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On convergence rate estimates for approximations of solution operators for linear non-autonomous evolution equations

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We improve some recent estimates of the rate of convergence for product approximations of solution operators for linear non-autonomous Cauchy problem. The Trotter product formula approximation is proved to converge to the solution operator in the operator-norm. We estimate the rate of convergence of this approximation. The result is applied to diffusion equation perturbed by a time-dependent potential.

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1. Introduction

The theory of equations of evolution plays an important role in various areas of pure and applied mathematics, physics and other natural sciences [1–3]. We focus on a non-autonomous linear Cauchy problem of the form:

$$\dot{u}(t) = -(A + B(t))u(t), \quad u(s) = u_s \in X, \quad 0 < s \le t \le T, \tag{1.1}$$

where $\{A+B(t),\ \mathrm{dom}(A)\cap\mathrm{dom}(B(t))\}_{t\in\mathcal{I}}$ is a family of closed linear operators on the separable Banach space X, $\mathcal{I} = [0,T] \subset \mathbb{R}$. Let $\mathcal{I}_0 = (0,T]$. The solution operator $\{U(t,s)\}_{(t,s)\in\Delta}$, i.e. $u(t) = U(t,s)u_s$ solves (1.1) in some sense, can be obtained using the Howland–Evans approach. The main idea of this approach is to reformulate the *non-autonomous* problem (1.1) on X in such a way that it becomes equivalent to an *autonomous* Cauchy problem on the Banach space $L^p(\mathcal{I},X)$ of p-summable functions on $\mathcal{I}$ with values in X. Then solutions of the autonomous and the non-autonomous Cauchy problem are in one-to-one correspondence, and therefore, it is equivalent which of them one has to solve. Once, the solution is obtained, the problem of a good approximation arises. The Trotter product formula [4] or [5, Theorem 3.5.8] provides approximation in the strong topology. In practice, a convergence in the operator-norm topology is more useful, especially, if the error bound for approximation can be estimated. Then, for example, independent of the initial condition, the smallness of the iteration steps and their number can be calculated in such a way that the error bound of the approximation will be smaller than a given accuracy.

We are going to analyze a linear non-autonomous Cauchy problem of the form (1.1) where the aim is to find for the problem (1.1) a so-called *solution operator* or *propagator*: $\{U(t,s)\}_{(t,s)\in\Delta}$, $\Delta = \{(t,s) \in \mathcal{I}_0 \times \mathcal{I}_0 : 0 < s \le t \le T\}$, $\mathcal{I}_0 = (0,T]$. It has the property that $u(t) = U(t,s)u_s$ for $(t,s) \in \Delta$ is a "solution" of the Cauchy problem (1.1) for an appropriate set of initial data u_s. By definition, the propagator $\{U(t,s)\}_{(t,s)\in\Delta}$ is a *strongly continuous* operator-valued function $U(\cdot,\cdot) : \Delta \to \mathcal{B}(X)$ satisfying:

$$U(t,t) = I \quad \text{for} \quad t \in \mathcal{I}_0,\ U(t,r)U(r,s) = U(t,s) \quad \text{for} \quad t,r,s \in \mathcal{I}_0 \quad \text{with} \quad s \le r \le t,$$
$$\|U\|_{\mathcal{B}(X)} := \sup_{(t,s)\in\Delta} \|U(t,s)\|_{\mathcal{B}(X)} < \infty.$$

Our goal is to find an approximation operator $\{U_n(t,s)\}_{(t,s)\in\Delta}$, $n \in \mathbb{N}$, for the solution operator $\{U(t,s)\}_{(t,s)\in\Delta}$, which approximates the solution operator in the operator-norm topology, and to estimate of its convergence rate. Such convergence rate estimates have been already found by Ichinose and Tamura for positive self-adjoint operators [6] in the Hilbert space setting. Recently (see [7]) the operator norm convergence and an error estimate were proved when the underlying space is a Banach space. In our paper [7], the main technical tool to construct such approximation is the Trotter product formula. We proved that under the assumptions formulated in this paper, the Trotter product formula converges not only in the strong but in the operator-norm topology. To lift the strong topology to the operator-norm, we used the Trotter product formula and the relation between solution operator and evolution semigroup.

Following the ideas of [7], we improve in the present paper the convergence rate estimate $O(1/n^{\beta-\alpha})$, $0 < \alpha < \beta < 1$, which was obtained there. We assume in [7] that the involved operators A and $B(t)$ verify conditions inspired by [6] in a Hilbert space, although we do not suppose that for each t the operator $B(t)$ generates a bounded holomorphic semigroup. This gives us an extension of results [6] for the rate $O(\ln(n)/n)$ in a Hilbert space to a Banach space. On the other hand, it is not surprising that the error bound estimate in [7] is weaker than $O(\ln(n)/n^{1-\alpha})$ obtained for the first time in a Banach space by [8] under the same conditions as in [7], but for the autonomous Cauchy problem. Note that below (Section 2.2) our conditions (A2) and (A3) are a bit stronger conditions than in [7] or in [8]. Despite that, we were unable to reproduce the strikingly fast convergence rate of [6] for the case of Banach spaces. Although these stronger conditions allow us to push β up to $\beta = 1$. So the obtained in the present paper rate $O(1/n^{1-\alpha})$ is improved compared to [7] and also to [8] by elimination of the $\ln(n)$.

2. Preliminaries and assumptions

2.1. Preliminaries

Throughout the paper, we are dealing with a separable Banach space $(X, \|\cdot\|_X)$. For a linear operator $A : \mathrm{dom}(A) \subset X \to X$, we define the resolvent by $R(\lambda, A) := (A - \lambda)^{-1} : X \to \mathrm{dom}(A)$. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called a strongly continuous (one-parameter) semigroup if it satisfies the functional equation:

$$T(0) = I, \quad T(t+s) = T(t)T(s), \quad t, s \geq 0,$$

and the orbit maps $[0, \infty) \ni t \mapsto T(t)x$ are continuous for every $x \in X$. In the following we simply call them semigroups. For a given semigroup we define its generator by:

$$Ax := \lim_{h\searrow 0} \frac{1}{h} (x - T(h)x),$$

with domain:

$$\mathrm{dom}(A) := \Big\{x \in X : \lim_{h\searrow 0} \frac{1}{h}(x - T(h)x) \text{ exists}\Big\}.$$

Note that the definition differs from the standard one by the minus sign. It is well-known that the generator of a semigroup is a closed and densely defined linear operator which uniquely determines the semigroup (see e.g. [5, Theorem I.1.4]). For a given generator A we will write $T(\tau) = e^{-\tau A}$, $\tau \geq 0$.

For any semigroup $\{T(t)\}_{t\geq 0}$ there are constants M_A, γ_A, such that it holds $\|T(t)\| \leq M_A e^{\gamma_A t}$ for all $t \geq 0$. Such semigroups are called of class $\mathcal{G}(M_A, \gamma_A)$ and we write $A \in \mathcal{G}(M_A, \gamma_A)$. If $\gamma_A \leq 0$, $\{T(t)\}_{t\geq 0}$ is called a bounded semigroup. If $\|T(t)\| \leq 1$, the semigroup is called contractive.

For any semigroup we can construct a bounded semigroup by adding some constant $\nu \geq \gamma_A$ to its generator: the operator $\tilde{A} := A + \nu$ generates a semigroup $\{\tilde{T}(t)\}_{t\geq 0}$ with $\|\tilde{T}(t)\| \leq M_A$. It is known that for a generator $A \in \mathcal{G}(M_A, \gamma_A)$, the open half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < \gamma_A\}$ is contained in the resolvent set $\varrho(A)$ of A and the estimate $\|R(\lambda, A)\| \leq M_A/(\mathrm{Re}(\lambda) - \gamma_A)$ holds. If $\tilde{A} = A + \nu$, then the open half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < \gamma_A - \nu\}$ is contained in the resolvent set of $\tilde{A}$.

The semigroup $\{T(t)\}_{t\geq 0}$ on X is called a bounded holomorphic semigroup if its generator A satisfies $\mathrm{ran}(T(t)) \subset \mathrm{dom}(A)$ for all $t > 0$ and $\sup_{t>0} \|tAT(t)\| < \infty$. It is well-known, that in this case the semigroup $\{T(t)\}_{t\geq 0}$ can be extended holomorphically to a sector $\{z \in \mathbb{C} : |\arg(z)| < \delta\} \cup \{0\} \subset \mathbb{C}$ of angle $\delta > 0$. For generators A of bounded holomorphic semigroups with $0 \in \varrho(A)$ one can define fractional powers A^α. Then, for $\alpha \in (0, 1)$, it holds $\mathrm{dom}(A) \subset \mathrm{dom}(A^\alpha) \subset \mathrm{dom}(A^0) = X$. In the following we need the well-known estimate for generators of a bounded holomorphic semigroup:

$$\sup_{t>0} \|t^\alpha A^\alpha T(t)\| = M_\alpha^A < \infty. \tag{2.1}$$

2.2. Assumptions

Below we made the following assumptions with respect to the operator A and the family $\{B(t)\}_{t\in\Delta}$.

Assumption 2.1.
(A1) *The operator A is a generator of a bounded holomorphic semigroup of class $\mathcal{G}(M_A, 0)$ and $0 \in \varrho(A)$. Let $\{B(t)\}_{t\in\mathcal{I}}$ be a family of generators on X belonging to the same class $\mathcal{G}(M_B, \beta)$. The function $\mathcal{I} \ni t \mapsto (B(t) + \xi)^{-1}x \in X$ is strongly measurable for any $x \in X$ and any $\xi > \beta$.*
(A2) *There is an $\alpha \in (1/2, 1)$ such that for a.e. $t \in \mathcal{I}$ it holds that $\mathrm{dom}(A^\alpha) \subset \mathrm{dom}(B(t))$ and $\mathrm{dom}((A^\alpha)^*) \subset \mathrm{dom}(B(t)^*)$. Moreover, it holds:*

$$C_\alpha := \operatorname*{ess\,sup}_{t\in\mathcal{I}} \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} < \infty \quad \textit{and} \quad C^*_\alpha := \operatorname*{ess\,sup}_{t\in\mathcal{I}} \|B(t)^*(A^{-\alpha})^*\|_{\mathcal{B}(X^*)} < \infty, \tag{2.2}$$

where A^* and $B(t)^*$ denote the adjoint operators of A and $B(t)$, respectively.
(A3) *There is a constant $L > 0$ such that estimate:*

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\|_{\mathcal{B}(X)} \le L|t - s|,$$

holds for a.e. $t, s \in \mathcal{I}$.

Remark 2.2.
(a) In [7], the assumptions are slightly weaker. It is assumed that the domains satisfy $\mathrm{dom}(A^*) \subset \mathrm{dom}(B(t)^*)$.
(b) The assumption $0 \in \varrho(A)$ is just for simplicity. Otherwise, the generator A can be shifted by a constant $\eta > 0$. One can prove that the domain of the fractional power of A does not change either.
(c) In [6] both operators A and $B(t)$ are assumed to be positive self-adjoint operators on a separable Hilbert space. The assumptions made in [6] yield that Assumption 2.1 is valid. We note that the first results in Banach spaces for *autonomous* Cauchy problem are due to [8]. The Trotter product approximation was proven there in the framework of Assumption 2.1: (A1), (A2).
(d) The assumptions above imply that for a.e. $t \in \mathcal{I}$ the operator $B(t)$ is infinitesimally small with respect to A. Indeed, fix $t \in \mathcal{I}$ and assuming (A1), (A2) we conclude:

$$\mathrm{dom}(A + \eta) = \mathrm{dom}(A) \subset \mathrm{dom}(A^\alpha) \subset \mathrm{dom}(B(t)),$$

for $\eta > 0$ and hence:

$$\|B(t)(A + \eta)^{-1}\|_{\mathcal{B}(X)} \le \|B(t)A^{-\alpha}\|_{\mathcal{B}(X)} \cdot \|A^\alpha(A + \eta)^{-1}\|_{\mathcal{B}(X)} \le \frac{C_\alpha C_0}{\eta^{1-\alpha}}.$$

Therefore for any $x \in \mathrm{dom}(A) \subset \mathrm{dom}(B(t))$, we get:

$$\|B(t)x\|_X \le \frac{C_\alpha C_0}{\eta^{1-\alpha}} \cdot \|(A + \eta)x\|_X \le C_\alpha C_0 \eta^\alpha \left(\frac{1}{\eta}\|Ax\|_X + \|x\|_X\right).$$

The relative bound can be chosen arbitrarily small by shifting $\eta > 0$. In particular, using standard perturbation results ([9, Corollary IX.2.5]), we conclude that $A + B(t)$ is the generator of a holomorphic semigroup, i.e. problem (1.1) is a parabolic evolution equation.

3. Construction of solution operators

We start by description of our strategy. Details can be found in [7]. Our approach to construct the solution operator $\{U(t,s)\}_{(t,s)\in\Delta}$ of (1.1) leads to a perturbation or extension problem for linear operators. It can be used in very general settings and it is quite flexible. The idea can be described as follows: The *non-autonomous* Cauchy problem in X can be reformulated as an *autonomous* Cauchy problem in a new Banach space $L^p(\mathcal{I}, X)$, $p \in [1, \infty)$, of p-summable functions on the interval $\mathcal{I}$ with values in the Banach space X. An operator family $\{C(t)\}_{t\in\mathcal{I}}$ on X induces an multiplication operator $\mathcal{C}$ on $L^p(\mathcal{I}, X)$ defined by:

$$(\mathcal{C}f)(t) := C(t)f(t),$$

$$\mathrm{dom}(\mathcal{C}) := \left\{ f \in L^p(\mathcal{I}, X) \ : \ \begin{array}{c} f(t) \in \mathrm{dom}(C(t)) \text{ for a.e. } t \in \mathcal{I} \\ \mathcal{I} \ni t \mapsto C(t)f(t) \in L^p(\mathcal{I}, X) \end{array} \right\}.$$

Theorem 3.1 ([7, Theorem 2.8]). *Let $\{C(t)\}_{t\in\mathcal{I}}$ be a family of generators on X such that for almost all $t \in \mathcal{I}$ it holds that $C(t) \in \mathcal{G}(M,\beta)$ for some $M \geq 1$ and $\beta \in \mathbb{R}$. If the function $\mathcal{I} \ni t \mapsto (C(t)+\xi)^{-1}x \in X$ is strongly measurable for $\xi > \beta$, $x \in X$, then the induced multiplication operator $\mathcal{C}$ is a generator in $L^p(\mathcal{J},X)$ and its semigroup is given by:*

$$\left(e^{-\tau\mathcal{C}}f\right)(t) = e^{-\tau C(t)}f(t), \quad f \in L^p(\mathcal{I},X),$$

for a.e $t \in \mathcal{I}$. In particular, for the operator-norms we get:

$$\|e^{-\tau\mathcal{C}}\|_{\mathcal{B}(L^p(\mathcal{I},X))} = \operatorname*{ess\,sup}_{t\in\mathcal{I}} \|e^{-\tau C(t)}\|_{\mathcal{B}(X)}.$$

So the generators $C(t)$ and $\mathcal{C}$ belong to the same class.

In particular in our case, the operator family $\{B(t)\}_{t\in\mathcal{I}}$ induces the generator $\mathcal{B}$ and A induces trivially the generator $\mathcal{A}$ on $L^p(\mathcal{I},X)$. Assuming (A1) and (A2) it turns out that the operators $\mathcal{B}\mathcal{A}^{-\alpha}$ and $\overline{\mathcal{A}^{-\alpha}\mathcal{B}}$ are bounded on $L^p(\mathcal{I},X)$ and it holds that $\|\mathcal{B}\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \leq C_\alpha$ and $\|\overline{\mathcal{A}^{-\alpha}\mathcal{B}}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \leq C_\alpha^*$.

Let us introduce the operator $D_0 := \partial_t$ on $L^p(\mathcal{I},X)$ defined by:

$$D_0 f(t) := \partial_t f(t), \quad \mathrm{dom}(D_0) := \{f \in W^{1,p}([0,T],X) : f(0) = 0\}.$$

Then, D_0 is a generator of class $\mathcal{G}(1,0)$ of the right-shift semigroup $\{S(\tau)\}_{\tau\geq 0}$ that has the form:

$$\left(e^{-\tau D_0}f\right)(t) = (S(\tau)f)(t) := f(t-\tau)\chi_{\mathcal{I}}(t-\tau), \quad f \in L^p(\mathcal{I},X), \text{ a.e. } t \in \mathcal{I}.$$

We note that the generator D_0 has empty spectrum since the semigroup $\{S(\tau)\}_{\tau\geq 0}$ is nilpotent and therefore the integral $\int_0^\infty d\tau\ e^{-\tau\lambda}S(\tau)f$ exists for any $\lambda \in \mathbb{C}$ and for any $f \in L^p(\mathcal{I},X)$.

Let us look at the operator sum D_0 and $\mathcal{A}$. Since A is time-independent, the operators $\mathcal{A}$ and D_0 commute, and, hence, also their semigroups commute. So, the operator family $\{e^{-\tau\mathcal{A}}e^{-\tau D_0}\}_{\tau\geq 0}$ defines a semigroup on $L^p(\mathcal{I},X)$. Its generator is denoted by $\mathcal{K}_0$. It is closure of the operator sum $D_0 + \mathcal{A}$, i.e. $\mathcal{K}_0 = \overline{D_0 + \mathcal{A}}$. We note that all the generators $\mathcal{K}_0$, $\mathcal{A}$, A belong to the same class.

Remark 3.2. By assumption $(A1)$ the operator $\mathcal{A}$ generates a holomorphic semigroup. Note that the operator $\mathcal{K}_0$ is not a generator of a holomorphic semigroup. Indeed, if we have:

$$\left(e^{-\tau\mathcal{K}_0}f\right)(t) = \left(e^{-\tau D_0}e^{-\tau\mathcal{A}}f\right)(t) = e^{-\tau A}f(t-\tau)\chi_{\mathcal{I}}(t-\tau), \quad f \in L^p(\mathcal{I},X).$$

Since the right-hand side is zero for $\tau \geq t$, the semigroup can not be extended holomorphically to the complex plane.

Now, look at the operator sum:

$$\widetilde{\mathcal{K}} = D_0 + \mathcal{A} + \mathcal{B}, \quad \mathrm{dom}(\widetilde{\mathcal{K}}) = \mathrm{dom}(D_0) \cap \mathrm{dom}(\mathcal{A}) \cap \mathrm{dom}(\mathcal{B}). \tag{3.1}$$

In [7], the following theorem is proved.

Theorem 3.3 ([7, Theorems 4.3 and 4.4]). *Assume (A1) and (A2). Then, the operator closure $\overline{\widetilde{\mathcal{K}}} =: \mathcal{K}$ is a generator on $L^p(\mathcal{I},X)$, and it holds:*

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{B}, \quad \mathrm{dom}(\mathcal{K}) = \mathrm{dom}(\mathcal{K}_0) \cap \mathrm{dom}(\mathcal{B}). \tag{3.2}$$

Moreover, it is an evolution generator, i.e. there is a unique propagator or solution operator $\{U(t,s)\}_{(t,s)\in\Delta}$ such that the representation:

$$\left(e^{-\tau\mathcal{K}}f\right)(t) = (\mathcal{U}(\tau)f)(t) = U(t,t-\tau)\chi_{\mathcal{I}}(t-\tau)f(t-\tau), \quad \tau \geq 0, \quad t \in \mathcal{I}.$$

holds.

We note that for the proof it is not necessary that the operators $B(t)$ themselves are generators. After proving the existence of the unique solution operator the goal is to approximate the solution operator $\{U(t,s)\}_{(t,s)\in\Delta}$. This will be done by proving an operator-norm convergence for the Trotter product formula for $\mathcal{K} = \mathcal{K}_0 + \mathcal{B}$.

4. Stability

Proving the Trotter product formula, it is important to establish stability conditions. Notice that stability is satisfied if the contractivity of the involved semigroups is assumed which might be too strong in applications. There are many stability conditions known for evolution equations. In particular, the Kato-stability is of interest, cf. [10, Definition 4.1], which is equivalent to a renormalizability conditions of the underlying Banach space, cf. [10]. We note that our following stability condition is weaker than Kato-stability.

Definition 4.1. Let A be a generator and let $\{B(t)\}_{t\in\mathcal{I}}$ be a family of generators in X. The family $\{B(t)\}_{t\in\mathcal{I}}$ is called A-stable if there is a constant $M > 0$ such that:

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta} \left\| \prod_{j=1}^{n\leftarrow} G_j(t,s;n) \right\|_{\mathcal{B}(X)} \le M \ ,$$

holds for any $n \in \mathbb{N}$ where $G_j(t,s;n) := e^{-\frac{t-s}{n}B(s+j\frac{t-s}{n})}e^{-\frac{t-s}{n}A}$, $j = 0,1,2,\ldots,n$, and the product is ordered increasingly in j from the right to the left.

Let us introduce the notion:

$$T(\tau) = e^{-\tau\mathcal{B}}e^{-\tau\mathcal{K}_0}, \quad \tau \ge 0.$$

Lemma 4.2 ([7, Lemma 5.8]). *If the operator family $\{B(t)\}_{t\in\mathcal{I}}$ is A-stable, then:*

$$\|T(\tau/n)^m\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le M,$$

for any $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $\tau \ge 0$. In particular, we have:

$$\|T(\tau)^m\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le M,$$

for any $m \in \mathbb{N}$ and $\tau \ge 0$.

5. Convergence in the operator-norm topology

Theorem 3.3 leads to the problem, how the semigroup of $\mathcal{K}$ can be approximated in terms of the semigroups generated by D_0, $\mathcal{A}$ and $\mathcal{B}$. The classical Trotter product formula gives an approximation in the strong topology. In this section, we establish an approximation in the operator-norm topology on $L^p(\mathcal{I},X)$. This is done in several steps. This approximation in $L^p(\mathcal{I},X)$ can be used to prove an convergence rate estimate in X for the propagators.

5.1. Technical lemmata

In this section, we state and prove all technical lemmas that we used to prove the convergence and estimate of the Trotter product formula in the operator-norm in $L^p(\mathcal{I},X)$. As above we set $T(\tau) := e^{-\tau\mathcal{B}}e^{-\tau\mathcal{K}_0}, \quad \tau \ge 0$. Note that $T(\tau) = 0$ for $\tau \ge T$. Similarly, $e^{-\tau\mathcal{K}} = 0$ for $\tau \ge T$.

Lemma 5.1. *Let the assumptions (A1) and (A2) be satisfied.*
(i) *Then,* $\operatorname{dom}(\mathcal{K}_0) \subset \operatorname{dom}(\mathcal{A}^\alpha)$ *and there is a constant $\Lambda_\alpha > 0$ such that:*

$$\|\mathcal{A}^\alpha e^{-\tau\mathcal{K}}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le \frac{\Lambda_\alpha}{\tau^\alpha}, \tag{5.1}$$

holds for $\tau > 0$.
(ii) *If $\{B(t)\}_{t\in\mathcal{I}}$ is A-stable, then there is a constant $\Pi_\alpha > 0$ such that the estimates:*

$$\|(T(\tau) - e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le \Pi_\alpha\tau, \tag{5.2}$$

$$\|\mathcal{A}^{-\alpha}(T(\tau) - e^{-\tau\mathcal{K}})\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le \Pi_\alpha\tau, \tag{5.3}$$

are valid for $\tau > 0$.
(iii) *If $\{B(t)\}_{t\in\mathcal{I}}$ is A-stable, then there is a constant $Y_\alpha > 0$ such that the estimate:*

$$\|\overline{T(\tau)^k\mathcal{A}^\alpha}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le Y_\alpha\left(\tau^{1-2\alpha} + \frac{1}{(k\tau)^\alpha}\right), \quad \tau > 0, \quad k \in \mathbb{N}, \tag{5.4}$$

holds for $\tau > 0$.

Proof. (i)–(ii) The assertions $\mathrm{dom}(\mathcal{K}_0) \subseteq \mathrm{dom}(\mathcal{A}^\alpha)$ as well as (5.1) and (5.2) follow from Lemma 7.3, Lemma 7.4 and Lemma 7.6 of [7]. To prove (5.3) one has slightly to modify the second part of the proof of Lemma 7.6 of [7].

(iii) For $k\tau \geq T$ we have $T(\tau)^k = 0$. Hence, one has to prove the estimate (5.4) only for $k\tau \leq T$. In fact, using Lemma 4.2, we get $\|T(\tau)^k\| \leq M$, $\tau \in [0, \infty)$. Hence:

$$\begin{aligned}\|T(\tau)^k \mathcal{A}^\alpha f\| &\leq \|(T(\tau)^k - e^{-k\tau\mathcal{K}_0})\mathcal{A}^\alpha f\| + \|e^{-k\tau\mathcal{K}_0}\mathcal{A}^\alpha f\| \\ &\leq \|\sum_{j=0}^{k-1} T(\tau)^{k-1-j}(e^{-\tau\mathcal{B}} - I)e^{-(j+1)\tau\mathcal{K}_0}\mathcal{A}^\alpha f\| + \|e^{-k\tau\mathcal{K}_0}\mathcal{A}^\alpha f\| \\ &\leq M\sum_{j=0}^{k-1}\int_0^\tau d\sigma \|e^{-\sigma\mathcal{B}}\mathcal{B}\mathcal{A}^{-\alpha}\| \, \|\mathcal{A}^{2\alpha}e^{-(j+1)\tau\mathcal{K}_0}f\| + \|e^{-k\tau\mathcal{K}_0}\mathcal{A}^\alpha f\|,\end{aligned}$$

where we have used $I - e^{-\tau\mathcal{B}} = \int_0^\tau \mathcal{B}e^{-\sigma\mathcal{B}}d\sigma$. Moreover, from (2.1) we get:

$$\|\mathcal{A}^{2\alpha}e^{-(j+1)\tau\mathcal{K}_0}f\| \leq \frac{M^A_{2\alpha}}{((j+1)\tau)^{2\alpha}}\|f\| \quad \text{and} \quad \|\mathcal{A}^\alpha e^{-k\tau\mathcal{K}_0}f\| \leq \frac{M^A_\alpha}{(k\tau)^\alpha}\|f\|,$$

for $\tau > 0$. Hence, using $\alpha > \frac{1}{2}$, we get:

$$\begin{aligned}\|T(\tau)^k\mathcal{A}^\alpha f\| &\leq \frac{MM^T_{\mathcal{B}}M^A_{2\alpha}C_\alpha\tau}{\tau^{2\alpha}}\sum_{j=0}^{k-1}\frac{1}{(j+1)^{2\alpha}}\|f\| + \frac{M^A_\alpha}{(k\tau)^\alpha}\|f\| \\ &\leq \frac{MM^T_{\mathcal{B}}M^A_{2\alpha}C_\alpha\zeta(2\alpha)}{\tau^{2\alpha-1}}\|f\| + \frac{M^A_\alpha}{(k\tau)^\alpha}\|f\|,\end{aligned}$$

for $\tau \in \mathcal{I}$, where $\zeta(\beta) := \sum_{j=1}^\infty 1/j^\beta$, $\beta > 1$, is the Riemann ζ-function and we have set $M^T_{\mathcal{B}} := \sup_{\tau\in\mathcal{I}}\|e^{-\tau\mathcal{B}}\|$. Using that $T(\tau)^k = 0$ for $\tau k \geq T$ we find:

$$\|T(\tau)^k\mathcal{A}^\alpha f\| \leq \frac{MM^T_{\mathcal{B}}M^A_{2\alpha}C_\alpha\zeta(2\alpha)}{\tau^{2\alpha-1}}\|f\| + \frac{M^A_\alpha}{(k\tau)^\alpha}\|f\|, \quad f \in \mathrm{dom}(\mathcal{A}),$$

for $\tau > 0$. Taking the supremum over the unit ball in $\mathrm{dom}(\mathcal{A})$, we prove (5.4). □

Lemma 5.2. *Let the assumptions (A1), (A2), and (A3) be satisfied. Then, there is a constant $Z_\alpha > 0$ such that:*

$$\|\mathcal{A}^{-\alpha}(T(\tau) - e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \leq Z_\alpha\tau^{1+\alpha}, \quad \tau \geq 0. \tag{5.5}$$

Proof. Let $f \in \mathrm{dom}(\mathcal{K}_0) = \mathrm{dom}(\mathcal{K})$. We have:

$$\begin{aligned}\frac{d}{d\sigma}T(\sigma)e^{-(\tau-\sigma)\mathcal{K}}f =& \frac{d}{d\sigma}e^{-\sigma\mathcal{B}}e^{-\sigma\mathcal{K}_0}e^{-(\tau-\sigma)\mathcal{K}}f \\ =& - e^{-\sigma\mathcal{B}}\mathcal{B}e^{-\sigma\mathcal{K}_0}e^{-(\tau-\sigma)\mathcal{K}}f - e^{-\sigma\mathcal{B}}e^{-\sigma\mathcal{K}_0}\mathcal{K}_0e^{-(\tau-\sigma)\mathcal{K}}f + e^{-\sigma\mathcal{B}}e^{-\sigma\mathcal{K}_0}Ke^{-(\tau-\sigma)\mathcal{K}}f \\ =& - e^{-\sigma\mathcal{B}}\mathcal{B}e^{-\sigma\mathcal{K}_0}e^{-(\tau-\sigma)\mathcal{K}}f + e^{-\sigma\mathcal{B}}e^{-\sigma\mathcal{K}_0}\mathcal{B}e^{-(\tau-\sigma)\mathcal{K}}f \\ =& e^{-\sigma\mathcal{B}}\{e^{-\sigma\mathcal{K}_0}\mathcal{B}f - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}e^{-(\tau-\sigma)\mathcal{K}}f,\end{aligned}$$

which yields:

$$T(\tau)f - e^{-\tau\mathcal{K}}f = \int_0^\tau \frac{d}{d\sigma}T(\sigma)e^{-(\tau-\sigma)\mathcal{K}}f d\sigma = \int_0^\tau d\sigma\; e^{-\sigma\mathcal{B}}\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}e^{-(\tau-\sigma)\mathcal{K}}f\;. \tag{5.6}$$

Now, we have the following identity:

$$\begin{aligned}&e^{-\sigma\mathcal{B}}\left(e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right)e^{-(\tau-\sigma)\mathcal{K}}f \\ &= (e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})f \\ &+ (e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}e^{-(\tau-\sigma)\mathcal{K}_0}f \\ &+ \{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})f + \{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}e^{-(\tau-\sigma)\mathcal{K}_0}f,\end{aligned}$$

which yields for $f = \mathcal{A}^{-\alpha} g$:

$$\begin{aligned}
&\mathcal{A}^{-\alpha} e^{-\sigma\mathcal{B}} \left(e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right) e^{-(\tau-\sigma)\mathcal{K}} \mathcal{A}^{-\alpha} g \\
&\quad = \mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha} g \\
&\quad + \mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}\mathcal{A}^{-\alpha} e^{-(\tau-\sigma)\mathcal{K}_0} g \\
&\quad + \mathcal{A}^{-\alpha}\big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha} g \\
&\quad + \mathcal{A}^{-\alpha}\big\{(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0})\mathcal{B} - \mathcal{B}(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0})\big\}e^{-(\tau-\sigma)\mathcal{K}_0}\mathcal{A}^{-\alpha} g \\
&\quad + \mathcal{A}^{-\alpha}(e^{-\sigma D_0}\mathcal{B} - \mathcal{B}e^{-\sigma D_0})\mathcal{A}^{-\alpha} e^{-(\tau-\sigma)\mathcal{K}_0} g.
\end{aligned} \tag{5.7}$$

In the following, we estimate the five terms separately.

Initially, we use the fact that $\mathcal{A}$ and $\mathcal{K}_0$ commute and conclude that:

$$(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha} g = \int_0^{\tau-\sigma} dr\, e^{-(\tau-\sigma-r)\mathcal{K}}\mathcal{B}\mathcal{A}^{-\alpha} e^{-r\mathcal{K}_0} g.$$

Thus, for the first term we get:

$$\begin{aligned}
&\mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha} g \\
&\quad = -\int_0^{\sigma} d\tau\, \mathcal{A}^{-\alpha}\mathcal{B}e^{-r\mathcal{B}}[e^{-\sigma\mathcal{K}_0}, \mathcal{B}]\mathcal{A}^{-\alpha} \int_0^{\tau-\sigma} dr\, \mathcal{A}^{\alpha} e^{-(\tau-\sigma-r)\mathcal{K}}\mathcal{B}\mathcal{A}^{-\alpha} e^{-r\mathcal{K}_0} g,
\end{aligned}$$

where:

$$[e^{-\sigma\mathcal{K}_0}, \mathcal{B}]f := \big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}f, \quad f \in \mathrm{dom}(\mathcal{K}_0), \quad \tau \geq 0\ .$$

Using Lemma 5.1, we obtain the estimate:

$$\begin{aligned}
&\|\mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha} g\| \\
&\leq \sigma\, 2C_\alpha^* C_\alpha^2 \Lambda_\alpha M_\mathcal{B}^T M_\mathcal{A}^2 \int_0^{\tau-\sigma} dr\, \frac{1}{(\tau-\sigma-r)^\alpha}\, \|g\| \leq \sigma(\tau-\sigma)^{1-\alpha}\, \frac{2C_\alpha^* C_\alpha^2 \Lambda_\alpha M_\mathcal{B}^T M_\mathcal{A}^2}{1-\alpha}\, \|g\|,
\end{aligned} \tag{5.8}$$

for $\sigma \in [0, \tau]$ and $\tau \geq 0$. For the second term, we get the estimate:

$$\|\mathcal{A}^{-\alpha}(e^{-\sigma\mathcal{B}} - I)\big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}\mathcal{A}^{-\alpha} e^{-(\tau-\sigma)\mathcal{K}_0} g\| \leq \sigma\, 2C_\alpha^* C_\alpha M_\mathcal{B}^T M_\mathcal{A}^2 \|g\|\ . \tag{5.9}$$

for $\sigma \in [0, \tau]$ and $\tau \geq 0$. Since we have:

$$e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0} h = \int_0^{\tau-\sigma} dr\, e^{-(\tau-r-\sigma)\mathcal{K}}\mathcal{B}e^{-r\mathcal{K}_0} h\ , \quad h \in \mathrm{dom}(\mathcal{K}_0),$$

one obtains for the third term the estimate:

$$\|\mathcal{A}^{-\alpha}\big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\big\}(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0})\mathcal{A}^{-\alpha} g\| \leq (\tau-\sigma)\, 2C_\alpha^* C_\alpha M_\mathcal{A}^2 M_\mathcal{K}\, \|g\|\ , \tag{5.10}$$

for $\sigma \in [0, \tau]$ and $\tau \geq 0$. Moreover, using:

$$e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0} h = -\int_0^{\sigma} dr\, e^{-r\mathcal{K}_0}\mathcal{A}e^{-(\sigma-r)D_0} h\ ,$$

we get for the fourth term:

$$\begin{aligned}
&\mathcal{A}^{-\alpha}\Big\{(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0})\mathcal{B} - \mathcal{B}(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0})\Big\}e^{-(\tau-\sigma)\mathcal{K}_0}\mathcal{A}^{-\alpha} g \\
&= \left(-\int_0^{\sigma} dr\, \mathcal{A}^{1-\alpha} e^{-r\mathcal{K}_0} e^{-(\sigma-r)D_0}\mathcal{B}\mathcal{A}^{-\alpha} + \mathcal{A}^{-\alpha}\mathcal{B}\int_0^{\sigma} dr\, e^{-r\mathcal{K}_0}\mathcal{A}^{1-\alpha} e^{-(\sigma-r)D_0}\right) e^{-(\tau-\sigma)\mathcal{K}_0} g,
\end{aligned}$$

which yields the estimate:

$$\begin{aligned}
&\|\mathcal{A}^{-\alpha}\Big\{(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0})\mathcal{B} - \mathcal{B}(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0})\Big\}e^{-(\tau-\sigma)\mathcal{K}_0}\mathcal{A}^{-\alpha}g\| \\
&\le C_\alpha M_\mathcal{A} M^A_{1-\alpha}\int_0^\sigma dr\,\frac{1}{r^{1-\alpha}}\,\|g\| + C^*_\alpha M_\mathcal{A}\, M^A_{1-\alpha}\int_0^\sigma dr\,\frac{1}{r^{1-\alpha}}\,\|g\| = \frac{(C_\alpha + C^*_\alpha)\, M_\mathcal{A} M^A_{1-\alpha}}{\alpha}\sigma^\alpha\|g\|
\end{aligned} \tag{5.11}$$

for $\sigma \in [0, \tau]$ and $\tau \ge 0$. To estimate the fifth term, we note that:

$$\begin{aligned}
(e^{-\sigma D_0}\mathcal{B} - \mathcal{B}e^{-\sigma D_0})f &= e^{-\sigma D_0}B(\cdot)f(\cdot) - \mathcal{B}\chi_\mathcal{I}(\cdot - \sigma)f(\cdot - \sigma) \\
&= \chi_\mathcal{I}(\cdot - \sigma)B(\cdot - \sigma)f(\cdot - \sigma) - B(\cdot)\chi_\mathcal{I}(\cdot - \sigma)f(\cdot - \sigma) \\
&= \chi_\mathcal{I}(\cdot - \sigma)\{B(\cdot - \sigma) - B(\cdot)\}f(\cdot - \sigma),
\end{aligned}$$

and therefore:

$$\begin{aligned}
\|\mathcal{A}^{-\alpha}(e^{-\sigma D_0}\mathcal{B} - \mathcal{B}e^{-\sigma D_0})e^{-(\tau-\sigma)\mathcal{K}_0}\mathcal{A}^{-\alpha}g\| &\le M_\mathcal{A}\|\mathcal{A}^{-\alpha}\{e^{-\sigma D_0}\mathcal{B} - \mathcal{B}e^{-\sigma D_0}\}\mathcal{A}^{-\alpha}g\| \\
&\le \operatorname*{ess\,sup}_{t\in\mathcal{I}} \|A^{-\alpha}\{B(t-\sigma) - B(t)\}A^{-\alpha}\|_{\mathcal{B}(X)}\,\|g\| \le L\sigma\|g\|,
\end{aligned} \tag{5.12}$$

for $\sigma \in [0, \tau]$ and $\tau \ge 0$. From (5.7) we find the estimate:

$$\begin{aligned}
&\|\mathcal{A}^{-\alpha}e^{-\sigma\mathcal{B}}\left(e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right)e^{-(\tau-\sigma)\mathcal{K}}\mathcal{A}^{-\alpha}g\| \\
&\quad\le \|\mathcal{A}^{-\alpha}\left(e^{-\sigma\mathcal{B}} - I\right)\left\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right\}\left(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}\right)\mathcal{A}^{-\alpha}g\| \\
&\quad+ \|\mathcal{A}^{-\alpha}\left(e^{-\sigma\mathcal{B}} - I\right)\left\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right\}\mathcal{A}^{-\alpha}e^{-(\tau-\sigma)\mathcal{K}_0}g\| \\
&\quad+ \|\mathcal{A}^{-\alpha}\Big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\Big\}\left(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}\right)\mathcal{A}^{-\alpha}g\| \\
&\quad+ \|\mathcal{A}^{-\alpha}\Big\{\left(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0}\right)\mathcal{B} - \mathcal{B}\left(e^{-\sigma\mathcal{K}_0} - e^{-\sigma D_0}\right)\Big\}e^{-(\tau-\sigma)\mathcal{K}_0}\mathcal{A}^{-\alpha}g\| \\
&\quad+ \|\mathcal{A}^{-\alpha}\left(e^{-\sigma D_0}\mathcal{B} - \mathcal{B}e^{-\sigma D_0}\right)\mathcal{A}^{-\alpha}e^{-(\tau-\sigma)\mathcal{K}_0}g\|,
\end{aligned}$$

for $\sigma \in [0, \tau]$ and $\tau \ge 0$. Taking into account (5.8), (5.9), (5.10), (5.11), and (5.12) we find:

$$\begin{aligned}
&\|\mathcal{A}^{-\alpha}e^{-\sigma\mathcal{B}}\left(e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right)e^{-(\tau-\sigma)\mathcal{K}}\mathcal{A}^{-\alpha}g\| \\
&\quad\le \Big\{\sigma(\tau-\sigma)^{1-\alpha}\frac{2C^*_\alpha C^2_\alpha\Lambda_\alpha M^T_\mathcal{B} M^2_\mathcal{A}}{1-\alpha} + \sigma\, 2C^*_\alpha C_\alpha M^T_\mathcal{B} M^2_\mathcal{A} \\
&\qquad + (\tau-\sigma)\, 2C^*_\alpha C_\alpha M^2_\mathcal{A} M_\mathcal{K} + \sigma^\alpha\,\frac{(C_\alpha + C^*_\alpha)\, M_\mathcal{A} M^A_{1-\alpha}}{\alpha} + \sigma\, L\Big\}\|g\|\ ,
\end{aligned}$$

for $\sigma \in [0, \tau]$ and $\tau \ge 0$. Setting:

$$Z_1 := \frac{2C^*_\alpha C^2_\alpha\Lambda_\alpha M^T_\mathcal{B} M^2_\mathcal{A}}{1-\alpha}, \qquad Z_2 := 2C^*_\alpha C_\alpha M^T_\mathcal{B} M^2_\mathcal{A} + L,$$

$$Z_3 := 2C^*_\alpha C_\alpha M^2_\mathcal{A} M_\mathcal{K}, \qquad Z_4 := \frac{(C_\alpha + C^*_\alpha)\, M_\mathcal{A} M^A_{1-\alpha}}{\alpha},$$

we obtain:

$$\begin{aligned}
&\|\mathcal{A}^{-\alpha}e^{-\sigma\mathcal{B}}\left(e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\right)e^{-(\tau-\sigma)\mathcal{K}}\mathcal{A}^{-\alpha}g\| \\
&\quad\le \Big\{Z_1\,\sigma(\tau-\sigma)^{1-\alpha} + Z_2\,\sigma + Z_3\,(\tau-\sigma) + Z_4\,\sigma^\alpha\Big\}\|g\|\ .
\end{aligned} \tag{5.13}$$

From (5.6) we derive the representation:

$$\mathcal{A}^{-\alpha}\left(T(\tau) - e^{-\tau\mathcal{K}}\right)\mathcal{A}^{-\alpha}g = \int_0^\tau d\sigma\,\mathcal{A}^{-\alpha}e^{-\sigma\mathcal{B}}\Big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\Big\}e^{-(\tau-\sigma)\mathcal{K}}\mathcal{A}^{-\alpha}g,$$

which yields the estimate:

$$\|\mathcal{A}^{-\alpha}(T(\tau) - e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}g\| \le \int_0^\tau d\sigma\,\|\mathcal{A}^{-\alpha}e^{-\sigma\mathcal{B}}\Big\{e^{-\sigma\mathcal{K}_0}\mathcal{B} - \mathcal{B}e^{-\sigma\mathcal{K}_0}\Big\}e^{-(\tau-\sigma)\mathcal{K}}\mathcal{A}^{-\alpha}g\|.$$

Inserting (5.13) into this estimate and using:

$$\int_0^\tau d\sigma\sigma\,(\tau-\sigma)^{1-\alpha} = \tau^{3-\alpha}\int_0^1 dx\, x(1-x)^{1-\alpha} = \tau^{3-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)},$$

we find the estimate:

$$\|\mathcal{A}^{-\alpha}(T(\tau)-e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}g\| \le Z_1\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}\,\tau^{3-\alpha} + \frac{Z_2+Z_3}{2}\,\tau^2 + \frac{Z_4}{1+\alpha}\,\tau^{1+\alpha},$$

for $\tau \ge 0$. We have:

$$\|\mathcal{A}^{-\alpha}(T(\tau)-e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}g\| \le \left(Z_1\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}\tau^{2-2\alpha} + \frac{Z_2+Z_3}{2}\tau^{1-\alpha} + Z_4\right)\,\tau^{1+\alpha},$$

for $\tau \ge 0$. Since $T(\tau)=0$ and $e^{-\tau\mathcal{K}}=0$ for $\tau \ge T$ we finally obtain:

$$\|\mathcal{A}^{-\alpha}(T(\tau)-e^{-\tau\mathcal{K}})\mathcal{A}^{-\alpha}g\| \le \left(Z_1\frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}T^{2-2\alpha} + \frac{Z_2+Z_3}{2}T^{1-\alpha} + Z_4\right)\,\tau^{1+\alpha},$$

which proves the lemma. □

Lemma 5.3. *Let $\alpha \in [0,1)$. Then the estimates:*

$$\sum_{m=1}^{n-1}\frac{1}{m^\alpha} \le \frac{n^{1-\alpha}}{1-\alpha} \quad \textit{and} \quad \sum_{m=1}^{n-1}\frac{1}{(n-m)^\alpha m^\alpha} \le \frac{2}{1-\alpha}n^{1-2\alpha}\ , \tag{5.14}$$

are valid for $n = 2,3,\ldots$.

Proof. The function $f(x)=x^{-\alpha}$, $x>0$, is decreasing. Hence:

$$\sum_{m=1}^{n-1}\frac{1}{m^\alpha} \le \int_0^{n-1} dx\,\frac{1}{x^\alpha} \le \frac{(n-1)^{1-\alpha}}{1-\alpha} \le \frac{n^{1-\alpha}}{1-\alpha},$$

for $n = 2,3,\ldots$. Further, we have:

$$\sum_{m=1}^{n-1}\frac{1}{(n-m)^\alpha m^\alpha} \le 2\frac{1}{n^\alpha}\sum_{m=1}^{n-1}\frac{1}{m^\alpha} \le 2\frac{1}{n^\alpha}\frac{n^{1-\alpha}}{1-\alpha} = \frac{2}{1-\alpha}n^{1-2\alpha},$$

and the claim follows. □

5.2. The Trotter product formula in operator-norm topology

Now, we are able to prove and to estimate the rate of operator-norm convergence of the Trotter product approximation.

Theorem 5.4. *Let the assumptions (A1), (A2), and (A3) be satisfied. If the family of generators $\{B(t)\}_{t\in\mathcal{I}}$ is A-stable, then there exists a (depending on $\alpha \in (1/2,\ 1)$ and on the compact interval $\mathcal{I}$) constant $C_{\alpha,\mathcal{I}} > 0$ such that:*

$$\|\left(e^{-\tau\mathcal{B}/n}e^{-\tau\mathcal{K}_0/n}\right)^n - e^{-\tau\mathcal{K}}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \le \frac{C_{\alpha,\mathcal{I}}}{n^{1-\alpha}}, \tag{5.15}$$

for $\tau \ge 0$ and $n = 2,3,\ldots$.

Proof. Let $T(\sigma) := e^{-\sigma\mathcal{B}}e^{-\sigma\mathcal{K}_0}$ and $U(\sigma) := e^{-\sigma\mathcal{K}}$, $\sigma \ge 0$. Then the following identity holds:

$$\begin{aligned}
T(\sigma)^n - U(\sigma)^n &= \sum_{m=0}^{n-1}T(\sigma)^{n-m-1}(T(\sigma)-U(\sigma))U(\sigma)^m \\
&= T(\sigma)^{n-1}(T(\sigma)-U(\sigma)) + (T(\sigma)-U(\sigma))U(\sigma)^{n-1} + \sum_{m=1}^{n-2}T(\sigma)^{n-m-1}(T(\sigma)-U(\sigma))U(\sigma)^m \\
&= T(\sigma)^{n-1}\mathcal{A}^\alpha\mathcal{A}^{-\alpha}(T(\sigma)-U(\sigma)) + (T(\sigma)-U(\sigma))\mathcal{A}^{-\alpha}\mathcal{A}^\alpha U(\sigma)^{n-1} \\
&\quad + \sum_{m=1}^{n-2}T(\sigma)^{n-m-1}\mathcal{A}^\alpha\mathcal{A}^{-\alpha}(T(\sigma)-U(\sigma))\mathcal{A}^{-\alpha}\mathcal{A}^\alpha U(\sigma)^m,
\end{aligned}$$

which yields the estimate:

$$\begin{aligned}&\|T(\sigma)^n - U(\sigma)^n\|\\ &\leq \|\overline{T(\sigma)^{n-1}\mathcal{A}^\alpha}\| \, \|A^{-\alpha}(T(\sigma) - U(\sigma))\| + \|(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \, \|\mathcal{A}^\alpha U(\sigma)^{n-1}\|\\ &\quad + \sum_{m=1}^{n-2} \|\overline{T(\sigma)^{n-m-1}\mathcal{A}^\alpha}\| \, \|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \, \|\mathcal{A}^\alpha U(\sigma)^m\|.\end{aligned}$$

From Lemma 5.1 we get the estimates:

$$\|\overline{T(\sigma)^{n-1}\mathcal{A}^\alpha}\| \leq Y_\alpha \left(\sigma^{1-2\alpha} + \frac{1}{((n-1)\sigma)^\alpha}\right), \quad n \geq 2,$$

as well as:

$$\|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\| \leq \Pi_\alpha \sigma \quad \text{and} \quad \|(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \leq \Pi_\alpha \, \sigma,$$

for $\sigma \in (0, \tau]$. Hence:

$$\|\overline{T(\sigma)^{n-1}\mathcal{A}^\alpha}\| \, \|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\| \leq \Pi_\alpha Y_\alpha \sigma^{1-\alpha} \left(\sigma^{1-\alpha} + \frac{1}{(n-1)^\alpha}\right),$$

and:

$$\|(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \, \|\mathcal{A}^\alpha U(\sigma)^{n-1}\| \leq \frac{\Pi_\alpha \Lambda_\alpha}{(n-1)^\alpha} \, \sigma^{1-\alpha},$$

where we have used (5.1). Since:

$$\|\mathcal{A}^{-\alpha}(T(\sigma) - e^{-\sigma\mathcal{K}})\mathcal{A}^{-\alpha}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \leq Z_\alpha \, \sigma^{1+\alpha}, \quad \tau \in [0, \tau_0),$$

by Lemma 5.2 we obtain:

$$\begin{aligned}&\|\overline{T(\sigma)^{n-m-1}\mathcal{A}^\alpha}\| \, \|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \, \|\mathcal{A}^\alpha U(\sigma)^m\|\\ &\leq Y_\alpha \left(\sigma^{1-2\alpha} + \frac{1}{((n-m-1)\sigma)^\alpha}\right) Z_\alpha \sigma^{1+\alpha} \, \Lambda_\alpha \frac{1}{(\sigma m)^\alpha}\\ &\leq Y_\alpha Z_\alpha \Lambda_\alpha \left(\sigma^{2-2\alpha} \frac{1}{m^\alpha} + \sigma^{1-\alpha} \frac{1}{(n-m-1)^\alpha m^\alpha}\right).\end{aligned}$$

Now, using Lemma 5.3 we get:

$$\begin{aligned}&\sum_{m=1}^{n-2} \|\overline{T(\sigma)^{n-m-1}\mathcal{A}^\alpha}\| \, \|\mathcal{A}^{-\alpha}(T(\sigma) - U(\sigma))\mathcal{A}^{-\alpha}\| \, \|\mathcal{A}^\alpha U(\sigma)^m\|\\ &\quad \leq Z_\alpha \Lambda_\alpha Y_\alpha \sigma^{2-2\alpha} \sum_{m=1}^{n-2} \frac{1}{m^\alpha} + Z_\alpha \Lambda_\alpha Y_\alpha \sigma^{1-\alpha} \sum_{m=1}^{n-2} \frac{1}{(n-m-1)^\alpha m^\alpha}\\ &\quad \leq \frac{Z_\alpha \Lambda_\alpha Y_\alpha}{1-\alpha} \left(n^{1-\alpha}\sigma^{2-2\alpha} + 2n^{1-2\alpha}\sigma^{1-\alpha}\right).\end{aligned}$$

Summing up, we find that:

$$\begin{aligned}\|T(\sigma)^n - U(\sigma)^n\| \leq &\Pi_\alpha \, Y_\alpha \sigma^{1-\alpha} \left(\sigma^{1-\alpha} + \frac{1}{(n-1)^\alpha}\right) + \frac{\Pi_\alpha \Lambda_\alpha}{(n-1)^\alpha} \, \sigma^{1-\alpha} +\\ &\frac{Z_\alpha \Lambda_\alpha Y_\alpha}{1-\alpha} n^{1-\alpha}\sigma^{2-2\alpha} + \frac{2Z_\alpha \Lambda_\alpha Y_\alpha}{1-\alpha} n^{1-2\alpha}\sigma^{1-\alpha}.\end{aligned}$$

Note that setting $\sigma := \tau/n$, one obtains:

$$\begin{aligned}&\|T(\tau/n)^n - U(\tau/n)^n\|\\ &\leq \frac{\Pi_\alpha \, \Lambda_\alpha \, T^{2-2\alpha}}{(n-1)^{2-2\alpha}} + \frac{\Pi_\alpha \, \Lambda_\alpha}{n-1} + \frac{\Pi_\alpha \, \Lambda_\alpha \, T^{1-\alpha}}{(n-1)} + \frac{Z_\alpha \Lambda_\alpha Y_\alpha T^{2-2\alpha}}{1-\alpha} \frac{1}{n^{1-\alpha}} + \frac{2Z_\alpha \Lambda_\alpha Y_\alpha T^{1-\alpha}}{1-\alpha} \frac{1}{n^\alpha},\end{aligned}$$

for $\tau \geq 0$ and $n = 2, 3, \ldots$. Hence, there exists a constant $C_{\alpha,\mathcal{I}} > 0$ such that: (5.15) holds. □

Remark 5.5. It is worth noting that this result depends only on the domains of the operators A and $B(t)$ and not on their concrete realization.

5.3. Operator-norm convergence of propagators

Theorem 5.4 allows to estimate the rate of approximation by the product formula of the solution operator (propagator) $\{U(t,s)\}_{(t,s)\in\Delta}$.

To this aim we note that due to Theorem 3.3 we have the identity:

$$\left(\left\{ \left(e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0}\right)^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)}\right\}g\right)(t) = \left\{U_n(t,t-\tau) - U(t,t-\tau)\right\}\chi_{\mathcal{I}}(t-\tau)g(s-\tau),$$

for $(t,t-\tau)\in\Delta$ and $g\in L^p(\mathcal{I},X)$, where:

$$U_n(t,s) := \prod_{j=1}^{n\leftarrow} G_j(t,s;n), \quad (t,s)\in\Delta,$$

where $G_j(t,s;n) := e^{-\frac{t-s}{n}B\left(s+j\frac{t-s}{n}\right)}e^{-\frac{t-s}{n}A}$ and the product is increasingly ordered from the right to the left. Next, we introduce on $L^p(\mathcal{I},X)$ the left-shift semigroup:

$$(L(\tau)f)(t) := \chi_{\mathcal{I}}(t+\tau)f(t+\tau), \quad f\in L^p(\mathcal{I},X).$$

Theorem 5.6. *Let the assumptions (A1), (A2), and (A3) be satisfied. If the family of generators $\{B(t)\}_{t\in\mathcal{I}}$ is A-stable, then there is a constant $C_{\alpha,\mathcal{I}}>0$:*

$$\operatorname*{ess\,sup}_{(t,s)\in\Delta}\|U_n(t,s)-U(t,s)\|_{\mathcal{B}(X)} \le \frac{C_{\alpha,\mathcal{I}}}{n^{1-\alpha}}, \quad n=2,3,\ldots, \tag{5.16}$$

where the constant $C_{\alpha,\mathcal{I}}$ coincides with that one of Theorem 5.4.

Proof. We set:

$$S_n(t,s) := U_n(t,s) - U(t,s), \quad (t,s)\in\Delta, \quad n\in\mathbb{N},$$

and:

$$\mathcal{S}_n(\tau) := L(\tau)\{(e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0})^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)}\} : L^p(\mathcal{I},X)\to L^p(\mathcal{I},X),$$

for $\tau\ge 0$ and $n=2,3,\ldots$. Then one gets:

$$(\mathcal{S}_n(\tau)g)(t) = S_n(t+\tau,t)\chi_{\mathcal{I}}(t+\tau)g(t), \quad t\in\mathcal{I}_0, \quad g\in L^p(\mathcal{I},X).$$

Hence, for any $\tau\in\mathcal{I}$ and $n\in\mathbb{N}$, the operator $\mathcal{S}_n(\tau)$ is a multiplication operator on $L^p(\mathcal{I},X)$ induced by the family $\{S_n(\cdot+\tau,\cdot)\chi_{\mathcal{I}}(\cdot+\tau)\}_{\tau\in\mathcal{I}}$ of bounded operators. Applying equation (7.27) of [7], we conclude that for $\tau\ge 0$ one has the identity:

$$\begin{aligned}\|\left(e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0}\right)^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)}\|_{\mathcal{B}(L^p(\mathcal{I},X))} &= \|L(\tau)\{\left(e^{-\frac{\tau}{n}\mathcal{B}}e^{-\frac{\tau}{n}\mathcal{K}_0}\right)^n - e^{-\tau(\mathcal{B}+\mathcal{K}_0)}\}\|_{\mathcal{B}(L^p(\mathcal{I},X))} \\ = \|\mathcal{S}_n(\tau)\|_{\mathcal{B}(L^p(\mathcal{I},X))} &= \operatorname*{ess\,sup}_{t\in\mathcal{I}_0}\|S_n(t+\tau,t)\chi_{\mathcal{I}}(t+\tau)\|_{\mathcal{B}(X)} \\ &= \operatorname*{ess\,sup}_{t\in\mathcal{I}_0}\|\{U_n(t+\tau,t)-U(t+\tau,t)\}\chi_{\mathcal{I}}(t+\tau)\|_{\mathcal{B}(X)} \\ &= \operatorname*{ess\,sup}_{t\in(0,T-\tau]}\|U_n(t+\tau,t)-U(t+\tau,t)\|_{\mathcal{B}(X)}.\end{aligned} \tag{5.17}$$

Now, taking into account Theorem 5.4, we find:

$$\operatorname*{ess\,sup}_{t\in(0,T-\tau]}\|U_n(t+\tau,t)-U(t+\tau,t)\|_{\mathcal{B}(X)} \le \frac{C_{\alpha,\mathcal{I}}}{n^{1-\alpha}}, \quad \tau\ge 0, \quad n\in 2,3,\ldots,$$

which yields (5.16). □

Remark 5.7.

(i) In the case of a Hilbert space Ichinose and Tamura proved in [6] that the convergence rate has order $O(\ln(n)/n)$ if one assumes that the operators A and $B(t)$ are positive and self-adjoint. On the other hand, the authors proved in [7] for Banach spaces that the convergence rate estimate is $O(n^{-(\beta-\alpha)})$ for any $\beta\in(\alpha,1)$, assuming $\mathrm{dom}(A^*)\subset\mathrm{dom}(B(t)^*)$. We comment here that under the same conditions for *autonomous* case ($B(t)=B$) the estimate in a Banach has the form $O(\ln(n)/n^{1-\alpha})$, $0<\alpha<1$, see [8, Theorem 3.6].

(ii) The key identity (5.17) that makes a contact between the evolution semigroup and the solution operator (propagator) approaches to non-autonomous Cauchy problems, also shows that estimates (5.15) and (5.16) are equivalent.

(iii) We note that a prior the operator family $\{U_n(t,s)\}_{(t,s)\in\Delta}$ do not define a propagator since the co-cycle equation is in general not satisfied. But one can check that:

$$U_n(t,s) = U_{n-k}\left(t, s + \frac{k}{n}(t-s)\right) U_k\left(s + \frac{k}{n}(t-s), s\right),$$

is satisfied for $0 < s \le t \le T$, $n \in \mathbb{N}$ and any $k \in \{0,1,\ldots,n\}$.

6. Example: diffusion equation perturbed by a time-dependent potential

We investigate the diffusion equation perturbed by a time-dependent potential. On the Banach space $X = L^q(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^2-boundary ($d \ge 2$) and $q \in (1,\infty)$, the equation reads:

$$\dot{u}(t) = \Delta u(t) - B(t)u(t), \quad u(s) = u_s \in L^q(\Omega), \quad t,s \in \mathcal{I}_0. \tag{6.1}$$

Δ denotes the Laplace operator on $L^q(\Omega)$ with Dirichlet boundary conditions defined on:

$$\Delta : \mathrm{dom}(\Delta) = H^2_q(\Omega) \cap \mathring{H}^1_q(\Omega) \to L^q(\Omega).$$

It turns out that $-\Delta$ is the generator of a holomorphic contraction semigroup on $L^q(\Omega)$ (cf. [11, Theorem 7.3.5/6]). $B(t)$ denotes a time-dependent scalar-valued multiplication operator given by:

$$(B(t)f)(x) = V(t,x)f(x), \quad \mathrm{dom}(B(t)) = \{f \in L^q(\Omega) : V(t,x)f(x) \in L^q(\Omega)\},$$

where:

$$V : \mathcal{I} \times \Omega \to \mathbb{C}, \quad V(t,\cdot) \in L^\varrho(\Omega).$$

For $\alpha \in (0,1)$, the fractional power of $-\Delta$ are defined on the domain:

$$(-\Delta)^\alpha : \mathring{H}^{2\alpha}_q(\Omega) \to L^q(\Omega).$$

Note, that for $2\alpha < 1/q$, it holds that $\mathring{H}^{2\alpha}_q(\Omega) = H^{2\alpha}_q(\Omega)$. The adjoint operator of $(-\Delta)^\alpha$ is defined on the domain $\mathrm{dom}(((-\Delta)^\alpha)^*) = \mathring{H}^{2\alpha}_{q'}(\Omega) \subset L^{q'}(\Omega)$, where $1/q + 1/q' = 1$. The operators $B(t)$ are scalar-valued and hence $B(t)^* = \overline{B(t)} : \mathrm{dom}(B(t)) \subset L^{q'}(\Omega) \to L^{q'}(\Omega)$. Moreover, one can show that $\mathcal{K}_0 = D_0 + \mathcal{A}$, i.e. the operator sum $D_0 + \mathcal{A}$ is already closed.

Now, we are going to verify the assumptions (A1)–(A3) in order to approximate the solution of (6.1). This means, we determine the required regularity in space and in time of the potential $V(\cdot,\cdot)$ to ensure the assumptions (A1)–(A3).

To guarantee that the operators $B(t)$ are generators, we assume that the potential $V(t,x)$ is positive, i.e.:

$$\mathrm{Re}(V(t,x)) \ge 0, \text{ for a.e. } (t,x) \in \mathcal{I} \times \Omega.$$

Then, for any $t \in \mathcal{I}$ the operator $V(t,x)$ is a generator of a contraction semigroup on $X = L^q(\Omega)$ (cf. [5, Theorem I.4.11–12]). In particular, the operator family $B(t)$ is A-stable.

For fixed $d \ge 2$ and $\alpha \in (1/2,\ 1)$, we define the following values for the parameters $\tilde{r}, \tilde{\rho}, \tau$:

	$q \in \left(1, \frac{d}{2\alpha}\right)$	$q \in \left[\frac{d}{2\alpha}, \infty\right)$
$q' \in \left(1, \frac{d}{2\alpha}\right)$	$\tilde{r} \in \left[\frac{d}{2\alpha}, \infty\right]$, $\tilde{\rho} \in \left[\frac{d}{2\alpha}, \infty\right]$, $\tau \in \left[\frac{d}{4\alpha}, \infty\right]$	$\tilde{r} \in (q,\infty]$, $\tilde{\rho} \in \left[\frac{d}{2\alpha}, \infty\right]$, $\tau \in \left[\frac{d}{2\alpha + dq}, \infty\right]$
$q' \in \left[\frac{d}{2\alpha}, \infty\right)$	$\tilde{r} \in \left[\frac{d}{2\alpha}, \infty\right]$, $\tilde{\rho} \in (q',\infty]$, $\tau \in \left[\frac{d}{2\alpha + dq'}, \infty\right]$	$\tilde{r} \in (q,\infty]$, $\tilde{\rho} \in (q',\infty]$, $\tau \in (1,\infty]$

Take $\tilde{r}$, $\tilde{\rho}$ from the table above and define r, ρ via:

$$\frac{1}{r} + \frac{1}{\tilde{r}} = \frac{1}{q}, \quad \frac{1}{\rho} + \frac{1}{\tilde{\rho}} = \frac{1}{q'}. \tag{6.2}$$

Using Sobolev embeddings of the form:

$$H^s_{\gamma_1}(\Omega) \subset L^{\gamma_2}(\Omega) \text{ for } \begin{cases} \gamma_2 \in \Big[\gamma_1, \dfrac{\frac{d}{s}\gamma_1}{\frac{d}{s}-\gamma_1}\Big], \text{ if } \gamma_1 \in \Big(1, \dfrac{d}{s}\Big); \\ \gamma_2 \in [\gamma_1, \infty), \text{ if } \gamma_1 \in \Big[\dfrac{d}{s}, \infty\Big); \end{cases} \tag{6.3}$$

it is not hard to show $H^{2\alpha}_q(\Omega) \subset L^r(\Omega)$ and $H^{2\alpha}_{q'}(\Omega) \subset L^\rho(\Omega)$ on the one hand, and $L^r(\Omega), L^\rho(\Omega) \subset \mathrm{dom}(B(t))$ on the other hand. This means, $\mathrm{dom}((-\Delta)^\alpha) \subset \mathrm{dom}(B(t))$ and $\mathrm{dom}(((-\Delta)^\alpha)^*) \subset \mathrm{dom}(B(t)^*)$. The operator $B(t)$ is a multiplication operator defined by $V(t,\cdot)$ and hence, following (6.2) the regularity of $V(t,\cdot)$ has to be at least $\varrho := \max\{\tilde r, \tilde\rho\}$. Hence, assuming $V \in L^\infty(\mathcal{I}, L^\varrho(\Omega))$, we have $\operatorname{ess\,sup}_{t\in\mathcal{I}} \|B(t)(-\Delta)^\alpha\| \le \infty$ and $\operatorname{ess\,sup}_{t\in\mathcal{I}} \|B(t)^*((-\Delta)^\alpha)^*\| \le \infty$. Hence, (A1) and (A2) are satisfied.

Moreover, let:

$$F(t) := (-\Delta)^{-\alpha} B(t) (-\Delta)^{-\alpha} : L^q(\Omega) \to \mathring{H}^{2\alpha}_q(\Omega).$$

For τ from the table above the relation:

$$\frac{1}{r} + \frac{1}{\tau} + \frac{1}{\rho} \le 1 \tag{6.4}$$

holds. One can show that each $t \in \mathcal{I}$ the operator $F(t)$ is bounded for $V(t,\cdot) \in L^\tau(\Omega)$. Indeed, let $f \in L^q(\Omega)$ and $g \in L^{q'}(\Omega)$. Define $\tilde f = \Delta^{-\alpha} f \in \mathring{H}^{2\alpha}_q(\Omega) \subset L^r(\Omega)$ and $\tilde g = (\Delta^{-\alpha})^* g = (\Delta^*)^{-\alpha} g \in \mathring{H}^{2\alpha}_{q'}(\Omega) \subset L^\rho(\Omega)$. Then, we have for $t \in \mathcal{I}$:

$$\langle F(t) f, g\rangle = \langle (-\Delta)^{-\alpha} B(t)(-\Delta)^{-\alpha} f, g\rangle = \langle (-\Delta)^{-\alpha} f, B(t)^* (-\Delta^*)^{-\alpha} g\rangle = \langle \tilde f, B(t)^* \tilde g\rangle.$$

The boundedness of $\langle \tilde f, B(t)^*\tilde g\rangle$ is satisfied for $V(t,\cdot) \in L^\tau(\Omega)$. Assuming V to be Lipschitz continuous in time, i.e. assuming $V \in C^{\mathrm{Lip}}(\mathcal{I}, L^\tau(\Omega))$, it follows that (A3) is satisfied. We remark that since we have $r \ge q$, it holds that $\tau \le \tilde\rho$ and hence,$\tau \le \varrho = \max\{\tilde r, \tilde\rho\}$.

The arguments that we collected above yield the following statement concerning our example (6.1):

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2-boundary, let $q \in (1,\infty)$ and let $\alpha \in (1/2,\ 1)$. Let $B(t) f = V(t,\cdot) f$ define a scalar valued multiplication operator on $L^q(\Omega)$ with:*

$$V \in L^\infty(\mathcal{I}, L^\varrho(\Omega)) \cap C^{\mathrm{Lip}}(\mathcal{I}, L^\tau(\Omega)), \tag{6.5}$$

where $\varrho = \max\{\tilde r, \tilde\rho\}$ and $\tilde r, \tilde\rho, \tau$ is chosen from the above table. Moreover, let $\mathrm{Re}(V(t,x)) \ge 0$ for $t \in \mathcal{I}$ and for a.e. $x \in \Omega$.

Then, the evolution problem (6.1) has a unique solution operator $U(t,s)$ which can be approximated in operator-norm by:

$$\sup_{(t,s)\in\Delta} ||U_n(t,s) - U(t,s)||_{\mathcal{B}(L^q(\Omega))} = O(n^{-(1-\alpha)}),$$

where:

$$U_n(t,s) = \overrightarrow{\prod_{j=1}^{n}} e^{-\frac{t-s}{n} V\left(\frac{n-j+1}{n} t + \frac{j-1}{n} s, \cdot\right)} e^{\frac{t-s}{n}\Delta}. \tag{6.6}$$

Proof. The claim follows, using Theorem 3.3 and Theorem 5.6. The "ess sup" becomes a "sup", since the solution operator and the approximating operator are continuous. □

Remark 6.2.

(i) In [12], the existence of a solution operator for equation (6.1) is shown assuming weaker regularity in space and time for the potential. We assumed uniform boundedness of the function $t \mapsto ||B(t)(-\Delta)^\alpha||_{\mathcal{B}(X)}$, which is indeed too strong but important for the considerations.

(ii) We focused on domains, which are bounded and have C^2-boundaries. Our considerations can be extended to other domains, too.

(iii) Although the approximating propagator $\{U_n(t,s)\}_{(t,s)\in\Delta}$ defined in (6.6) looks elaborate, it has a simple structure. The semigroup of the Laplace operator on $L^q(\mathbb{R}^d)$ is given by the Gauss–Weierstrass semigroup (see for example [5, Chapter 2.13]) defined via:

$$(e^{t\Delta} u)(x) = (T(t)u)(x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u(y)\,dy.$$

The terms $e^{-\tau V(t_j)}$ are scalar valued and can be easily computed.

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