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# Functional equations for the Potts model with competing interactions on a Cayley tree

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In this paper, we consider an infinite system of functional equations for the Potts model with competing interactions of radius $r = 2$ and countable spin values $0, 1, \ldots$, and non-zero-filled, on a Cayley tree of order two. We describe conditions on $h_x$ guaranteeing compatibility of distributions $\mu^{(n)}(\sigma_n)$.

**Keywords:** Cayley tree, Potts model, Gibbs measures, functional equations.

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## 1. Introduction

The Potts model is related to and generalized by several other models, including the $XY$ model, the Heisenberg model and the $N$-vector model. The infinite-range Potts model is known as the Kac model. When the spins are taken to interact in a non-Abelian manner, the model is related to the flux tube model, which is used to discuss confinement in quantum chromodynamics. Generalizations of the Potts model have also been used to model grain growth in metals and coarsening in foams. A further generalization of these methods by James Glazier and Francois Graner, known as the cellular Potts model, has been used to simulate static and kinetic phenomena in foam and biological morphogenesis. In this model, introduced by Askin and Teller (1943) and Potts (1952), the energy between two adjacent spins at vertices i and j is taken to be zero if the spins are the different and equal to a constant $J_{ij}$ if they are same.

In [1], the Potts model with countable set $\Phi$ of spin values on $Z^d$ was considered and it was proved that with respect to Poisson distribution on $\Phi$, the set of limiting Gibbs measure is not empty. In [2], the Potts model with a *countable* set of spin values on a Cayley tree was considered and it was shown that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independent of parameters for the Potts model with countable set of spin values on the Cayley tree. This is a crucial difference from models with a finite set of spin values, since those may have more than one translation-invariant Gibbs measures.

The work initiated in [4] was continued in [3] and a model was considered with nearest-neighbor interactions and local state space given by the uncountable set $[0, 1]$ on a Cayley tree $\Gamma^k$ of order $k \geq 2$. The translation-invariant Gibbs measures are studied via a non-linear functional equation and we prove the non-uniqueness of translation-invariant Gibbs measures in the right parameter regime for all $k \geq 2$ and not only for $k \in \{2, 3\}$ as in [3]. In [5], models (Hamiltonians) with-nearest-neighbor interactions and with the set $[0, 1]$ of spin values, on a Cayley tree $\Gamma^k$ of order $k \geq 1$ were studied.

In this letter, we consider Potts model with competing interactions and countable spin values and we derive an infinite system of functional equations for the Potts model on a second order Cayley tree.

## 2. Preliminaries

The Cayley tree (Bethe lattice) $\Gamma^k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k + 1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where $V$ is the set of vertices and $L$ the set of edges. Two vertices $x$ and $y$ are called *nearest neighbors* if there exists an edge $l \in L$ connecting them and we denote $l = \langle x, y\rangle$. A collection of nearest neighbor pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from $x$ to $y$. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from $x$ and $y$.

For a fixed $x^0 \in V$, called the root, we set:

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^{n} W_m,$$

and denote:

$$S(x) = \{y \in W_{n+1} : d(x,y) = 1\}, \quad x \in W_n,$$

the set of direct successors of $x$.

The vertices $x$ and $y$ are called *next-nearest-neighbor* (NNN) which is denoted by $\rangle x, y\langle$, if there exists a vertex $z \in V$ such that $x, z$ and $y, z$ are nearest-neighbor. We consider NNN $\rangle x, y\langle$, for which there exists $n$ such that $x \in W_n$ and $y \in W_{n+2}$, this kind of NNN is considered with the three states Potts model (see [6]).

We consider a Potts model with competing nearest-neighbor and prolonged next-nearest-neighbor interactions on a Cayley tree where the spin takes values in the set $\Phi := 0, 1, 2, ....$ A configuration $\sigma$ on $V$ is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; the set of all configurations is $\Phi^V$.

The Hamiltonian for the Potts model with competing interactions has the form:

$$H(\sigma) = -J \sum_{\substack{\langle x,y\rangle \\ x,y \in V}} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{\substack{\rangle x,y\langle \\ x,y \in V}} \delta_{\sigma(x)\sigma(y)}, \tag{2.1}$$

where $J, J_1 \in R$ are coupling constants and $\delta$ is the Kroneker's symbol.

Let $\lambda$ be the Lebesgue measure on $[0,1]$. For the set of all configurations on $A$, the a priori measure $\lambda_A$ is introduced as the $|A|$ fold product of the measure $\lambda$. Here and subsequently, $|A|$ denotes the cardinality of $A$. We consider a standard sigma-algebra $\mathbb{B}$ of subsets of $\Omega = [0,1]^V$ generated by the measurable cylinder subsets. A probability measure $\mu$ on $(\Omega, \mathbb{B})$ is called a Gibbs measure (with Hamiltonian $H$) if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equation, namely for any $n = 1, 2, \ldots,$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu\left(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}\right) = \int_{\Omega} \mu(d\omega) \nu^{V_n}_{\omega|W_{n+1}},$$

where $\nu^{V_n}_{\omega|W_{n+1}}$ is the conditional Gibbs density:

$$\nu^{V_n}_{\omega|W_{n+1}}(\sigma_n) = \frac{1}{Z_n(\omega|_{W_{n+1}})} \exp\Big\{-\beta H(\sigma_n \mid |\omega|_{W_{n+1}})\Big\},$$

and $\beta = \dfrac{1}{T}$, $T > 0$ is the temperature.

Let $L_n = \{\langle x, y\rangle \in L : x, y \in V_n\}$ and $\Omega_{V_n}$ is the set of configurations in $V_n$ (and $\Omega_{W_n}$ that in $W_n$). Furthermore, $\sigma|_{V_n}$ and $\omega|_{W_n}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to $V_n$ and $W_{n+1}$, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in $V_n$ and $H\left(\sigma_n||\omega|_{W_{n+1}}\right)$ is defined as the sum $H(\sigma_n) + U\left(\sigma_n, \omega|_{W_{n+1}}\right)$ where:

$$H(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \xi_{\sigma_n(x)\sigma_n(y)},$$

$$U\left(\sigma_n, \omega|_{W_{n+1}}\right) = -J \sum_{\substack{\langle x,y\rangle: \\ x \in V_n,\ y \in W_{n+1}}} \xi_{\sigma_n(x)\omega(y)}.$$

Finally, $Z_n\left(\omega|_{W_{n+1}}\right)$ represents the partition function in $V_n$, with the boundary condition $\omega|_{W_{n+1}}$:

$$Z_n\left(\omega|_{W_{n+1}}\right) = \int_{\Omega_{V_n}} \exp\Big\{-\beta H\left(\widetilde{\sigma}_n \,||\, \omega|_{W_{n+1}}\right)\Big\} \lambda_{V_n}(d\widetilde{\sigma}_n).$$

We write $x < y$ if the path from $x^0$ to $y$ goes through $x$. We call vertex $y$ a direct successor of $x$ if $y > x$ and $x, y$ are nearest neighbors. We denote by $S(x)$ the set of direct successors of $x$ and observe that any vertex $x \neq x^0$ has $k$ direct successors and $x^0$ has $k+1$.

Let $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in \mathbb{R}^{[0,1]}$ be a mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by:

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right\}. \tag{2.2}$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp\left\{-\beta H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x}\right\} \lambda_{V_n}(d\widetilde{\sigma}_n). \tag{2.3}$$

The probability distributions $\mu^{(n)}$ are called compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int\limits_{\Omega_{W_n}} \mu^{(n)} \left(\sigma_{n-1} \vee \omega_n\right) \lambda_{W_n} \left(d(\omega_n)\right) = \mu^{(n-1)} \left(\sigma_{n-1}\right). \tag{2.4}$$

Here, $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. In this case, because of the Kolmogorov extension theorem, there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$. Such a measure is called a splitting Gibbs measure corresponding to Hamiltonian (2.1) and function $x \mapsto h_x$, $x \neq x^0$.

The following theorem describes conditions on $h_x$ guaranteeing compatibility of distributions $\mu^{(n)}(\sigma_n)$.

## 3. Functional Equations

**Theorem 3.1** Probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \dots$, in (2.2), for a Cayley tree of order two, are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$h^*_{i,x} = F_i(h^*_y, h^*_z, \beta, J), \quad i = 1, 2, \dots, \tag{3.1}$$

where $S(x) = \{y, z\}$, $h^*_x = \left(h_{1,x} - h_{0,x} + \ln \dfrac{\nu(1)}{\nu(0)},\ h_{2,x} - h_{0,x} + \ln \dfrac{\nu(2)}{\nu(0)},\ \dots\right)$ and

$$F_i(h^*_y, h^*_z, \beta, J) = \ln \frac{1 + \sum\limits_{\substack{p,q=0\\p+q\neq 0}}^{\infty} \exp\left\{\beta J(\delta_{ip} + \delta_{iq}) + J_1\beta\delta_{pq} + h^*_{p,y} + h^*_{q,z}\right\}}{1 + \sum\limits_{\substack{p,q=0\\p+q\neq 0}}^{\infty} \exp\left\{\beta J(\delta_{0p} + \delta_{0q}) + J_1\beta\delta_{pq} + h^*_{p,y} + h^*_{q,z}\right\}}.$$

**Proof**. *Necessity* Assume that (2.4) holds; we will prove (3.1). Substituting (2.2) in (2.4), obtain that for any configurations $\sigma_{n-1} : x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in \Phi$:

$$\frac{1}{Z_n} \sum_{\sigma^{(n)} \in \Phi^{W_n}} \exp\left\{ - \beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right\} \times \prod_{y \in W_n} \nu(\sigma(y))$$
$$= \frac{1}{Z_{n-1}} \exp\left\{ - \beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x} \right\}.$$

$$\frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)} \in \Phi^{W_n}} \exp\left\{ -\beta H_{n-1}(\sigma_{n-1}) + J\beta \sum_{\substack{x \in W_{n-1}\\ y,z \in S(x)}} \left(\delta_{\sigma(x)\sigma(y)} + \delta_{\sigma(x)\sigma(z)}\right) + J_1\beta \sum_{\substack{x \in W_{n-1}\\ y,z \in S(x)}} \delta_{\sigma(y)\sigma(z)} + \sum_{x \in W_n} h_{\sigma(x),x} \right\}$$
$$\times \prod_{y \in W_n} \nu(\sigma(y)) = \exp\left\{ - \beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x} \right\}.$$

After some abbreviations, we obtain:

$$\frac{Z_{n-1}}{Z_n} \prod_{x \in W_{n-1}} \sum_{\sigma^{(n)}_x = \{\sigma(y), \sigma(z)\}} \exp\left\{ J\beta \left(\delta_{\sigma(x)\sigma(y)} + \delta_{\sigma(x)\sigma(z)}\right) + J_1\beta\delta_{\sigma(y)\sigma(z)} + h_{\sigma(y),y} + h_{\sigma(z),z} + \ln\nu(\sigma(y)) + \ln\nu(\sigma(z)) \right\}$$
$$= \prod_{x \in W_{n-1}} \exp\left\{ h_{\sigma_{n-1}(x),x} \right\}.$$

Consequently, for any $i \in \Phi$,

$$\frac{\exp\left\{h_{0,y} + h_{0,z} + 2\ln\nu(0)\right\} + \sum\limits_{\substack{p,q=0\\p+q\neq 0}}^{\infty} \exp\left\{J\beta(\delta_{ip} + \delta_{iq}) + J_1\beta\delta_{pq} + h_{p,y} + h_{q,z} + \ln\nu(p) + \ln\nu(q)\right\}}{\exp\left\{h_{0,y} + h_{0,z} + 2\ln\nu(0)\right\} + \sum\limits_{\substack{p,q=0\\p+q\neq 0}}^{\infty} \exp\left\{J\beta(\delta_{0p} + \delta_{0q}) + J_1\beta\delta pq + h_{p,y} + h_{q,z} + \ln\nu(p) + \ln\nu(q)\right\}}$$
$$= \exp\left\{h_{i,x} - h_{0,x}\right\},$$

such that:

$$h^*_{i,x} = \ln \frac{1 + \sum\limits_{\substack{p,q=0\\ p+q\neq 0}}^{\infty} \exp\Big\{J\beta(\delta_{ip}+\delta_{iq}) + J_1\beta\delta_{pq} + h^*_{p,y} + h^*_{q,z}\Big\}}{1 + \sum\limits_{\substack{p,q=0\\ p+q\neq 0}}^{\infty} \exp\Big\{J\beta(\delta_{0p}+\delta_{0q}) + J_1\beta\delta_{pq} + h^*_{p,y} + h^*_{q,z}\Big\}},$$

where:

$$h^*_{i,x} = h_{i,x} - h_{0,x} + \ln\frac{\nu(i)}{\nu(0)}.$$

*Sufficiency.* Let (3.1) is satisfied we will prove (2.4).

$$\sum_{p,q=0}^{\infty} \exp\Big\{J\beta(\delta_{ip}+\delta_{iq}) + J_1\beta\delta_{pq} + h_{p,y} + h_{q,z} + \ln\nu(p) + \ln\nu(q)\Big\} = a(x)\exp\Big\{h_{i,x}\Big\}, \tag{3.2}$$

here $i = 0, 1, \dots$.

We have:

$$\text{LHS of (2.4)} = \frac{1}{Z_n}\exp\Big\{-\beta H_{n-1}(\sigma_{n-1})\Big\}\prod_{x\in W_{n-1}}\nu(\sigma(x))\times$$

$$\sum_{\substack{x\in W_{n-1}\\ y,z\in S(x)}} \exp\Big\{J\beta(\delta_{\sigma(x)\sigma(y)}+\delta_{\sigma(x)\sigma(z)}) + J_1\beta\delta_{\sigma(y)\sigma(z)} + h_{\sigma(y),y} + h_{\sigma(z),z}\Big\}. \tag{3.3}$$

Substituting (3.2) into (3.3) and denoting $A_n = \prod\limits_{x\in W_{n-1}} a(x)$, we get:

$$\text{RHS of (3.3)} = \frac{A_{n_1}}{Z_n}\exp\Big\{-\beta H_{n-1}(\sigma_{n-1})\Big\}\prod_{x\in W_{n-1}} h_{\sigma_{n-1}(x),x}. \tag{3.4}$$

Since $\mu^{(n)}$, $n \geq 1$ is a probability, we should have:

$$\sum_{\sigma_{n-1}}\sum_{\sigma}^{(n)} \mu^{(n)}\left(\sigma_{n-1}, \sigma^{(n-1)}\right) = 1.$$

Hence, from (3.4) we obtain $Z_{n-1}A_{n-1} = Z_n$, and (2.4) holds.

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