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Waveguides with fast oscillating boundary

G. Cardone

University of Sannio, Department of Engineering,
Corso Garibaldi, 107, 82100 Benevento, Italy

gcardone@unisannio.it

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We consider an elliptic operator in a planar waveguide with a fast oscillating boundary where we impose Dirichlet, Neumann or Robin boundary conditions assuming that both the period and the amplitude of the oscillations are small. We describe the homogenized operator, establish the norm resolvent convergence of the perturbed resolvent to the homogenized one, and prove the estimates for the rate of convergence. It is shown that under the homogenization, the type of the boundary condition can change.

Keywords: elliptic operator, unbounded domain, norm resolvent convergence.

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1. Introduction

We study the problem of homogenization of boundary value problems in domains with a fast oscillating boundary when such boundary is given by the graph of the function $x_2 = \eta(\varepsilon)b(x_1\varepsilon^{-1})$, where ε is a small positive parameter, $\eta(\varepsilon)$ is a positive function tending to zero as $\varepsilon \to +0$, and b is a smooth periodic function. The parameter ε describes the period of the boundary oscillations while $\eta(\varepsilon)$ is their amplitude.

In previous results, the weak or strong resolvent convergence of the solutions was proved and the resolvents were also treated in various possible norms. In some cases, the estimates for the convergence rate were proven. It was also shown that when constructing the next terms of the asymptotics for the perturbed solutions, one can get estimates for the convergence rate or improve it [1–8]. In some cases, complete asymptotic expansions were constructed [9–12].

One more type of established results is the uniform resolvent convergence for the problems. Such convergence was established just for few models, see [13, Ch. III, Sec. 4], [8]. The estimates for the rates of convergence were also established. In both papers, the amplitude and the period of oscillations were of the same order. At the same time, the uniform resolvent convergence for the models considered in the homogenization theory provided quite strong results. Moreover, recently a series of papers by M. Sh. Birman, T. A. Suslina, V. V. Zhikov and S. E. Pastukhova have stimulated interest in this aspect (see [14–27] and references therein and further papers by these authors). The uniform resolvent convergence was shown to hold true for the elliptic operators with fast oscillating coefficients and the estimates for the rates of convergence were obtained. There are also similar results for some problems in bounded domains, see [26]. Similar results but for the boundary homogenization were established in [28–32]. Here, the Laplacian in a planar straight infinite strip with frequently alternating boundary conditions was considered. Such boundary conditions were imposed by partitioning the boundary into small segments where Dirichlet and Robin conditions were imposed in turn. The homogenized problem involves one of the classical boundary conditions instead of the alternating ones. For all possible homogenized problems, the uniform resolvent and the estimates for the rates of convergence were proven and the asymptotics for the spectra were constructed.

In the present paper, we also consider the boundary homogenization for the elliptic operators in unbounded domains but the perturbation is a fast oscillating boundary. As the domain, we choose a planar straight infinite strip with a periodic fast oscillating boundary; the operator is a general self-adjoint second order elliptic operator. The operator is regarded as an unbounded one in an appropriate L_2 space. On the oscillating boundary, we impose Dirichlet, Neumann, or Robin conditions. Apart from a mathematical interest in this problem, as a physical motivation, we can mention a model of a planar quantum or acoustic waveguide with a fast oscillating boundary.

Our main result is the form of the homogenized operator and the uniform resolvent convergence of the perturbed operator to the homogenized one. This convergence is established in the sense of the norm of the operator acting from L_2 into W_2^1. The estimates for the rate of convergence are provided. Most of the estimates are sharp. In the case of the Dirichlet or Neumann conditions on the oscillating boundary, the homogenized problem involves the same condition on the mollified boundary no matter how the period and amplitude of the oscillations behave. Provided the amplitude is not greater than the period (in order), the Robin conditions on the oscillating boundary leads us to a similar condition but with an additional term in the coefficient. If the amplitude is greater than the period, the homogenization transforms the Robin conditions into those of Dirichlet. The last result is in a good accordance with a similar case, treated in [33]. The difference is that in [33], the strong resolvent convergence was proven provided the coefficient in the Robin conditions was positive, while we succeeded to prove the uniform resolvent convergence provided the coefficient is either positive or non-negative and vanishing on the set of zero measure. All the results stated in this paper are proved in [34].

2. Problem and main results

Let $x = (x_1, x_2)$ be the Cartesian coordinates in $\mathbb{R}^2$, ε be a small positive parameter, $\eta = \eta(\varepsilon)$ be a non-negative function uniformly bounded for sufficiently small ε, $b = b(t)$ be a non-negative 1-periodic function belonging to $C^2(\mathbb{R})$. We define two domains, cf. Fig. 1:

$$\Omega_0 := \{x : 0 < x_2 < d\}, \quad \Omega_\varepsilon := \{x : \eta(\varepsilon)b(x_1\varepsilon^{-1}) < x_2 < d\},$$

where $d > 0$ is a constant, and its boundaries are indicated as:

$$\Gamma := \{x : x_2 = d\}, \quad \Gamma_0 := \{x : x_2 = 0\}, \quad \Gamma_\varepsilon := \{x : x_2 = \eta(\varepsilon)b(x_1\varepsilon^{-1})\}.$$

By $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, $A_0 = A_0(x)$, $i, j = 1, 2$, we denote the functions defined on Ω_0 and satisfying the belongings $A_{ij} \in W^2_\infty(\Omega_0)$, $A_j \in W^1_\infty(\Omega_0)$, $A_0 \in L_\infty(\Omega_0)$. Functions A_{ij}, A_j are assumed to be complex-valued, while A_0 is real-valued. In addition, functions A_{ij} satisfy the ellipticity condition:

$$A_{ij} = \overline{A}_{ji}, \quad \sum_{i,j=1}^{2} A_{ij} z_i \overline{z}_j \geqslant c_0(|z_1|^2 + |z_2|^2), \quad x \in \Omega_0, \quad z_j \in \mathbb{C}. \tag{2.1}$$

By $a = a(x)$ we denote a real function defined on $\{x : 0 < x_2 < \delta\}$ for some small fixed δ, and it is supposed that $a \in W^1_\infty(\{x : 0 < x_2 < d\})$.

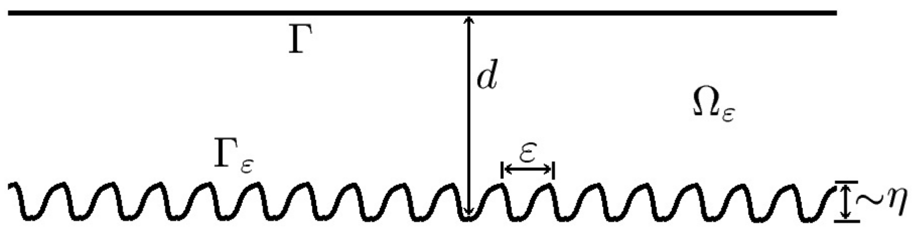

FIG. 1. Domain with oscillating boundary

The main object of our study is the operator:

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 \quad \text{in} \quad L_2(\Omega_\varepsilon), \tag{2.2}$$

subject to Dirichlet conditions on Γ. On the other boundary, we choose either Dirichlet conditions:

$$u = 0 \quad \text{on} \quad \Gamma_\varepsilon,$$

or Robin conditions:

$$\left(\frac{\partial}{\partial \nu^\varepsilon} + a\right) u = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad \frac{\partial}{\partial \nu^\varepsilon} = -\sum_{i,j=1}^{2} A_{ij} \nu_j^\varepsilon \frac{\partial}{\partial x_i} - \sum_{j=1}^{2} \overline{A}_j \nu_j^\varepsilon,$$

where $\nu^\varepsilon = (\nu_1^\varepsilon, \nu_2^\varepsilon)$ is the outward normal to Γ_ε. In the case of Dirichlet conditions on Γ_ε we denote this operator as $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$, while for Robin conditions it is $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$. The former also includes the case of Neumann conditions since the function a can be identically zero.

Rigorously, we introduce $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ as the lower-semibounded self-adjoint operator in $L_2(\Omega_\varepsilon)$ associated with the closed symmetric lower-semibounded sesquilinear form:

$$\lambda\mathfrak{h}^{\mathrm{D}}_{\varepsilon,\eta}(u, v) := \sum_{i,j=1}^{2} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2} \left(A_j \frac{\partial u}{\partial x_j}, v\right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2} \left(u, A_j \frac{\partial v}{\partial x_j}\right)_{L_2(\Omega_\varepsilon)} + (A_0 u, v)_{L_2(\Omega_\varepsilon)},$$

in $L_2(\Omega_\varepsilon)$ with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{D}}_{\varepsilon,\eta}) := W^1_{2,0}(\Omega_\varepsilon, \partial\Omega_\varepsilon)$. Hereinafter $\mathfrak{D}(\cdot)$ is the domain of a form or an operator, and $W^j_{2,0}(\Omega, S)$ denotes the Sobolev space consisting of the functions in $W^j_2(\Omega)$ with zero trace on a curve S lying in a domain $\Omega \subset \mathbb{R}^2$. The operator $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$ is introduced in the same way via the sesquilinear form:

$$\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(u,v) := \sum_{i,j=1}^{2} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2} \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2} \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} + (A_0 u, v)_{L_2(\Omega_\varepsilon)} + (au, v)_{L_2(\Gamma_\varepsilon)},$$

with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}) := W^1_{2,0}(\Omega_\varepsilon, \Gamma)$.

The main aim of the paper is to study the asymptotic behavior of the resolvents of $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ and $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$ as $\varepsilon \to +0$. To formulate the main results we first introduce some additional operators.

By $\mathcal{H}^{\mathrm{D}}_0$ we denote operator (2.2) in $L_2(\Omega_0)$ subject to Dirichlet conditions. We introduce it by analogy with $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ as associated with the form:

$$\mathfrak{h}^{\mathrm{D}}_0(u,v) := \sum_{i,j=1}^{2} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_0)} + \sum_{j=1}^{2} \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_0)} + \sum_{j=1}^{2} \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_0)} + (A_0 u, v)_{L_2(\Omega_0)}, \tag{2.3}$$

in $L_2(\Omega_0)$ with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{D}}_0) := W^1_{2,0}(\Omega_0, \partial\Omega_0)$. The domain of operator $\mathcal{H}^{\mathrm{D}}_0$ is $W^2_{2,0}(\Omega_0, \partial\Omega_0)$ that can be shown by analogy with [35, Ch. Ⅲ, Sec. 7,8], [36, Lm. 2.2].

Our first main result (proved in section 2 in [34]) describes the uniform resolvent convergence for $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$.

Theorem 2.1. *Let $f \in L_2(\Omega_0)$. For sufficiently small ε, the estimate:*

$$\|(\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta} - \mathrm{i})^{-1} f - (\mathcal{H}^{\mathrm{D}}_0 - \mathrm{i})^{-1} f\|_{W^1_2(\Omega_\varepsilon)} \leqslant C\eta^{1/2} \|f\|_{L_2(\Omega_0)},$$

holds true, where C is a constant independent of ε and f.

The next four theorems describe the resolvent convergence for operator $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$. Given $a_0 \in W^1_\infty(\Gamma_0)$, let $\mathcal{H}^{\mathrm{R}}_0$ be the self-adjoint operator in $L_2(\Omega_0)$ associated with the lower-semibounded sesquilinear symmetric form:

$$\mathfrak{h}^{\mathrm{R}}_0(u,v) := \sum_{i,j=1}^{2} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_0)} + \sum_{j=1}^{2} \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_0)} + \sum_{j=1}^{2} \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_0)} + (A_0 u, v)_{L_2(\Omega_0)} + (a_0 u, v)_{L_2(\Gamma_0)},$$

with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{R}}_0) := W^1_{2,0}(\Omega_0, \Gamma)$. It can be shown by analogy with [35, Ch. Ⅲ, Sec. 7,8], [36, Lm. 2.2] that the domain of $\mathcal{H}^{\mathrm{R}}_0$ consists of the functions $u \in W^2_{2,0}(\Omega_0, \Gamma)$ satisfying Robin conditions:

$$\left(\frac{\partial}{\partial \nu^0} + a_0 \right) u = 0 \quad \text{on} \quad \Gamma_0, \quad \frac{\partial}{\partial \nu^0} := -\sum_{i=1}^{2} A_{i2} \frac{\partial}{\partial x_i} - \overline{A}_2. \tag{2.4}$$

First, we consider the particular case of Neumann conditions on Γ_ε, i.e., $a = 0$. Operator $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$ and associated quadratic form $\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}$ are re-denoted in this case by $\mathcal{H}^{\mathrm{N}}_{\varepsilon,\eta}$ and $\mathfrak{h}^{\mathrm{N}}_{\varepsilon,\eta}$. By $\mathcal{H}^{\mathrm{N}}_0$, we denote the self-adjoint lower-semibounded operator in $L_2(\Omega_0)$ associated with the sesquilinear form $\mathfrak{h}^{\mathrm{N}}_0$ which is $\mathfrak{h}^{\mathrm{R}}_0$ taken for $a_0 \equiv 0$. Its domain is the set of the functions in $W^2_{2,0}(\Omega_0, \Gamma)$ satisfying boundary conditions (2.4) with $a_0 = 0$. The resolvent convergence in this case is given in following theorem (for the proof see section 3 in [34]).

Theorem 2.2. *Let $f \in L_2(\Omega_\varepsilon)$. Then for sufficiently small ε the estimate*

$$\|(\mathcal{H}^{\mathrm{N}}_{\varepsilon,\eta} - \mathrm{i})^{-1} f - (\mathcal{H}^{\mathrm{N}}_0 - \mathrm{i})^{-1} f\|_{W^1_2(\Omega_\varepsilon)} \leqslant C\eta^{1/2} \|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f.

Assume now $a \not\equiv 0$. Here we consider separately two cases:

$$\varepsilon^{-1}\eta(\varepsilon) \to \alpha = \text{const} \geqslant 0, \quad \varepsilon \to +0, \tag{2.5}$$

$$\varepsilon^{-1}\eta(\varepsilon) \to +\infty, \qquad \varepsilon \to +0. \tag{2.6}$$

The first assumption means that the amplitude of the oscillation of curve Γ_ε is of the same order (or smaller) as the period. The other assumption corresponds to the case when the amplitude is much greater than the period. In what follows, the first case is referred to as a relatively slow oscillating boundary Γ_ε while the other describes relatively high oscillating boundary Γ_ε.

We begin with the slowly oscillating boundary. We denote:

$$a_0(x_1) := a(x_1, 0) \int\limits_0^1 \sqrt{1 + \alpha^2 \big(b'(t)\big)^2}\, dt. \tag{2.7}$$

The proof of the following theorem is given in section 3 in [34].

Theorem 2.3. *Suppose (2.5) and let $f \in L_2(\Omega_\varepsilon)$. Then, for sufficiently small ε, the estimate*

$$\|(\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta} - \mathrm{i})^{-1} f - (\mathcal{H}^{\mathrm{R}}_0 - \mathrm{i})^{-1} f\|_{W_2^1(\Omega_\varepsilon)} \leqslant C(\eta^{1/2}(\varepsilon) + |\varepsilon^{-2}\eta^2(\varepsilon) - \alpha^2|)\|f\|_{L_2(\Omega_0)}$$

holds true, where function a_0 in (2.4) is defined in (2.7), and C is a constant independent of ε and f.

We proceed to the case of the highly oscillating boundary Γ_ε. Here, the homogenized operator happens to be quite sensitive to the sign of a and zero level set of this function. In the paper, we describe the resolvent convergence as a is non-negative. We first suppose that a is bounded from below by a positive constant. Surprisingly, but here the homogenized operator has the Dirichlet condition on Γ_0 as in Theorem 2.1. The proof of the following Theorem is given in section 4 in [34].

Theorem 2.4. *Suppose (2.6),*

$$a(x) \geqslant c_1 > 0, \quad c_1 = \text{const}, \tag{2.8}$$

and that the function b is not identically constant. Let $f \in L_2(\Omega_0)$. Then, for sufficiently small ε, the estimate:

$$\|(\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta} - \mathrm{i})^{-1} f - (\mathcal{H}^{\mathrm{D}}_0 - \mathrm{i})^{-1} f\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\big(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2}\big)\|f\|_{L_2(\Omega_0)} \tag{2.9}$$

holds true, where C is a constant independent of ε and f.

In the next theorem, that is proved in section 4 in [34], we still suppose that a is non-negative but can have zeroes. An essential assumption is that zero level set of a is of zero measure. We let $b_* := \max\limits_{[0,1]} b$.

Theorem 2.5. *Suppose (2.6),*

$$a \geqslant 0, \tag{2.10}$$

and that the function b is not identically constant. Assume also that for all sufficiently small δ, the set $\{x : a(x) \leqslant \delta,\ 0 < x_2 < (b_* + 1)\eta\}$ is contained in an at most countable union of the rectangles $\{x : |x_1 - X_n| < \mu(\delta),\ 0 < x_2 < (b_* + 1)\eta\}$, where $\mu(\delta)$ is a some nonnegative function such that $\mu(\delta) \to +0$ as $\delta \to +0$, and numbers X_n, $n \in \mathbb{Z}$, are independent of δ, are taken in the ascending order, and satisfy uniform in n and m estimate:

$$|X_n - X_m| \geqslant c > 0, \quad n \neq m. \tag{2.11}$$

Let $f \in L_2(\Omega_0)$. Then, for sufficiently small ε, the estimate:

$$\begin{aligned}\|(\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta} - \mathrm{i})^{-1} f - &(\mathcal{H}^{\mathrm{D}}_0 - \mathrm{i})^{-1} f\|_{W_2^1(\Omega_\varepsilon)} \\ &\leqslant C\big(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2} + \mu^{1/2}(\delta)|\ln \mu(\delta)|^{1/2}\big)\|f\|_{L_2(\Omega_0)}\end{aligned} \tag{2.12}$$

holds true, where C is a constant independent of ε and f, and $\delta = \delta(\varepsilon)$ is any function tending to zero as $\varepsilon \to +0$.

Let us discuss the main results. We first observe that under the hypotheses of all theorems we have the corresponding spectral convergence, namely, the convergence of the spectrum and the associated spectral projectors – see, for instance, [37, Thms. VIII.23, VIII.24]. We also stress that in all Theorems 2.1–2.5 the resolvent convergence is established in the sense of the uniform norm of bounded operator acting from $L_2(\Omega_0)$ into $W_2^1(\Omega_\varepsilon)$.

In the case of the Dirichlet conditions on Γ_ε, the homogenized operator has the same condition on Γ_0 no matter how the boundary Γ_ε oscillates, slowly or highly. The estimate for the rate of convergence is also universal being $\mathcal{O}(\eta^{1/2})$. Despite here we consider a periodically oscillating boundary, in the proof of Theorem 2.1 this fact is not used. This is why its statement is also valid for a periodically oscillating boundary described by the equation $x_2 = \eta b(x_1, \varepsilon)$, where b is an arbitrary function bounded uniformly in ε and such that $b(\cdot, \varepsilon) \in C(\mathbb{R})$. The estimate is Theorem 2.1 is sharp, see the discussion in the end of Sec. 2 in [34].

A similar situation occurs if we have Neumann conditions on Γ_ε. Here, Theorem 2.2 says that the homogenized operator is subject to Neumann conditions on Γ_0 and the rate of the uniform resolvent convergence is the same as in Theorem 2.1, namely, $\mathcal{O}(\varepsilon^{1/2})$. This estimate is again sharp, as the example in the end of Sec. 3 in [34] shows.

Once we have Robin conditions on Γ_ε, the situation is completely different. If the boundary oscillates slowly, the homogenized operator still has Robin conditions on Γ_0, but the coefficient depends on the geometry of the original oscillations, cf. (2.7). The estimate for the rate of the resolvent convergence in this case involves an additional term in comparison with the Dirichlet or Neumann cases, cf. Theorem 2.3. The estimate in this theorem is again sharp, see the example in the end of Sec. 3 in [34].

As boundary Γ_ε oscillates relatively highly, the resolvent convergence changes dramatically. If coefficient a is strictly positive, the homogenized operator has Dirichlet conditions on Γ_0. A new term, $\varepsilon^{1/2}\eta^{-1/2}$, appears in the estimate for the rate of the uniform resolvent convergence, cf. Theorem 2.4. We are able to prove that this term is sharp, see the discussion in the end of Sec. 4 in [34].

Provided function a is non-negative and vanishes only on a set of zero measure, the homogenized operator still has Dirichlet conditions on Γ_0, but the estimate for the rate of the uniform resolvent convergence becomes worse. Namely, the behavior of a in a vicinity of its zeroes becomes important. This is reflected by functions $\mu(\delta)$ and δ in (2.12). The latter should be chosen so that $\delta \to +0$, $\varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2} \to +0$, $\varepsilon \to +0$, that is always possible. The optimal choice of δ is so that:

$$\begin{aligned} &\mu^{1/2}(\delta)|\ln \mu(\delta)|^{1/2} \sim \varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2}, \\ &\delta\mu(\delta)|\ln \mu(\delta)| \sim \varepsilon\eta^{-1}. \end{aligned} \tag{2.13}$$

As we see, the choice of δ depends on a particular structure of $\mu(\delta)$. The most typical case is $\mu(\delta) \sim \delta^{1/2}$, i.e., the function a vanishes by the quadratic law in a vicinity of its zeroes. In this case, condition (2.13) becomes:

$$\delta^{3/2}|\ln \delta| \sim \varepsilon\eta^{-1},$$

which implies:

$$\delta \sim \varepsilon^{2/3}\eta^{-2/3}|\ln \varepsilon\eta^{-1}|^{-2/3}.$$

Then, the estimate for the resolvent convergence in Theorem 2.5 is of order $\mathcal{O}\big((\eta^{1/2} + \varepsilon^{1/6}\eta^{-1/6}|\ln \varepsilon\eta^{-1}|^{1/3}\big)$.

We are not able to prove the sharpness of estimate (2.12), but in the end of Sec. 4 in [34] we provide some arguments showing that estimate (2.12) is rather close to being optimal.

In conclusion, we discuss the case of Robin conditions on highly oscillating Γ_ε when the coefficient a does not satisfy the hypotheses of Theorems 2.4, 2.5. If it is still non-negative but vanishes for a set of non-zero values, and at the end-points of this set the vanishing happens with certain rate like in Theorem 2.5, we conjecture that the homogenized operator involves mixed Dirichlet and Neumann conditions on Γ_0. Namely, if $a(x_1, 0) \equiv 0$ on Γ_0^N and $a(x_1, 0) > 0$ on Γ_0^D, $\Gamma_0 = \Gamma_0^N \cup \Gamma_0^D$, it is natural to expect that the homogenized operator has Neumann conditions on Γ_0^N and Dirichlet one on Γ_0^D. This conjecture can be regarded as the mixture of the statements of Theorems 2.2 and 2.5. The main difficulty of proving this conjecture is that the domain of such homogenized operator is no longer a subset of $W_2^2(\Omega_0)$ because of the mixed boundary conditions. At the same time, this fact was essentially used in all our proofs. An even more complicated situation occurs once a is negative or sign-indefinite. If a is negative on a set of non-zero measure, it can be shown that the bottom of the spectrum of the perturbed operator goes to $-\infty$ as $\varepsilon \to +0$. In such cases, one should study the resolvent convergence near this bottom, i.e., for the spectral parameter tending to $-\infty$. This makes the issue quite troublesome. We stress that under the hypotheses of all Theorems 2.1–2.5, the bottom of the spectrum is lower-semibounded uniformly in ε.

References

[1] Friedman A., Hu B., Liu Y. A boundary value problem for the poisson equation with multi-scale oscillating Boundary. *J. Diff. Eq.*, 1997, **137**, P. 54–93.

[2] Friedman A., Hu B. A Non-stationary Multi-scale Oscillating Free Boundary for the Laplace and Heat Equations. *J.Diff. Eq.*, 1997, **137**, P. 119–165.

[3] Belyaev A.G., Pyatnitskii A.L., Chechkin G.A. Asymptotic behavior of a solution to a boundary value problem in a perforated domain with oscillating boundary. *Siberian Math. J.*, 1998, **39**, P. 621–644.

[4] Kozlov V.A., Nazarov S.A. Asymptotics of the spectrum of the Dirichlet problem for the biharmonic operator in a domain with a deeply indented boundary. *St. Petersburg Math. J.*, 2011, **22**, P. 941–983.

[5] Chechkin G.A., Chechkina T.P. On homogenization of problems in domains of the “Infusorium” type. *J. Math. Sci., New York*, 2004, **120**, P. 1470–1482.

[6] Mikelić A. *Rough boundaries and wall laws. Qualitative properties of solutions to partial differential equations, Lecture notes of Necas Center for mathematical modeling*, ed. by E. Feireisl, P. Kaplicky and J. Malek, Vol. 5, Matfyzpress, Publishing House of the Faculty of Mathematics and Physics Charles University in Prague, Prague, 2009, P. 103–134.

[7] Nazarov S.A. The two terms asymptotics of the solutions of spectral problems with singular perturbations. *Math. USSR-Sb.*, 1991, **69**, P. 307–340.

[8] Nazarov S.A. Dirichlet problem in an angular domain with rapidly oscillating boundary: Modeling of the problem and asymptotics of the solution. *St. Petersburg Math. J.*, 2008, **19**, P. 297–326.

[9] Amirat Y., Chechkin G.A., Gadyl'shin R.R. Asymptotics for eigenelements of Laplacian in domain with oscillating boundary: multiple eigenvalues. *Appl. Anal.*, 2007, **86**, P. 873–897.

[10] Amirat Y., Chechkin G.A., Gadylshin R.R. Asymptotics of simple eigenvalues and eigenfunctions for the Laplace operator in a domain with oscillating boundary. *Computational Mathematics and Mathematical Physics*, 2006, **46**, P. 97–110.

[11] Nazarov S.A. Asymptotics of solutions and modeling the problems of elasticity theory in domains with rapidly oscillating boundaries. *Izv. Math.*, 2008, **72**.

[12] Gobbert M.K., Ringhofer C.A. An Asymptotic Analysis for a Model of Chemical Vapor Deposition on a Microstructured Surface. *SIAM Jour. Appl. Math.*, 1998, **58**, P. 737–752.

[13] Olejnik O.A., Shamaev A.S., Yosifyan G.A. *Mathematical problems in elasticity and homogenization.* Studies in Mathematics and its Applications. **26**, North-Holland, Amsterdam etc., 1992.

[14] Bunoiu R., Cardone G., Suslina T. Spectral approach to homogenization of an elliptic operator periodic in some directions. *Math. Meth. Appl. Sci.*, 2011, **34**, P. 1075–1096.

[15] Cardone G., Pastukhova S.E., Zhikov V.V. Some estimates for nonlinear homogenization. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.*, 2005, **29**, P. 101–110.

[16] Cardone G., Pastukhova S.E., Perugia C. Estimates in homogenization of degenerate elliptic equations by spectral method. *Asympt. Anal.*, 2013, **81**, P. 189–209.

[17] Pastukhova S.E., Tikhomirov R.N. Operator Estimates in Reiterated and Locally Periodic Homogenization. *Dokl. Math.*, 2006, **76**, P. 548–553.

[18] Pastukhova S.E. Some Estimates from Homogenized Elasticity Problems. *Dokl. Math.*, 2006, **73**, P. 102–106.

[19] BirmanM.S. On homogenization procedure for periodic operators near the edge of an internal gap. *St. Petersburg Math. J.*, 2004, **15**, P. 507–513.

[20] Birman M.S., Suslina T.A. Homogenization of a multidimensional periodic elliptic operator in a neighbourhood of the edge of the internal gap. *J. Math. Sciences*, 2006, **136**, P. 3682–3690.

[21] Birman M.S., Suslina T.A. Homogenization with corrector term for periodic elliptic differential operators. *St. Petersburg Math. J.*, 2006, **17**, P. 897–973.

[22] Birman M.S., Suslina T.A. Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$. *St. Petersburg Math. J.*, 2007, **18**, P. 857–955.

[23] Vasilevskaya E.S., Suslina T.A. Threshold approximations for a factorized selfadjoint operator family with the first and second correctors taken into account. *St. Petersburg Math. J.*, 2012, **23**, P. 275–308.

[24] Zhikov V.V. On operator estimates in homogenization theory. *Dokl. Math.*, 2005, **72**, P. 534–538.

[25] Zhikov V.V. Spectral method in homogenization theory. *Proc. Steklov Inst. Math.*, 2005, **250**, P. 85–94.

[26] Zhikov V.V. Some estimates from homogenization theory. Dokl. Math., 2006, **73**, P. 96–99.

[27] Zhikov V.V., Pastukhova S.E., Tikhomirova S.V. On the homogenization of degenerate elliptic equations. *Dokl. Math.*, 2006, **74**, P. 716–720.

[28] Borisov D., Bunoiu R., Cardone G. On a waveguide with frequently alternating boundary conditions: homogenized Neumann condition. *Ann. H. Poincaré*, 2010, **11**, P. 1591–1627.

[29] Borisov D., Bunoiu R., Cardone G. On a waveguide with an infinite number of small windows. *C.R. Mathematique*, 2011, **349**, P. 53–56.

[30] Borisov D., Bunoiu R., Cardone G. Homogenization and asymptotics for a waveguide with an infinite number of closely located small windows. *J. of Math. Sci.*, 2009, **176**, P. 774–785.

[31] Borisov D., Bunoiu R., Cardone G. Waveguide with non-periodically alternating Dirichlet and Robin conditions: homogenization and asymptotics. *Z. Ang. Math. Phys.*, 2013, **64** (3), P. 439–472.

[32] Borisov D., Cardone G. Homogenization of the planar waveguide with frequently alternating boundary conditions. *J. of Phys. A: Mathematics and General*, 2009, **42**, P. 365205 (21 pp.).

[33] Chechkin G.A., Friedman A., Piatnitski A.L. The Boundary-value Problem in Domains with Very Rapidly Oscillating Boundary. *J. Math. Anal. Appl.*, 1999, **231**, P. 213–234.

[34] Borisov D., Cardone G., Faella L., Perugia C. Uniform resolvent convergence for a strip with fast oscillating boundary. *J. Diff. Eq.*, 2013, **255**, P. 4378–4402.

[35] Ladyzhenskaya O.A., Uraltseva N.N. *Linear and quasilinear elliptic equations*. Academic Press, New York, 1968.

[36] Borisov D. Asymptotics for the solutions of elliptic systems with fast oscillating coefficients. *St. Petersburg Math. J.*, 2009, **20**, P. 175–191.

[37] Reed M., Simon B. *Methods of mathematical physics. Functional analysis*, Academic Press, 1980.