^t K_1\alpha(\eta)d\eta+\int\limits_0^t K_2\beta(\eta)d\eta=\\&=\tfrac{1}{\Gamma(\frac{1}{3})}D^{2/3}_{(0,t)}[F_3(0,t)-F_1(0,t)].\end{aligned} \tag{13}$$

We take derivatives from (10) – (11) to obtain:

$$f(0)\alpha'(t)+\varphi(0)\beta'(t)+\frac{1}{\Gamma(\frac{1}{3})}\int\limits_0^t K_3\varphi_1(\eta)d\eta=\frac{1}{\Gamma(\frac{1}{3})}D_{(0,t)}^{2/3}\frac{d}{dt}[\phi_0(t)-F_1(0,t)], \tag{14}$$

$$f'(0)\alpha'(t)+\varphi'(0)\beta'(t)+\frac{1}{\Gamma(\frac{2}{3})}\int\limits_0^t K_4\varphi_1(\eta)d\eta=\frac{1}{\Gamma(\frac{2}{3})}D_{(0,t)}^{1/3}\frac{d}{dt}[\phi_1(t)-F_{1x}(0,t)]. \tag{15}$$

From conditions (4) and (5), it follows:

$$\begin{aligned}&f'(0)\varphi_1(t)+b_2f'(0)\varphi_2(t)-b_2\varphi'(0)\psi_2(t)+\tfrac{1}{\Gamma(\frac{2}{3})}\int\limits_0^t K_5\alpha(\eta)d\eta+\tfrac{1}{\Gamma(\frac{2}{3})}\int\limits_0^t K_6\beta(\eta)d\eta=\\&=\tfrac{1}{\Gamma(\frac{2}{3})}D_{(0,t)}^{1/3}[b_2F_{2x}(0,t)-F_{1x}(0,t)],\end{aligned} \tag{16}$$

$$\begin{aligned}&f'(0)\varphi_1(t)+b_3f'(0)\varphi_3(t)-b_3\varphi'(0)\psi_3(t)+\tfrac{1}{\Gamma(\frac{2}{3})}\int\limits_0^t K_5\alpha(\eta)d\eta+\tfrac{1}{\Gamma(\frac{2}{3})}\int\limits_0^t K_6\beta(\eta)d\eta=\\&=\tfrac{1}{\Gamma(\frac{2}{3})}D_{(0,t)}^{1/3}[b_3F_{3x}(0,t)-F_{1x}(0,t)],\end{aligned} \tag{17}$$

$$\begin{aligned}&-\tfrac{\pi}{3}\varphi_1(t)-\tfrac{1}{a_2}\tfrac{2\pi}{3}\varphi_2(t)-\tfrac{1}{a_3}\tfrac{2\pi}{3}\varphi_3(t)+\int\limits_0^t K_7\alpha'(\eta)d\eta+\int\limits_0^t K_8\beta'(\eta)d\eta=\\&=\tfrac{1}{a_2}F_{2xx}(0,t)+\tfrac{1}{a_3}F_{3xx}(0,t)-F_{1xx}(0,t),\end{aligned} \tag{18}$$

where the kernels of integral operators defined as:

$$K_1=\int\limits_\eta^t\frac{1}{(t-\tau)^{2/3}(\tau-\eta)^{1/3}}f'\left(-\frac{1}{(\tau-\eta)^{1/3}}\right)d\tau,$$

$$K_2=\int\limits_\eta^t\frac{1}{(t-\tau)^{2/3}(\tau-\eta)^{1/3}}\varphi'\left(-\frac{1}{(\tau-\eta)^{1/3}}\right)d\tau,$$

$$K_3=\int\limits_\eta^t\frac{1}{(t-\tau)^{2/3}(\tau-\eta)^{2/3}}f''\left(-\frac{1}{(\tau-\eta)^{1/3}}\right)d\tau,$$

$$K_4=\int\limits_\eta^t\frac{1}{(t-\tau)^{1/3}(\tau-\eta)^{2/3}}f'''\left(-\frac{1}{(\tau-\eta)^{1/3}}\right)d\tau,$$

$$K_5=\int\limits_\eta^t\frac{1}{(t-\tau)^{1/3}(\tau-\eta)^{1/3}}f''\left(-\frac{1}{(\tau-\eta)^{1/3}}\right)d\tau,$$

$$K_6=\int\limits_\eta^t\frac{1}{(t-\tau)^{1/3}(\tau-\eta)^{1/3}}\varphi''\left(-\frac{1}{(\tau-\eta)^{1/3}}\right)d\tau,$$

$$K_7=\int\limits_0^x U(y+1;t-\eta)dy \quad , \quad K_8=\int\limits_0^x V(y+1;t-\eta)dy.$$

We obtained the system of integral equations (12) – (18) with respect to unknowns $(\alpha'(t),\ \beta'(t),\ \varphi_1(t),\ \varphi_2(t),\ \varphi_3(t),\ \psi_2(t),\ \psi_3(t))$. The matrix of the coefficients M of these unknowns on the off integral part of the system has a determinant:

$$\det M = \frac{\pi^4 b_2 b_3}{81 a_2 a_3}\left(a_2^2 + a_3^2 + a_2^2 a_3^2 + \frac{a_2}{b_3} + \frac{a_3}{b_2}\right).$$

Under conditions of the theorem this determinant is not singular.

According to the asymptotes of Airy functions the kernels of the integral operators are integrable (see [14, 15]). Hence, it follows from the uniqueness theorem and Fredholm alternatives that the system of equations has a unique solution. Thus the solvability of the problem is proved.

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