NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2016, 7 (5), P. 888–892

Characterization of the normal subgroups of finite index for the group representation of a Cayley tree

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DOI 10.17586/2220-8054-2016-7-5-888-892

In this paper we give a characterization of normal subgroups for the group representation of the Cayley tree.

Keywords: Cayley tree, G_k-group, normal subgroup, homomorphism, epimorphism.

Received: 4 August 2016

Revised: 29 August 2016

1. Introduction

The method used for the description of Gibbs measures on Cayley trees is the method of Markov random field theory and recurrent equations of this theory, however, the modern theory of Gibbs measures on trees uses new tools such as group theory, information flows on trees, node-weighted random walks, contour methods on trees, nonlinear analysis. A very recently published book [1] discusses all above-mentioned methods for describing Gibbs measures on trees. In the configuration of physical system is located on a lattice (in our case on the graph of a group), then the configuration can be considered as a function defined on the lattice. There are many works devoted to several kind of partitions of groups (lattices) (see e.g. [1–5, 7]).

One of the central problems in the theory of Gibbs measures is to study periodic Gibbs measures corresponding to a given Hamiltonian. For any normal subgroups H of the group G_k, we define H-periodic Gibbs measures.

In Chapter 1 of [1] several normal subgroups were constructed for the group representation of the Cayley tree. In [6], we found full description of normal subgroups of index four and six for the group. In this paper, we continue this investigation and construct all normal subgroups of index eight and ten for the group representation of the Cayley tree.

Cayley tree. A Cayley tree (Bethe lattice) Γ^k of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L that of edges (arcs).

A group representation of the Cayley tree. Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ...a_{k+1}$, respectively.

A one to one correspondence is known to exist between the set of vertices V of the Cayley tree Γ^k and the group G_k(see [1]).

To obtain this correspondence, we fix an arbitrary element $x_0 \in V$ and let it correspond to the unit element e of the group G_k. Using $a_1, ..., a_{k+1}$, we numerate the nearest-neighbors of element e, moving by positive direction. Next, we give the numeration for the nearest-neighbors of each $a_i, i = 1, ..., k+1$ by $a_i a_j, j = 1, ..., k+1$. Since all a_i have the common neighbor e, we give to it $a_i a_i = a_i^2 = e$. Other neighbors are numerated starting from $a_i a_i$ by the positive direction. We numerate the set of all the nearest-neighbors of each $a_i a_j$ by words $a_i a_j a_q, q = 1, ..., k+1$, starting from $a_i a_j a_j = a_i$ by the positive direction. Iterating this argument, one gets a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

Any(minimally represented) element $x \in G_k$ has the following form: $x = a_{i_1} a_{i_2} ... a_{i_n}$, where $1 \leq i_m \leq k+1, m = 1, ..., n$. The number n is called the length of the word x and is denoted by $l(x)$. The number of letters $a_i, i = 1, ..., k+1$, that enter the non-contractible representation of the word x is denoted by $w_x(a_i)$.

Proposition 1. *[8] Let φ be homomorphism of the group G_k with the kernel H. Then H is a normal subgroup of the group G_k and $\varphi(G_k) \simeq G_k/H$, (where G_k/H is a quotient group) i.e., the index $|G_k : H|$ coincides with the order $|\varphi(G_k)|$ of the group $\varphi(G_k)$.*

Let H be a normal subgroup of a group G. The natural homomorphism g from G onto the quotient group G/H is given by the formula $g(a) = aH$ for all $a \in G$. Then, $\mathrm{Ker}\,\varphi = H$.

Definition 1. *Let $M_1, M_2, ..., M_m$ be some sets and $M_i \neq M_j$, for $i \neq j$. We call the intersection $\cap_{i=1}^{m} M_i$ contractible if there exists $i_0 (1 \leq i_0 \leq m)$ such that:*

$$\cap_{i=1}^{m} M_i = \left(\cap_{i=1}^{i_0-1} M_i\right) \cap \left(\cap_{i=i_0+1}^{m} M_i\right).$$

Let $N_k = \{1, ..., k+1\}$. The following Proposition describes several normal subgroups of G_k.
Put

$$H_A = \left\{ x \in G_k \mid \sum_{i \in A} \omega_x(a_i) \text{ is even} \right\}, \quad A \subset N_k. \tag{1.1}$$

Proposition 2. *[1] For any $\emptyset \neq A \subseteq N_k$, the set $H_A \subset G_k$ satisfies the following properties: (a) H_A is a normal subgroup and $|G_k : H_A| = 2$;*
(b) $H_A \neq H_B$, for all $A \neq B \subseteq N_k$;
(c) Let $A_1, A_2, ..., A_m \subseteq N_k$. If $\cap_{i=1}^{m} H_{A_i}$ is non-contractible, then it is a normal subgroup of index 2^m.

Theorem 1. *[6]*
1. The group G_k does not have normal subgroups of odd index ($\neq 1$).
2. The group G_k has a normal subgroups of arbitrary even index.

2. New normal subgroups of finite index

2.1. The case of index eight

Definition 2. *A group G is called a **dihedral** group of degree 4 ($i.e., D_4$) if G is generated by two elements a and b satisfying the relations:*

$$o(a) = 4, \ o(b) = 2, \ ba = a^3 b.$$

Definition 3. *A group G is called a **quaternion** group ($i.e., Q_8$) if G is generated by two elements $a, \ b$ satisfying the relation:*

$$o(a) = 4, a^2 = b^2, ba = a^3 b.$$

Remark 1. [8] *D_4 is not isomorph to Q_8.*

Definition 4. *A commutative group G is called a **Klein 8-group** ($i.e., K_8$) if G is generated by three elements a, b and c satisfying the relations: $o(a) = o(b) = o(c) = 2$.*

Proposition 3. [8] *There exist (up to isomorphism) only two noncommutative nonisomorphic groups of order 8*

Proposition 4. *Let φ be a homomorphism of the group G_k onto a group G of order 8. Then, $\varphi(G_k)$ is isomorph to either D_4 or K_8.*

Proof. `Case 1` Let $\varphi(G_k)$ be isomorph to any noncommutative group of order 8. By Proposition 1, $\varphi(G_k)$ is isomorph to either D_4 or Q_8. Let $\varphi(G_k) \simeq Q_8$ and e_q be an identity element of the group Q_8. Then, $e_q = \varphi(e) = \varphi(a_i^2) = (\varphi(a_i))^2$ where $a_i \in G_k, \ i \in N_k$. Hence, for the order of $\varphi(a_i)$, we have $o(\varphi(a_i)) \in \{1, 2\}$. It is easy to check there are only two elements of the group Q_8 which order of element less than two. This is contradict.

`Case 2` Let $\varphi(G_k, *)$ be isomorph to any commutative group $(G, *_1)$ of order 8. Then, there exist distinct elements $a, b \in G$ such that $o(a) = o(b) = 2$. Let $H = \{e, a, b, ab\}$. It's easy to check that $(H, *_1)$ is a normal subgroup of the group $(G, *_1)$. For $c \notin H$ we have $H \neq cH$ ($cH = c *_1 H$). Hence $\varphi(G_k, *)$ is isomorph to only one commutative group $(cH \cup H, *_1)$. Clearly $(cH \cup H, *_1) \simeq K_8$. □

The group G has finit generators of the order two and r is the minimal number of such generators of the group G and without loss of generality, we can take these generators to be $b_1, b_2, ...b_r$. Let e_1 be an identity element of the group G. We define homomorphism from G_k onto G. Let $\Xi_n = \{A_1, \ A_2, ..., A_n\}$ be a partition of $N_k \backslash A_0, \ 0 \leq |A_0| \leq k+1-n$. Then, we consider homomorphism $u_n : \{a_1, a_2, ..., a_{k+1}\} \rightarrow \{e_1, b_1 ..., b_m\}$ as

$$u_n(x) = \begin{cases} e_1, & \text{if } \ x = a_i, i \in A_0 \\ b_j, & \text{if } \ x = a_i, i \in A_j, j = \overline{1, n}. \end{cases} \tag{2.1}$$

For $b \in G$, we denote that $R_b[b_1, b_2, ..., b_m]$ is a representation of the word b by generators $b_1, b_2, ..., b_r,\ r \leq m$. We define the homomorphism $\gamma_n : G \to G$ by the formula

$$\gamma_n(x) = \begin{cases} e_1, & \text{if } x = e_1 \\ b_i, & \text{if } x = b_i, i = \overline{1,r} \\ R_{b_i}[b_1, ..., b_r], & \text{if } x = b_i, i \neq \overline{1,r} \end{cases} \tag{2.2}$$

We set:

$$H^{(p)}_{\Xi_n}(G) = \{x \in G_k |\ l(\gamma_n(u_n(x))) \text{ is divisible by } 2p\},\ 2 \leq n \leq k-1. \tag{2.3}$$

Let $\gamma_n(u_n(x))) = \tilde{x}$. We introduce the following equivalence relation on the set $G_k : x \sim y$ if $\tilde{x} = \tilde{y}$. This relation is readily confirmed to be reflexive, symmetric and transitive.

Proposition 5. *Let $\Xi_n = \{A_1, A_2, ..., A_n\}$ be a partition of $N_k \backslash A_0,\ 0 \leq |A_0| \leq k+1-n$. Then $H^{(p)}_{\Xi_p}(G)$ is a normal subgroup of index $2p$ of the group G_k.*

Proof. For $x = a_{i_1} a_{i_2} ... a_{i_n} \in G_k$ it's sufficient to show that $x^{-1} H^{(p)}_{\Xi_n}(G)\ x \subseteq H^{(p)}_{\Xi_n}(G)$. Suppose that there exist $y \in G_k$ such that $l(\tilde{y}) \geq 2p$. Let $\tilde{y} = b_{i_1} b_{i_2} ... b_{i_q},\ q \geq 2p$ and $S = \{b_{i_1}, b_{i_1} b_{i_2}, ..., b_{i_1} b_{i_2} ... b_{i_q}\}$. Since $S \subseteq G$ there exist $x_1, x_2 \in S$ such that $x_1 = x_2$. But this contradicts $\tilde{y}$, which is a non-contractible. Thus we have proved that $l(\tilde{y}) < 2p$. Hence, for any $x \in H^{(p)}_{\Xi_n}(G)$ we have $\tilde{x} = e_1$. Next, we take any element z from the set $x^{-1} H^{(p)}_{\Xi_n}(G)\ x$. Then, $z = x^{-1} h\ x$ for some $h \in H^{(p)}_{\Xi_n}(G)$. We have $\tilde{z} = \gamma_n(v_n(z)) = \gamma_n\left(v_n(x^{-1} h\ x)\right) = \gamma_n\left(v_n(x^{-1}) v_n(h) v_n(x)\right) = \gamma_n\left(v_n(x^{-1})\right) \gamma_n\left(v_n(h)\right) \gamma_n\left(v_n(x)\right)$. From $\gamma_n\left(v_n(h)\right) = e_1$, we get $\tilde{z} = e_1$ i.e., $z \in H^{(p)}_{\Xi_n}(G)$. This completes the proof. □

Since $A_1, A_2, A_3 \subset N_k$ and $\cap_{i=1}^3 H_{A_i}$ is non-contractible we denote the following set:

$$\Re = \{\cap_{i=1}^3 H_{A_i} |\ A_1, A_2, A_3 \subset N_k\}.$$

Theorem 2. *For the group G_k, the following statement is hold:*

$$\{H|\ H \text{ is a normal subgroup of } G_k \text{ with } |G_k : H| = 8\} =$$

$$= \{H^{(4)}_{C_0 C_1 C_2}(D_4)|\ C_1, C_2 \text{ is a partition of } N_k \setminus C_0\} \cup \Re.$$

Proof. Let ϕ be a homomorphism with $|G_k : \operatorname{Ker} \phi| = 8$. Then by Proposition 2 we have $\phi(G_k) \simeq K_8$ and $\phi(G_k) \simeq D_4$.

Let $\phi : G_k \to K_8$ be an epimorphism. For any nonempty sets $A_1, A_2, A_3 \subset N_k$, we give a one to one correspondence between
operatornameKerφ$|\ \phi(G_k) \simeq K_8\}$ and $\Re$. Let $a_i \in G_k, i \in N_k$. We define following homomorphism corresponding to the set A_1, A_2, A_3:

$$\phi_{A_1 A_2 A_3}(a_i) = \begin{cases} a, & \text{if } i \in A_1 \setminus (A_2 \cup A_3) \\ b, & \text{if } i \in A_2 \setminus (A_1 \cup A_3) \\ c, & \text{if } i \in A_3 \setminus (A_1 \cup A_2) \\ ab, & \text{if } i \in (A_1 \cap A_2) \setminus (A_1 \cap A_2 \cap A_3) \\ ac, & \text{if } i \in (A_1 \cap A_3) \setminus (A_1 \cap A_2 \cap A_3) \\ bc, & \text{if } i \in (A_2 \cap A_3) \setminus (A_1 \cap A_2 \cap A_3) \\ abc, & \text{if } i \in A_1 \cap A_2 \cap A_3 \\ e, & \text{if } i \in N_k \setminus (A_1 \cup A_2 \cup A_3). \end{cases}$$

If $i \in \emptyset$, then we'll accept that there is no index $i \in N_k$ for which that condition is not satisfied. It is easy to check $\operatorname{Ker} \phi_{A_1 A_2 A_3} = H_{A_1} \cap H_{A_2} \cap H_{A_3}$. Hence $\{\operatorname{Ker} \phi|\ \phi(G_k) \simeq K_8\} = \Re$.

Now, we'll consider the case $\phi(G_k) \simeq D_4$. Let $\phi : G_k \to D_4$ be epimorphisms. Put

$$C_0 = \{i|\ \phi(a_i) = e\},\quad C_1 = \{i|\ \phi(a_i) = b\},\ C_2 = \{i|\ \phi(a_i) = ab\}.$$

One can construct following homomorphism (corresponding to C_0, C_1, C_2)

$$\phi_{C_0C_1C_2}(x) = \begin{cases} e, \text{ if } \tilde{x} = e \\ a, \text{ if } \tilde{x} = b_2b_1 \\ a^2, \text{ if } \tilde{x} = b_2b_1b_2b_1 \\ a^3, \text{ if } \tilde{x} = b_2b_1b_2b_1b_2b_1 \\ b, \text{ if } \tilde{x} = b_1 \\ ab, \text{ if } \tilde{x} = b_2 \\ a^2b, \text{ if } \tilde{x} = b_2b_1b_2 \\ a^3b, \text{ if } \tilde{x} = b_2b_1b_2b_1b_2. \end{cases}$$

Immediately, we conclude $\mathrm{Ker}(\phi_{C_0C_1C_2}) = H^{(4)}_{C_0C_1C_2}(D_4)$. We have constructed all homomorphisms ϕ on the group G_k which $|G_k : \mathrm{Ker}\,\phi| = 8$. Thus by Proposition 1, one gets:

$$\{H|\ |G_k : H| = 8\} \subseteq \{H^{(4)}_{C_0C_1C_2}(D_4)|\ C_1, C_2 \text{ is a partition of } N_k \setminus C_0\} \cup \Re.$$

By Proposition 2 and Proposition 5, we can easily see that:

$$\Re \cup \{H^{(4)}_{C_0C_1C_2}(D_4)|\ C_1, C_2 \text{ is a partition of } N_k \setminus C_0\} \subseteq \{H|\ |G_k : H| = 8\}.$$

The theorem is proved. □

Corollary 1. *The number of all normal subgroups of index* 8 *for the group* G_k *is equal to:* $8^{k+1} - 6 \cdot 4^{k+1} + 3^{k+1} + 9 \cdot 2^{k+1} - 5$.

Proof. Number of elements of the set $H_A \subset G_k, \emptyset \neq A \subset N_k$ is $2^{k+1}-1$. Then $|\Re| = (2^{k+1})(2^{k+1}-2)(2^{k+2}-3)$. Let $C_0 \subset N_k$ be a fixed set and $|C_0| = p$. If C_1, C_2 is a partition of $N_k \setminus C_0$ then there are $2^{k-p+1} - 2$ ways to choose the sets C_1 and C_2. Hence the cardinality of $\{H^{(4)}_{C_0C_1C_2}(D_4)|\ C_1, C_2 \text{ is a partition of } N_k \setminus C_0\}$ is equal to

$$(2^{k+1} - 2)C^0_{k+1} + (2^k - 2)C^1_{k+1} + ... + 2C^{k-1}_{k+1} = 3^{k+1} - 2^{k+2} + 1.$$

Since $\Re$ and $\{H^{(4)}_{C_0C_1C_2}(D_4)\}|\ C_1, C_2 \subset N_k$ are disjoint sets, the cardinality of the union of these sets is $8^{k+1} - 6 \cdot 4^{k+1} + 3^{k+1} + 9 \cdot 2^{k+1} - 5$. □

2.2. Case of index ten

Let the group R_{10} be generated by the permutations:

$$\pi_1 = (1,2)(3,4)(5,6),\ \ \pi_2 = (2,3)(4,5).$$

Proposition 6. *Let* φ *be a homomorphism of the group* G_k *onto a group* G *of order 10. Then,* $\varphi(G_k)$ *is isomorph to* R_{10}.

Proof. Let $(G, *)$ be a group and $|G| = 10$. Suppose there exist an epimorphism from G_k onto G. It is easy to check that there are at least two elements $a, b \in G_k$ such that $o(a) = o(b) = 2$. If $a * b = b * a$, then $(H, *)$ is a subgroup of the group $(G, *)$, where $H = \{e, a, b, a*b\}$. Then, by Lagrange's theorem, $|G|$ is divided by $|H|$ but 10 is not divided by 4. Hence, $a*b \neq b*a$. We have $\{e, a, b, a*b, b*a\} \subset G$ If G is generated by three elements, then there exist an element c such that $c \notin \{e, a, b, a*b, b*a\}$. Then, the set $\{e, a, b, a*b, b*a, c, c*a, c*b, c*a*b, c*b*a\}$ must be equal to G. Since G is a group, we get $a * b * a = b$ but from $o(a) = 2$ the last equality is equivalent to $a * b = b * a$. This is a contradiction. Hence, by Lagrange's theorem it is easy to see:

$$G = \{e, a, b, a*b, b*a, a*b*a, b*a*b, a*b*a*b, b*a*b*a, a*b*a*b*a\},$$

where $o(a * b) = 5$. Namely $G \simeq R_{10}$. This completes the proof. □

Theorem 3. *For the group* G_k*, the following statement is holds:*

$$\{H|\ |\ H \text{ is a normal subgroup of } G_k \text{ with } |G_k : H| = 10\} =$$

$$= \{H^{(5)}_{B_0B_1B_2}(R_{10})|\ B_1, B_2 \text{ is a partition of the set } N_k \setminus B_0\}.$$

Proof. Let ϕ be a homomorphism with $|G_k : \mathrm{Ker}\,\phi| = 10$. By Proposition 6 $\phi(G_k) \simeq R_{10}$ and by Proposition 5 we can easily see:

$$\{H^{(5)}_{B_0B_1B_2}(R_{10})|\ B_1, B_2 \text{ is a partition of the set } N_k \setminus B_0\} \subset \{H|\ |G_k : H| = 10\}.$$

Let $\varphi : G_k \to R_{10}$ be epimorphisms. We denote:

$$B_0 = \{i|\ \varphi(a_i) = e\},\quad B_1 = \{i|\ \varphi(a_i) = a,\ B_2 = \{i|\ \varphi(a_i) = b\}.$$

Then, we can show this homomorphism (corresponding to $B_1,\ B_2,\ B_3$), i.e.,

$$\phi_{B_0B_1B_2}(x) = \begin{cases} e, \text{ if } \tilde{x} = e \\ a, \text{ if } \tilde{x} = b_1 \\ b, \text{ if } \tilde{x} = b_2 \\ a*b, \text{ if } \tilde{x} = b_1b_2 \\ b*a, \text{ if } \tilde{x} = b_2b_1 \\ a*b*a, \text{ if } \tilde{x} = b_1b_2b_1 \\ b*a*b, \text{ if } \tilde{x} = b_2b_1b_2 \\ a*b*a*b, \text{ if } \tilde{x} = b_1b_2b_1b_2 \\ b*a*b*a, \text{ if } \tilde{x} = b_2b_1b_2b_1 \\ a*b*a*b*a, \text{ if } \tilde{x} = b_1b_2b_1b_2b_1. \end{cases}$$

We have constructed all homomorphisms ϕ on the group G_k which $|G_k : \mathrm{Ker}\,\phi| = 10$. Hence:

$$\{\mathrm{Ker}\,\phi|\ |G_k : \mathrm{Ker}\,\phi| = 10\} \subset \{H^{(5)}_{B_0B_1B_2}(R_{10})|\ B_1, B_2 \text{ is a partition of the set } N_k \setminus B_0\}.$$

By Proposition 1:

$$\{H|\ |G_k : H| = 10\} = \{H^{(5)}_{B_0B_1B_2}(R_{10})|\ B_1, B_2 \text{ is a partition of the set } N_k \setminus B_0\}.$$

The theorem is proved. □

Corollary 2. *The number of all normal subgroups of index* 10 *for the group* G_k *is equal to* $3^{k+1} - 2^{k+2} + 1$.

Proof. The proof of this Corollary is similar to proof of Corollary 1. □

Acknowledgements

I am deeply grateful to Professor U. A. Rozikov for the attention to my work.

References

[1] U.A. Rozikov. *Gibbs measures on a Cayley trees*, World Sci. Pub, Singapore, 2013.
[2] Cohen D.E., Lyndon R.C. Free bases for normal subgroups of free groups, *Trans. Amer. Math. Soc.*, 1963, **108**, P. 526–537.
[3] Ganikhodjaev N.N., Rozikov U.A. Description of periodic extreme Gibbs measures of some lattice model on the Cayley tree, *Theor. Math. Phys.*, 1997, 111, P. 480–486.
[4] G.I. Botirov. Functional equations for the Potts model with competing interactions on a Cayley tree. *Nanosystems: physics, chemistry, mathematics*, 2016, **7**(3), P. 401–404.
[5] Young J.W. On the partitions of a group and the resulting classification, *Bull. Amer. Math. Soc.*, 1927, **33**, P. 453–461.
[6] U.A. Rozikov, F.H. Haydarov. Normal subgroups of finite index for the group represantation of the Cayley tree, *TWMS Jour. Pure. Appl. Math.*, 2014, **5**, P. 234–240.
[7] Kurosh A.G. *Group theory*, Akademic Verlag, Berlin, 1953.
[8] D.S. Malik, J.N. Mordeson, M.K. Sen. *Fundamentals of Abstract Algebra*, McGraw-Hill Com., 1997.