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Spectral properties of a two–particle hamiltonian on a *d*-dimensional lattice

M. I. Muminov[1], A. M. Khurramov[2]

[1]Universiti Teknologi Malaysia (UTM), 81310 Skudai, Johor Bahru, Malaysia

[2]Department of Mechaniical and Mathematics, Samarkand State University, Uzbekistan

mmuminov@mail.ru, xurramov@mail.ru

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A system of two arbitrary quantum particles moving on d-dimensional lattice interacting via some attractive potential is considered. The number of eigenvalues of the family $h(k)$ is studied depending on the interaction energy of particles and the total quasi-momentum $k \in \mathbb{T}^d$ ($\mathbb{T}^d$ – d-dimensional torus). Depending on the interaction energy, the conditions for $h(\mathbf{0})$ that has simple or multifold virtual level at 0 are found.

Keywords: two-particle hamiltonian, virtual level, multiplicity of virtual level.

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1. Introduction

Lattice two-particle Hamiltonians have been investigated in [1–3]. In [1], the problem of the two-particle bound states for the transfer-matrix in a wide class of Gibbs fields on the lattices in the high temperature domains of ($t \gg 1$), as well in [2] the appearance of bound state levels standing in a definite distance from the essential spectrum has been shown for some quasi-momenta values. The spectral properties of the two-particle operator depending on total quasi-momentum have been studied in [3].

In [4], it was proven that if the operator $h(\mathbf{0})$ has a virtual level at the lower edge of essential spectrum, then the discrete spectrum of $h(k)$ lying below the essential spectrum is always nonempty for any $k \in \mathbb{T}^d \setminus \{\mathbf{0}\}$. In [5], assuming that dispersion relations $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are linearly dependent, it was proven that the positivity of $h(\mathbf{0})$ implies the positivity of $h(k)$ for all k.

In recent work [6], conditions were obtained for the discrete two-particle Schrödinger operator with zero-range attractive potential to have an embedded eigenvalue in the essential spectrum depending on the dimension of the lattice. In [7], the discrete spectra of one-dimensional discrete Laplacian with short range attractive perturbation were studied.

In [8], a system of two arbitrary particles in a three-dimensional lattice with some dispersion relation was considered. Particles interact via an attractive potential only on the neighboring knots of lattice. The existence and absence of eigenvalues of the family $h(k)$ depending on the energy of interaction and quasi-momentum $k \in \mathbb{T}^3$ ($\mathbb{T}^3$ – three dimensional torus) have been investigated. Moreover, depending on the interaction energy, the conditions were found for $h(\mathbf{0})$ to have a simple, two-fold, or three-fold virtual level at 0. In [9], the two-particle Schrödinger operator $h(k)$, $k \in \mathbb{T}^3$, associated with a system of two particles on the three-dimensional lattice, was considered. Here, some $6N$-dimensional integral operator is taken as the potential and the dispersion relation is chosen depending on N. In this work, the existence or absence of eigenvalues has also been studied for the family $h(k)$ depending on the interaction energy and total quasi-momentum k. Moreover, dependending on the interaction energy, conditions were found for the operator $h(\mathbf{0})$ that has $3N$-fold eigenvalue and a $3N$-fold virtual level.

The current work is a generalization of [8]. In this work, we consider the system of two arbitrary quantum particles moving on the d-dimensional lattice and interacting via an attractive potential. For all values of $k \in \mathbb{T}^d$ ($\mathbb{T}^d$ – d-dimensional torus) the dependence of the number of eigenvalues of the family $h(k)$ on the interaction energy is studied. The conditions for that $h(\mathbf{0})$ has simple or multifold virtual level (eigenvalue) at 0 are found for $d = 3$, 4 ($d \geq 5$).

2. Statement of the Main Result

Let $L_2(\mathbb{T}^d)$ be the Hilbert space of square-integrable functions defined on d-dimensional lattice $\mathbb{T}^d$.

Consider the two-particle Schrödinger operator $h(k)$, $k \in \mathbb{T}^d$, associated with the direct integral expansion of Hamiltonian of the system of two arbitrary particles, interacting via short-range pair potential [8], acting in $L_2(\mathbb{T}^d)$ as

$$h(k) = h_0(k) - \mathbf{v},$$

here $h_0(k)$ – multiplication operator by a function:

$$\mathcal{E}_k(p) = \varepsilon_1(p) + \varepsilon_2(k-p)$$

and $\mathbf{v}$ is an integral operator with kernel

$$v(p-s) = \mu_0 + \sum_{\alpha=1}^{d} \mu_\alpha \cos(p_\alpha - s_\alpha), \quad \mu_\alpha > 0.$$

Assumption 1. *Additionally, we assume that ε_l, $l = 1, 2$ are real-valued, continuous, even and periodic functions with period π in every variable.*

Please note that the Weyl theorem on the essential spectrum [10] implies that the essential spectrum $\sigma_{ess}(h(k))$ of the operator $h(k)$ coincides with the spectrum of the unperturbed operator $h_0(k)$:

$$\sigma_{ess}(h(k)) = \sigma(h_0(k)) = [m(k), M(k)],$$

where $m(k) = \min_{p\in\mathbb{T}^d} \mathcal{E}_k(p)$, $M(k) = \max_{p\in\mathbb{T}^d} \mathcal{E}_k(p)$.

Since $\mathbf{v} \geq 0$, one has:

$$\sup(h(k)f, f) \leq \sup(h_0(k)f, f) = M(k)(f, f), \quad f \in L_2(\mathbb{T}^d),$$

and, thus, $h(k)$ does not have eigenvalues lying above the essential spectrum:

$$\sigma(h(k)) \cap (M(k), +\infty) = \emptyset.$$

We set:

$$\mu_i^{\pm}(k; z) = \frac{c_i(k;z) + s_i(k;z) \pm \sqrt{(c_i(k;z) - s_i(k;z))^2 + 4\xi_i^2(k;z)}}{2[c_i(k;z)s_i(k;z) - \xi_i^2(k;z)]},$$

where

$$c_i(k;z) = \int\limits_{T^d} \frac{\cos^2 s_i \, ds}{\mathcal{E}_k(s) - z}, \quad s_i(k;z) = \int\limits_{T^d} \frac{\sin^2 s_i \, ds}{\mathcal{E}_k(s) - z},$$

$$\xi_i(k;z) = \int\limits_{T^d} \frac{\sin s_i \, \cos s_i \, ds}{\mathcal{E}_k(s) - z}, \quad z \leq m(k).$$

Recall that $c_i(k;z)s_i(k;z) - \xi_i^2(k;z) \geq 0$.

There exist (finite or infinite) limits:

$$\lim_{z\to m(k)-0} b(k;z), \quad \lim_{z\to m(k)-0} c_i(k;z), \quad \lim_{z\to m(k)-0} s_i(k;z), \quad \lim_{z\to m(k)-0} \xi_i^2(k;z),$$

where

$$b(k;z) = \int\limits_{\mathbb{T}^d} \frac{ds}{\mathcal{E}_k(s) - z}.$$

Lemma 1. *For any $k \in \mathbb{T}^d$ there exists finite limits:*

$$\mu^0(k) = \lim_{z\to m(k)-0} \frac{1}{b(k;z)}, \tag{2.1}$$

$$\mu_i^{\pm}(k) = \lim_{z\to m(k)-0} \mu_i^{\pm}(k;z), \quad i = 1, \ldots, d. \tag{2.2}$$

Moreover,

$$\mu_i^-(k) \leq \mu_i^+(k) \quad \textit{for all} \quad k \in \mathbb{T}^d, \quad i = 1, \ldots, d.$$

Let us define the functions:

$$\alpha(\mu;k) = \begin{cases} 0 & \text{if } \ \mu_0 \in (0; \mu^0(k)], \\ 1 & \text{if } \ \mu_0 \in (\mu^0(k); \infty), \end{cases} \tag{2.3}$$

$$\beta_i(\mu;k) = \begin{cases} 0 & \text{if } \ \mu_i \in (0; \mu_i^-(k)], \\ 1 & \text{if } \ \mu_i \in (\mu_i^-(k); \mu_i^+(k)], \\ 2 & \text{if } \ \mu_i \in (\mu_i^+(k); \infty) \end{cases} \tag{2.4}$$

for all $i = 1, \ldots, d$.

Theorem 1. *Let $\mu = (\mu_0, \cdots, \mu_d) \in \mathbb{R}_+^{d+1}$. Then, counting multiplicity, $h(k)$ has exactly:*

$$\alpha(\mu; k) + \sum_{i=1}^{d} \beta_i(\mu; k)$$

eigenvalues below the essential spectrum.

Assumption 2. *Assume that $m(\mathbf{0}) = \min_{p \in \mathbb{T}^d} \mathcal{E}_{\mathbf{0}}(p) = 0$ and*

$$\mathcal{M} = \{p \in \mathbb{T}^d: \ m(\mathbf{0}) = 0\} = \{p_1, \cdots, p_n\}, \quad n < \infty.$$

Moreover, assume that around points of $\mathcal{M}$ $\mathcal{E}_{\mathbf{0}}(p)$ is of order $\rho > 0$:

$$c|p - p_l|^\rho \leq \mathcal{E}_{\mathbf{0}}(p) \leq c_1 |p - p_l|^\rho \qquad as \quad p \to p_l, \quad l = 1, \ldots, n.$$

Let $C(\mathbb{T}^d)$ be a Banach space of continuous periodic functions on $\mathbb{T}^d$ and $G(k; z)$ denote the (Birman-Schwinger) integral operator in $L_2(\mathbb{T}^d)$ with the kernel:

$$G(k; z; p, q) = v(p - q)(\mathcal{E}_k(q))^{-1}, \ p, q \in \mathbb{T}^d.$$

Definition 1. *We say that the operator $h(\mathbf{0})$ has a virtual level at 0 (lower edge of essential spectrum) if 1 is an eigenvalue of $G(\mathbf{0}; 0)$ with some associated eigenfunction $\psi \in L_2(\mathbb{T}^d)$ satisfying:*

$$\frac{\psi(\cdot)}{\mathcal{E}_{\mathbf{0}}(\cdot)} \in L_1(\mathbb{T}^d) \setminus L_2(\mathbb{T}^d).$$

The number of such linearly independent vectors ψ is called the multiplicity of virtual level of $h(\mathbf{0})$.

We set:

$$\mu_\alpha^0 = \min\left\{\frac{1}{c_\alpha(\mathbf{0}; 0)}, \frac{1}{s_\alpha(\mathbf{0}; 0)}\right\}, \ \alpha = 1, \ldots, d.$$

We define the following sets depending on $c_\alpha(\mathbf{0}; 0)$ and $s_\alpha(\mathbf{0}; 0)$:

$$L_{\alpha 1} = \left\{\mu_\alpha^0: \ \frac{1}{c_\alpha(\mathbf{0}; 0)} > \mu_\alpha^0\right\},$$

$$L_{\alpha 2} = \left\{\mu_\alpha^0: \ \frac{1}{c_\alpha(\mathbf{0}; 0)} = \mu_\alpha^0, \, p_i^\alpha = \frac{\pi}{2} \text{ or } p_i^\alpha = -\frac{\pi}{2} \quad \text{for all} \quad i = 1, \cdots, n\right\},$$

$$L_{\alpha 3} = \left\{\mu_\alpha^0: \ \frac{1}{c_\alpha(\mathbf{0}; 0)} = \mu_\alpha^0, \, p_i^\alpha \neq \frac{\pi}{2} \text{ or } p_i^\alpha \neq -\frac{\pi}{2} \quad \text{at least one} \quad i = 1, \ldots, n\right\},$$

$$M_{\alpha 1} = \left\{\mu_\alpha^0: \ \frac{1}{s_\alpha(\mathbf{0}; 0)} > \mu_\alpha^0\right\},$$

$$M_{\alpha 2} = \left\{\mu_\alpha^0: \ \frac{1}{s_\alpha(\mathbf{0}; 0)} = \mu_\alpha^0, \, p_i^\alpha = 0 \text{ or } p_i^\alpha = \pi \quad \text{for all} \quad i = 1, \ldots, n\right\},$$

$$M_{\alpha 3} = \left\{\mu_\alpha^0: \ \frac{1}{s_\alpha(\mathbf{0}; 0)} = \mu_\alpha^0, \, p_i^\alpha \neq 0 \text{ or } p_i^\alpha \neq \pi \quad \text{at least one} \quad i = 1, \ldots, n\right\},$$

where p_i^α – α-th coordinate of minimum point p_i of $\mathcal{E}_{\mathbf{0}}(\cdot)$.

Let us define the following functions:

$$\beta(\mu_0) = \begin{cases} 0 & \text{if} \quad \mu_0 \in (0; \mu^0(\mathbf{0})), \\ 1 & \text{if} \quad \mu_0 = \mu^0(\mathbf{0}), \end{cases}$$

$$\gamma(\alpha) = \begin{cases} 0 & \text{if} \quad \mu_\alpha \in (0; \mu_\alpha^0) \quad \text{or}\, \mu_\alpha \in L_{\alpha 1} \cup L_{\alpha 2}, \\ 1 & \text{if} \quad \mu_\alpha \in L_{\alpha 3}, \end{cases}$$

$$\overline{\gamma}(\alpha) = \begin{cases} 0 & \text{if} \quad \mu_\alpha \in (0; \mu_\alpha^0) \quad \text{or} \quad \mu_\alpha \in L_{\alpha 1} \cup L_{\alpha 3}, \\ 1 & \text{if} \quad \mu_\alpha \in L_{\alpha 2}, \end{cases}$$

$$\eta(\alpha) = \begin{cases} 0 & \text{if} \quad \mu_\alpha \in (0; \mu_\alpha^0) \quad \text{or} \quad \mu_\alpha \in M_{\alpha 1} \cup M_{\alpha 2}, \\ 1 & \text{if} \quad \mu_\alpha \in M_{\alpha 3}, \end{cases}$$

$$\overline{\eta}(\alpha) = \begin{cases} 0 & \text{if} \quad \mu_\alpha \in (0; \mu_\alpha^0) \quad \text{or} \quad \mu_\alpha \in M_{\alpha 1} \cup M_{\alpha 3}, \\ 1 & \text{if} \quad \mu_\alpha \in M_{\alpha 2}. \end{cases}$$

Theorem 2. *(i) Let $\rho = 2$, $\mu_0 \in (0;\mu^0(\mathbf{0})]$, $\mu_\alpha \in (0,\mu^0_\alpha]$, $\alpha = 1,\dots,d$. Then*

1) if $d = 3,4$, then 0 is

$$\beta(\mu_0) + \sum_{\alpha=1}^{d} [\gamma(\alpha) + \eta(\alpha)]$$

– fold virtual level of $h(\mathbf{0})$. In addition, if $\bigcup_{\alpha=1}^{d} L_{\alpha 2} \cap M_{\alpha 2} \neq \emptyset$, then 0 is simultaneously

$$\sum_{\alpha=1}^{d} [\bar{\gamma}(\alpha) + \bar{\eta}(\alpha)]$$

– fold eigenvalue of $h(\mathbf{0})$.

2) if $d \geq 5$, then 0 is

$$\beta(\mu_0) + \sum_{\alpha=1}^{d} [\gamma(\alpha) + \eta(\alpha)]$$

– fold eigenvalue of $h(\mathbf{0})$.

(ii) Let $\rho \in (\frac{d}{2}, d)$, $d > 3$, $\mu_0 \in (0;\mu^0(\mathbf{0})]$, $\mu_\alpha \in (0,\mu^0_\alpha]$, $\alpha = 1,\dots,d$. Then 0 is at least

$$\beta(\mu_0) + \sum_{\alpha=1}^{d} [\gamma(\alpha) + \eta(\alpha)]$$

-fold virtual level of $h(\mathbf{0})$.

Remark 1. *1) By definition of sets $L_{\alpha 2}$ and $M_{\alpha 2}$ for each $\alpha = 1,\dots,d$ one has $L_{\alpha 3} \cup M_{\alpha 3} \neq \emptyset$. Moreover, in this case, the multiplicity of the virtual level of $h(\mathbf{0})$ is always not less than d if $\mu_\alpha = \mu^0_\alpha$, $\alpha = 1,\dots,d$.*

2) For $\rho = 2$ the function

$$\mathcal{E}_{\mathbf{0}}(\cdot) = \mathcal{E}_{\mathbf{0}}(p) = \varepsilon_1(p) + \varepsilon_2(p), \quad \varepsilon_1(p) = \varepsilon_2(p) = \cos^2 p_1 + \sum_{i=1}^{d} (1 + \cos 2p_i)$$

satisfies the assumptions of Theorem 2 with $\bigcup_{\alpha=1}^{d} L_{\alpha 2} \cap M_{\alpha 2} \neq \emptyset$. In addition, $L_{12} \neq \emptyset$.

3) For $\rho \in \left(\frac{d}{2}, d\right)$ the function:

$$\mathcal{E}_{\mathbf{0}}(p) = \varepsilon_1(p) + \varepsilon_2(p), \quad \varepsilon_1(p) = \varepsilon_2(p) = \left(\sum_{i=1}^{d} (1 - \cos 2p_i)\right)^{\rho/2}$$

satisfies the assumptions of Theorem 2.

3. Eigenvalues of $h(k)$

Proof of Lemma 1. Note that proof of (2.1) is obvious.

By definition $\mu^-_\alpha(k;z) < \mu^+_\alpha(k;z)$ for any $z < m(k)$ and $k \in \mathbb{T}^d$. Notice that:

$$c_\alpha(k;z)s_\alpha(k;z) - \xi^2_\alpha(k;z) = \int\limits_{\mathbb{T}^d} \frac{\cos^2 s_\alpha ds}{\mathcal{E}_k(s) - z} \int\limits_{\mathbb{T}^d} \frac{\sin^2 t_\alpha dt}{\mathcal{E}_k(t) - z} - \int\limits_{\mathbb{T}^d} \frac{\sin s_\alpha \cos s_\alpha ds}{\mathcal{E}_k(s) - z} \int\limits_{\mathbb{T}^d} \frac{\sin t_\alpha \cos t_\alpha dt}{\mathcal{E}_k(t) - z}$$

$$= \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \frac{\frac{1}{2}\cos^2 s_\alpha \sin^2 t_\alpha ds dt}{(\mathcal{E}_k(s) - z)(\mathcal{E}_k(t) - z)} - \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \frac{\sin s_\alpha \cos s_\alpha \sin t_\alpha \cos t_\alpha ds dt}{(\mathcal{E}_k(s) - z)(\mathcal{E}_k(t) - z)} + \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \frac{\frac{1}{2}\cos^2 t_\alpha \sin^2 s_\alpha ds dt}{(\mathcal{E}_k(s) - z)(\mathcal{E}_k(t) - z)}$$

$$= \frac{1}{2} \int\limits_{\mathbb{T}^d}\int\limits_{\mathbb{T}^d} \frac{\sin^2(s_\alpha - t_\alpha) ds dt}{(\mathcal{E}_k(s) - z)(\mathcal{E}_k(t) - z)}. \quad (3.1)$$

Hence, $c_\alpha(k;z)s_\alpha(k;z) - \xi^2_\alpha(k;z) > 0$ for all $z < m(k)$ and $k \in \mathbb{T}^d$.

The function $\mu_\alpha^+(k;z)$ we estimate as follows:

$$\mu_\alpha^+(k;z) = \frac{c_\alpha(k;z)+s_\alpha(k;z)+\sqrt{(c_\alpha(k;z)-s_\alpha(k;z))^2+4\xi_\alpha^2(k;z)}}{2[c_\alpha(k;z)s_\alpha(k;z)-\xi_\alpha^2(k;z)]}$$
$$= \frac{c_\alpha(k;z)+s_\alpha(k;z)+\sqrt{(c_\alpha(k;z)+s_\alpha(k;z))^2-4[c_\alpha(k;z)s_\alpha(k;z)-\xi_\alpha^2(k;z)]}}{2[c_\alpha(k;z)s_\alpha(k;z)-\xi_\alpha^2(k;z)]}$$
$$< \frac{c_\alpha(k;z)+s_\alpha(k;z)}{c_\alpha(k;z)s_\alpha(k;z)-\xi_\alpha^2(k;z)}. \quad (3.2)$$

Since $\dfrac{\sin^2(s_\alpha - t_\alpha)}{\mathcal{E}_k(t)-z} > 0$ for any $z < m(k)$ and for a.e. $k, s, t \in \mathbb{T}^d$, there exists $\delta > 0$ such that:

$$\min_{k,z} \int_{\mathbb{T}^d} \frac{\sin^2(s_\alpha - t_\alpha)dsdt}{\mathcal{E}_k(t)-z} \geq \delta.$$

From here and from (3.1) we get:

$$c_\alpha(k;z)s_\alpha(k;z) - \xi_\alpha^2(k;z) > \frac{\delta}{2}\int_{\mathbb{T}^d} \frac{ds}{\mathcal{E}_k(s)-z}.$$

Since

$$c_\alpha(k;z)+s_\alpha(k;z) = \int_{\mathbb{T}^d} \frac{ds}{\mathcal{E}_k(s)-z}$$

from (3.2) we get uniform upper estimate:

$$\mu_\alpha^+(k;z) < \frac{1}{2\delta}.$$

From here we get (2.2).

Lemma is proved.

Lemma 2. *$z < m(k)$ is an eigenvalue of $h(k)$ if and only if $\Delta(k;z) = 0$, where*

$$\Delta(k;z) = (1-\mu_0 b(k;z))\prod_{\alpha=1}^{d}\Big([1-\mu_\alpha c_\alpha(k;z)][1-\mu_\alpha s_\alpha(k;z)] - \mu_\alpha^2\xi_\alpha^2(k;z)\Big). \quad (3.3)$$

Proof. Let $z < m(k)$ be an eigenvalue of $h(k)$ with associated eigenfunction $f \neq 0$. Then $h(k)f = zf$ and so:

$$f = r_0(z)\mathbf{v}f, \quad (3.4)$$

where $r_0(z)$ is a resolvent of $h_0(k)$. Introduce the following notations:

$$\varphi_0 = \int_{\mathbb{T}^d} f(s)ds, \quad (3.5)$$

$$\varphi_\alpha = \int_{\mathbb{T}^d} \cos s_\alpha f(s)ds, \quad (3.6)$$

$$\psi_\alpha = \int_{\mathbb{T}^d} \sin s_\alpha f(s)ds, \quad \alpha = 1,2,3,...d. \quad (3.7)$$

Then, (3.4) is rewritten as:

$$f(p) = \frac{\mu_0\varphi_0}{\mathcal{E}_k(p)-z} + \frac{1}{\mathcal{E}_k(p)-z}\sum_{\alpha=1}^{d}\mu_\alpha[\cos p_\alpha\varphi_\alpha + \sin p_\alpha\psi_\alpha]. \quad (3.8)$$

From the π-periodicity of $\mathcal{E}_k(\cdot)$ in each argument, it follows that:

$$\int_{\mathbb{T}^d} \frac{\cos s_\alpha ds}{\mathcal{E}_k(s)-z} = \int_{\mathbb{T}^d} \frac{\cos s_\alpha \cos s_\beta ds}{\mathcal{E}_k(s)-z} = \int_{\mathbb{T}^d} \frac{\cos s_\alpha \sin s_\beta ds}{\mathcal{E}_k(s)-z} = \int_{\mathbb{T}^d} \frac{\sin s_\alpha \sin s_\beta ds}{\mathcal{E}_k(s)-z} = 0,\ \alpha \neq \beta. \quad (3.9)$$

Putting (3.8) in the relations (3.5)–(3.7) and using (3.9), we get that $\varphi_0, \varphi_1, ..., \varphi_d, \psi_1, \psi_2, ..., \psi_d$ satisfy the system of $(2d+1)$-linear equations:

$$\begin{aligned} &\varphi_0 = \mu_0 b(k;z)\varphi_0, \\ &\varphi_\alpha = \mu_\alpha c_\alpha(k;z)\varphi_\alpha + \mu_\alpha \xi_\alpha(k;z)\psi_\alpha, \quad \alpha = 1, ..., d \\ &\psi_\alpha = \mu_\alpha \xi_\alpha(k;z)\varphi_\alpha + \mu_\alpha s_\alpha(k;z)\psi_\alpha, \quad \alpha = 1, ..., d. \end{aligned} \tag{3.10}$$

This system of equations has a nonzero solution $(\varphi_0, \ldots, \varphi_d, \psi_1, \ldots \psi_d)$ if and only if its determinant is zero, i.e. $\det D(k;z) = 0$. It is easy to see that $\det D(k;z) = \Delta(k;z)$.

Conversely, let $\Delta(k;z) = 0$, $z < m(k)$. Then, at least one of the equalities $1 - \mu_0 b(k;z) = 0$, $[1 - \mu_\alpha c_\alpha(k;z)][1 - \mu_\alpha s_\alpha(k;z)] - \mu_\alpha^2 \xi_\alpha^2(k;z) = 0$, $\alpha \in \{1, \ldots, d\}$ holds. Thus, the vector $\mathbf{c} = (c_0, \cdots, c_{2d})$ where $c_0 = 1$, $c_\alpha = \varphi_\alpha$, $c_{d+\alpha} = \psi_\alpha$, is a solution of (3.10). Consequently, one of the functions:

$$\frac{1}{\mathcal{E}_k(p) - z}, \;\; \frac{1}{\mathcal{E}_k(p) - z}\mu_\alpha[\varphi_\alpha \cos p_\alpha + \psi_\alpha \sin p_\alpha]$$

is an eigenfunction of $h(k)$ associated with eigenvelue $z < m(k)$.

Observe that $\Delta(k;\cdot)$ is the Fredholm determinant of the operator $I - r_0(z)\mathbf{v}$, i.e. $\Delta(k;z) = \det(I - r_0(z)\mathbf{v})$. Moreover, it is well-known [11] that geometric multiplicity of eigenvalue 1 of $r_0(z)\mathbf{v}$ coincides with the multiplicity of zero z of $\Delta(k;\cdot)$. Since the multiplicities of eigenvalues 1 and z of operators respectively $r_0(z)\mathbf{v}$ and $h(k)$ are the same, we get that multiplicity of zeros of $\Delta(k;\cdot)$ is equal to the multiplicity of eigenvalues of $h(k)$. The lemma is thus proved.

Proof of Theorem 1. Notice that the function:

$$\Delta_\alpha(k;z) = [1 - \mu_\alpha c_\alpha(k;z)][1 - \mu_\alpha s_\alpha(k;z)] - \mu_\alpha^2 \xi_\alpha^2(k;z),$$

is a Fredholm determinant associated with the operator $I - r_0(z)\mathbf{v}_\alpha$, where $\mathbf{v}_\alpha$ – is an integral operator with kernel $v_\alpha(p-s) = \mu_\alpha \cos(p_\alpha - s_\alpha)$.

Since $\mathbf{v}_\alpha$ is a two-dimensional operator, number of zeros $\beta_\alpha(\mu;k)$ with multiplicities of the function $\Delta_\alpha(k;\cdot)$, lying below $m(k)$, is not more than 2. Function $\Delta_\alpha(k;\cdot)$ can be represented as:

$$\Delta_\alpha(k;z) = [c_\alpha(k;z)s_\alpha(k;z) - \xi_\alpha^2(k;z)]\Big(\mu_\alpha - \mu_\alpha^-(k;z)\Big)\Big(\mu_\alpha - \mu_\alpha^+(k;z)\Big). \tag{3.11}$$

Since:

$$\lim_{z\to m(k)-0} \mu_\alpha^\pm(k;z) = \mu_\alpha^\pm(k) < \infty,$$

one has:

$$\mu_\alpha - \mu_\alpha^\pm(k;m(k)) = \begin{cases} \geq 0 & \text{if} \quad \mu_\alpha \in (0, \mu_\alpha^\pm(k)], \\ < 0 & \text{if} \quad \mu_\alpha \in (\mu_\alpha^\pm(k), \infty). \end{cases}$$

Consequently, from (3.11) and (3.1) it can be deduced that:

$$\beta_\alpha(\mu;k) = \begin{cases} 0 & \text{if} \quad \mu_\alpha \in (0, \mu_\alpha^-(k)], \\ 1 & \text{if} \quad \mu_\alpha \in (\mu_\alpha^-(k), \mu_\alpha^+(k)], \\ 2 & \text{if} \quad \mu_\alpha \in (\mu_\alpha^+(k), \infty). \end{cases}$$

Observe that the function $1 - \mu_0 b(k;\cdot)$ is monotonously decreasing in $(\infty, m(k))$. Thus for the number of zeros $\alpha(\mu;k)$ of the function $\Delta_\alpha(k;\cdot)$ below $m(k)$ it holds:

$$\alpha(\mu;k) = \begin{cases} 0 & \text{if} \quad \mu_0 \in (0; \mu^0(k)], \\ 1 & \text{if} \quad \mu_0 \in (\mu^0(k); \infty). \end{cases}$$

If $\mu^0(k) = 0$, then $\lim\limits_{z\to m(k)-0} b(k;z) = +\infty$. Obviously, in this case $\alpha(\mu;k) = 1$ for any $\mu_0 > 0$.

The aforementioned facts imply that if: $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in R_+^{d+1}$, then the function $\Delta(k;\cdot)$ has exactly:

$$\alpha(\mu;k) + \sum_{i=1}^{d} \beta_i(\mu;k)$$

zeros (counting multiplicities) below $m(k)$.

Then, from Lemma 1, we get that for $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in R_+^{d+1}$ the operator $h(k)$ exactly:

$$\alpha(\mu;k) + \sum_{i=1}^{d} \beta_i(\mu;k)$$

zeros (counting multiplicities) below $m(k)$.

This finishes the proof.

Proof of Theorem 2. We shall study the equation:

$$G(\mathbf{0};0)\varphi=\varphi.$$

Notice that the function $\Delta(k;z)$, defined as (3.3) is the Fredholm determinant of $I-G(k;z)$. From Hypothesis 2, the function $\Delta(k;z)$ is defined for $k=\mathbf{0}$, $m(\mathbf{0})=0$. Since $\mathcal{E}_{\mathbf{0}}(\cdot)$ is even, the function

$$\xi_i(\mathbf{0};z)=\int\limits_{\mathbb{T}^d}\frac{\sin s_i\cos s_i ds}{\mathcal{E}_{\mathbf{0}}(s)-z}=0,\quad z\le 0.$$

Consequently, the function $\Delta(\mathbf{0};z)$ can be represented as:

$$\Delta(\mathbf{0};z)=(1-\mu_0 b(\mathbf{0};z))\prod_{\alpha=1}^{d}\Big([1-\mu_\alpha c_\alpha(\mathbf{0};z)][1-\mu_\alpha s_\alpha(\mathbf{0};z)]\Big).$$

The following lemma can be proved analogously to Lemma 2.

Lemma 3. *The number $\lambda=1$ is an eigenvlue of $G(\mathbf{0};0)$ if and only if $\Delta(\mu)=\Delta(\mathbf{0};0)=0$. In this case if $1-\mu_0 b(\mathbf{0};0)=0$ $\Big(1-\mu_\alpha c_\alpha(\mathbf{0};0)=0$ or $1-\mu_\alpha s_\alpha(\mathbf{0};0)=0\Big)$, then the function $\varphi_0=1$ $\Big(\varphi_\alpha(p)=\cos p_\alpha$ or $\psi_\alpha(p)=\sin p_\alpha\Big)$ is an eigenfucntion of the operator $G(\mathbf{0};0)$, associated with 1.*

Obviously, $\Delta(\mu)>0$ if $\mu_0\in(0;\mu^0(\mathbf{0}))$, $\mu_\alpha\in(0;\mu^0_\alpha)$, $\alpha=1,\dots,d$. By Lemma 3 $\lambda=1$ is not eigenvalue of $G(\mathbf{0};0)$. Hence 0 is not an eigenvalue of $h(\mathbf{0})$ for $\mu_0\in(0;\mu^0(\mathbf{0}))$, $\mu_\alpha\in(0;\mu^0_\alpha)$, $\alpha=1,\dots,d$.

Further, consider the equation $G(\mathbf{0};0)\varphi=\varphi$ for $\mu_0=\mu^0(\mathbf{0})$, $\mu_\alpha=\mu^0_\alpha$, $\alpha=1,\dots,d$.

(i) a) Let $\rho=2$, $\mu_0=\mu^0(\mathbf{0})$.

According to Lemma 3, $\lambda=1$ is an eigenvalue of $G(\mathbf{0};0)$, with associated eigenfunction $\varphi_0(p)=1$.

It is easy to check that if $d=3,4$, then:

$$F_0(\cdot)\in L_1(\mathbb{T}^d)\setminus L_2(\mathbb{T}^d),$$

and if $d\ge 5$, then:

$$F_0(\cdot)\in L_2(\mathbb{T}^d),$$

where

$$F_0(p)=\frac{1}{\mathcal{E}_{\mathbf{0}}(p)}.$$

It means that $z=0$ is virtual level of $h(\mathbf{0})$ for $d=3$, 4, and eigenvalue for $d\ge 5$.

b) Let $\mu_\alpha=\mu^0_\alpha$, $\alpha=1,\dots,d$. Then μ_α belongs one and only one of the sets $L_{\alpha1}$, $L_{\alpha2}$, $L_{\alpha3}$ $M_{\alpha1}$, $M_{\alpha2}$, $M_{\alpha3}$.

If $\mu_\alpha\in L_{\alpha1}\Big(\mu_\alpha\in M_{\alpha1}\Big)$, then $1-\mu_\alpha c_\alpha(\mathbf{0};0)>0$ $\Big(1-\mu_\alpha s_\alpha(\mathbf{0};0)>0\Big)$. If $\mu_\alpha\in L_{\alpha2}\Big(\mu_\alpha\in M_{\alpha2}\Big)$, then $\cos p_i^{(\alpha)}=0$ $\Big(\sin p_i^{(\alpha)}=0\Big)$ for all $i=1,\dots,d$. In this case

$$F_\alpha(\cdot)\in L_2(\mathbb{T}^d),\quad \Big(\Phi_\alpha(\cdot)\in L_2(\mathbb{T}^d)\Big),\quad d\ge 3,$$

where

$$F_\alpha(p)=\frac{\cos p_\alpha}{\mathcal{E}_{\mathbf{0}}(p)},\ \Phi_\alpha(p)=\frac{\sin p_\alpha}{\mathcal{E}_{\mathbf{0}}(p)},\ \alpha=1,...,d,$$

and, so, $z=0$ is not virtual level of $h(\mathbf{0})$ for $d\ge 3$, but is an eigenvalue of this operator.

If $\mu_\alpha\in L_{\alpha3}\Big(\mu_\alpha\in M_{\alpha3}\Big)$, then $\cos p_i^{(\alpha)}\ne 0$ $\Big(\sin p_i^{(\alpha)}\ne 0\Big)$ at least one of $i=\{1,\dots,d\}$. Consequently,

$$F_\alpha(\cdot)\in L_1(\mathbb{T}^d)\setminus L_2(\mathbb{T}^d),\quad \Big(\Phi_\alpha(\cdot)\in L_1(\mathbb{T}^d)\setminus L_2(\mathbb{T}^d)\Big)\quad\text{for}\quad d=3,4,$$

$$F_\alpha(\cdot)\in L_2(\mathbb{T}^d),\quad \Big(\Phi_\alpha(\cdot)\in L_2(\mathbb{T}^d)\Big)\quad\text{for}\quad d>4,$$

i.e. $z=0$ is a virtual level (eigenvalue) of the operator $h(\mathbf{0})$ for $d=3,4$ ($d>4$).

From a) and b) we deduce the following:

if $\mu_0=\mu^0(\mathbf{0})$, then $z=0$ is virtual level (eigenvalue) of $h(\mathbf{0})$ for $d=3,4$ ($d>4$);

if $\mu_\alpha\in L_{\alpha1}\cup L_{\alpha2}$, then $z=0$ is not virtual level of $h(\mathbf{0})$ for $d\ge 3$;

if $\mu_\alpha\in L_{\alpha3}$, then $z=0$ is virtual level (eigenvalue) of the operator $h(\mathbf{0})$ for $d=3,4$ ($d>4$);

if $\mu_\alpha \in L_{\alpha 2}$, then $z = 0$ is eigenvalue of the operator $h(\mathbf{0})$ for $d \geq 3$;
if $\mu_\alpha \in M_{\alpha 1} \cup M_{\alpha 2}$, then $z = 0$ is not virtual level of $h(\mathbf{0})$ for $d \geq 3$;
if $\mu_\alpha \in M_{\alpha 3}$, then $z = 0$ is a virtual level (eigenvalue) of $h(\mathbf{0})$ for $d = 3, 4$ ($d > 4$);
if $\mu_\alpha \in M_{\alpha 2}$, then $z = 0$ is eigenvalue of $h(\mathbf{0})$ for $d \geq 3$.
Part (i) of Theorem 2 is proved.
Part (ii) of Theorem 2 is proved analogously.

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