NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2017, 8 (3), P. 310–316

Finiteness of discrete spectrum of the two-particle Schrödinger operator on diamond lattices

M. I. Muminov, C. Lokman

Department of Mathematic of Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

mukhiddin@utm.my

DOI 10.17586/2220-8054-2017-8-3-310-316

We consider a two-particle Schrödinger operator H on the $d-$dimensional diamond lattice. We find a sufficiency condition of finiteness for discrete spectrum eigenvalues of H.

Keywords: two-particle Hamiltonian on lattice, Birman–Schwinger principle, discrete spectrum.

Received: 21 March 2017

Revised: 5 May 2017

1. Introduction

The spectrum of the many particle Schrödinger operator is closely connected to the spectrum of two-particle Schrödinger operator. To obtain the two-particle Schrödinger operator (in the continuous case) from the total Hamiltonian, we can separate the energy of motion of the center of mass such that the one-particle "bound states" are eigenvectors of the energy operator with separated total momentum (in this case, such an operator is indeed independent of the total momentum values) [1]. On the lattice case, the "separation of the center of mass" of a system is associated with the realization of the Hamiltonian as a "laminated operator", i.e., as the direct integral of the family $h(k), k \in \mathbb{T}^d$ (where $\mathbb{T}$ is a one dimensional torus), of the energy operators of two particles, where k is the value of the total quasi-momentum [2].

Conditions for the finiteness of the negative spectrum and for the absence of positive eigenvalues of the two-particle continuous Schrödinger operator H were presented in [3]. The finiteness of the number of bound states for two-particle cluster operators at some values of the clustering parameter was established in [4]. The sufficient condition of finiteness of discrete spectrum of two-particle lattice Schrödinger operators was given in [5]. An example one-dimensional lattice Schrödinger operator having at the same time an infinite number of discrete and embedded eigenvalues was given in the paper [6]. The existence conditions for eigenvalues of the family $h(k)$ depending on the energy of interaction and quasi-momentum k have been investigated in [7].

The models which can be obtained investigating differential operators on graphs have already been used by physicists, a good review of such publications can be found, for example, in [8,9]. Two particle scattering theory on graphs was studied in [9]. The obtained results are applied to the qualitative description of a simple three-electrode nanoelectronic device. In [10] and [11], the problem of quantum particle storage in a nanolayered structure was considered. The authors numerically solved an eigenvalue problem of the corresponding Hamiltonian.

K. Ando et al. [12] described the Schrödinger operators on square, triangular, hexagonal, Kagome, diamond, subdivision lattices and the spectral properties of these Schrödinger operators were studied with compactly supported potentials. Conditions for the finiteness of the discrete spectrum and the non-existence of embedded eigenvalues of these Schrödinger operators with compactly supported potentials were given. The inverse scattering for discrete Schrödinger operators with compactly supported potentials on $\mathbb{Z}^d$ and on the hexagonal lattice were studied in [13,14] in part, in these papers, the discreteness of embedded eigenvalues of these operators was proved.

We consider a discrete Schrödinger operator H on the $d-$dimensional diamond lattice with any continuous potential Q, i.e. a perturbation of discrete Laplacian with compact operator. The aim of the present paper is to prove the finiteness of the discrete spectrum of H. To show this, we use the technique proposed in [15]. First, we, using the well-known Birman-Schwinger principle, we reduce the study number of discrete spectrum $N_-(z)(N_+(z))$ of H, lying to the left (right) from z to the study of the number of eigenvalues $n(1, T_\mp(z))$ of the compact operator $T_\mp(z)$, lying to the right from 1, i.e. we prove the equalities $N_\mp(z) = n(1, T_\mp(z))$. Further, we show that the operator-valued function $T_\pm(\cdot)$ is well defined at the limits of the essential spectrum and apply the Weyl inequality.

2. Statement of the Main Results

We first give descriptions of the $d-$dimensional diamond lattice and a discrete Schrödinger operator on the $d-$dimensional diamond lattice [12].

Discrete Laplacian on the graph. We denote by $G = (V(G), E(G))$ the graph that consists of a vertex set $V(G)$, whose cardinality is at most countable, and an edge set $E(G)$, each element of which connects a pair of vertices. We assume that the graph is simple, i.e. there are neither self-loop, which is an edge connecting a vertex to itself, nor multiple edges, which are two or more edges connecting the same vertices. Let $v, u \in V(G)$, and $e \in E(G)$. We denote by $v \sim u$, when v is adjacent to u by e; by N_v the set of vertices which are adjacent to v, i. e. $N_v = \{u \in V(G) : u \sim v\}$. We denote by $\deg(v) = \sharp N_v$ the degree of v. We assume that the graph G is connected, which implies that $\deg(v) > 0$ for any $v \in V(G)$.

The discrete Laplacian $\triangle_d$ on G is defined as (see [16])

$$(\triangle_d \hat{f})(v) = \frac{1}{\deg(v)} \sum_{u \in N_v} [\hat{f}(u) - \hat{f}(v)],$$

for the function $\hat{f}$ on $V(G)$. It is well known that $-\triangle_d$ is bounded, self-adjoint on

$$\ell_2(G) = \Big\{\hat{f} : \sum_{v \in v \in V(G)} |\hat{f}(v)|^2 \deg(v) < \infty\Big\}.$$

Higher-dimensional diamond lattice. Let $\mathbb{Z}^d, d \geq 2$, be a d-dimensional integer lattice, $(\mathbb{Z}^d)^2 = \mathbb{Z}^d \times \mathbb{Z}^d$ be the Cartesian power of $\mathbb{Z}^d$, and $l_2((\mathbb{Z}^d)^2)$ be the Hilbert space of square-integrable functions defined on $(\mathbb{Z}^d)^2$.

Let A_d be a subset of $\mathbb{Z}^{d+1}$ defined as follows

$$A_d = \Big\{x \in \mathbb{Z}^{d+1} : \sum_{i=1}^{d+1} x_i = 0\Big\}$$

and $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_{d+1} = (0, \ldots, 0, 1)$, $\mathbf{v}_i = \mathbf{e}_{d+1} - \mathbf{e}_i$, $i = 1, \ldots, d$. Then, A_d is a lattice (see [12]) of rank d in $\mathbb{R}^d$ with basis $\mathbf{v}_i$, $i = 1, \cdots, d$, i.e.

$$A_d = \Big\{\mathbf{v}(n) : \quad \mathbf{v}(n) = \sum_{j=1}^{d} n_j \mathbf{v}_j,\, n = (n_1, \cdots, n_d) \in \mathbb{Z}^d\Big\}.$$

The lattice A_d is called $d-$dimensional diamond lattice.

We put

$$\mathcal{V} = A_d \cup (p + A_d), \quad p = \frac{1}{d+1}(\mathbf{v}_1 + \cdots + \mathbf{v}_d).$$

The set $\mathcal{V}$ is vertex set of $d-$dimensional lattice A_d.

The set of adjacent points of $\mathbf{v}(n) \in A_d$ and $p + \mathbf{v}(n') \in P + A_d$ are defined by

$$N_{\mathbf{v}(n)} = \big\{p + \mathbf{v}(n') : n - n' = (0, \ldots, 0), (1, 0, \ldots, 0), \ldots (0, \ldots, 0, 0)\big\},$$

$$N_{p+\mathbf{v}(n')} = \big\{\mathbf{v}(n) : n - n' = (0, \ldots, 0), (1, 0, \ldots, 0), \ldots (0, \ldots, 0, 0)\big\}.$$

Hence $\deg(v) = d + 1$ for $v \in N_{\mathbf{v}(n)}$ or $v \in N_{p+\mathbf{v}(n)}$.

Discrete Laplacian on higher-dimensional diamond lattice. Using the definition of Discrete Laplacian on Graph from the definition of adjacent sets on $\mathcal{V}$ Discrete Laplacian $\triangle_d$ on $\mathcal{V}$ is defined by

$$((d+1)(\triangle_d + 1)\hat{f})(v) = (\hat{g}_1(n), \hat{g}_2(n)),$$

where

$$\hat{g}_1(n) = \hat{f}_2(n) + \hat{f}_2(n - \mathbf{e}_1) + \cdots + \hat{f}_2(n - \mathbf{e}_d),$$

$$\hat{g}_2(n) = \hat{f}_1(n) + \hat{f}_1(n + \mathbf{e}_1) + \cdots + \hat{f}_1(n + \mathbf{e}_d).$$

Any function $\hat{f}$ on $\mathcal{V}$ is written as $\hat{f}(n) = (\hat{f}_1(n), \hat{f}_1(n))$, $n \in \mathbb{Z}^d$, where $\hat{f}_1(n) := \hat{f}_1(\mathbf{v}(n))$, $f_2(n) := \hat{f}_2(p + \mathbf{v}(n))$. Hence $\ell_2(\mathcal{V})$ is the Hilbert space equipped with the inner product

$$(\hat{f}, \hat{g})_{\ell_2(\mathcal{V})} := \sum_{v \in A_d} \hat{f}_1(v)\overline{\hat{g}_1(v)}\deg(v) + \sum_{v \in (p+A_d)} \hat{f}_2(v)\overline{\hat{g}_2(v)}\deg(v).$$

The discrete Schrödinger operator. Let $\mathbb{T}^d = \mathbb{R}^d/(2\pi)^d$. We denote by $L_2^{(2)}(\mathbb{T}^d)$ the Hilbert space with inner product

$$(f,g)_{L_2^{(2)}} = (f_1,g_1) + (f_2,g_2), \quad (f_j,g_j) = \int\limits_{\mathbb{T}^d} f_j(x)\overline{g_j(x)}dx.$$

We then define a unitary operator $\mathcal{U} : \ell_2(\mathcal{V}) \to L_2^{(2)}(\mathbb{T}^d)$

$$(\mathcal{U}\hat{f})_j = (2\pi)^{-d/2}\sqrt{d+1}\sum_{n\in\mathbb{Z}^d}\hat{f}_j(n)e^{n\cdot x}.$$

Passing to the Fourier series, we rewrite $-(\triangle_d+1)$ into the following form:

$$(\mathcal{U}(-(\triangle_d+1))\mathcal{U}^{-1}f)(x) = (\mathbf{H}_0 f)(x), \quad f \in L_2^{(s)}(\mathbb{T}^d),$$

where $\mathbf{H}_0$ is a matrices operator for a 2×2 matrix $\mathbf{H}_0(x)$

$$\mathbf{H}_0(x) = \begin{pmatrix} 0 & E(x) \\ \overline{E(x)} & 0 \end{pmatrix},$$

$$E(x) = \frac{1}{d+1}\left(1 + e^{ix_1} + \dots e^{ix_d}\right).$$

Note that

$$|E(x)|^2 = \frac{1}{(d+1)^2}\left[d+1+2\sum_{j=1}\cos x_1 + 2\sum_{i<j}\cos(x_i - x_j)\right], \quad j = 1,\cdots,d.$$

Hence

$$\min_p |E(p)| = 0, \quad \max_p |E(p)| = 1,$$

and the point $\mathbf{0} = (0,\dots,0) \in \mathbb{T}^d$ is a unit degenerated maximum point of function $|E(\cdot)|^2$.

Let $\widehat{\mathbf{Q}}$ be the potential on $\ell_2(\mathcal{V})$ defined as multiplication operator by real valued, diagonal 2×2 matrices

$$(\hat{\mathbf{Q}}\hat{f})(n) = \begin{pmatrix} \hat{Q}_1(n) & 0 \\ 0 & \hat{Q}_2(n) \end{pmatrix}\begin{pmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \end{pmatrix},$$

where

$$\hat{Q}_1(n) := \hat{Q}_1(\mathbf{v}(n)), \quad \hat{Q}_2(n) := \hat{Q}_2(p+\mathbf{v}(n)), \quad n\in\mathbb{Z}^d.$$

Throughout the paper, we shall assume that

$$\sum_{n\in\mathbb{Z}^d}|\hat{Q}_j(n)| < \infty, \quad j = 1,2. \tag{2.1}$$

The discrete Schrödinger operator is denoted by

$$\hat{\mathbf{H}} = -(d+1)(\Delta_d + 1) + \widehat{\mathbf{Q}}.$$

Passing to the Fourier series, we rewrite $\hat{H}$ into the following form

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{Q},$$

where

$$(\mathbf{Q}f)(x) = \begin{pmatrix} (Q_1 f_1)(x) \\ (Q_2 f_2)(x) \end{pmatrix}, \quad f \in L_2^{(2)}(\mathbb{T}^d),$$

$$(Q_j f_j)(x) = \int\limits_{\mathbb{T}^d} Q_j(x-t)f_j(t)dt, \quad j = 1,2,$$

$$Q_j(x) = (\mathcal{U}\hat{Q}_j)(x), \quad j = 1,2.$$

The Main Results. Note that from (2.1), it follows that the function $Q_j(\cdot)$ is continuous on $\mathbb{T}^d$. Hence the operator $\mathbf{Q}$ is a compact operator. By the Weyl theorem, the essential spectrum $\sigma_{ess}(\mathbf{H})$ of the operator $\mathbf{H}$ coincides with the spectrum of the unperturbed operator $\mathbf{H}_0$.

Lemma 2.1. The spectrum $\sigma(\mathbf{H}_0)$ of $\mathbf{H}_0$ coincides with the set

$$\{\lambda : |E(x)|^2 = \lambda^2 \quad \text{for some} \quad x \in \mathbb{T}^d\},$$

i.e.,

$$\sigma(\mathbf{H}_0) = [-1, 1].$$

Theorem 2.1. *Let* $\widehat{\mathbf{Q}}$ *satisfy* (2.1). *Then, the number of eigenvalues of* $\mathbf{H}$ *lying in* $(-\infty, -1) \cup (1, \infty)$ *is finite, i.e. the discrete spectrum of* $\mathbf{H}$ *is a finite set.*

The proof of Theorem 2.1 implies the following theorem.

Theorem 2.2. *Let* $v_{ij}, i, j = 1, 2$ *be continuous functions on* $(\mathbb{T}^d)^2$ *and* $\mathbf{V} = \Big(V_{ij}\Big)_{i,j=1}^{2}$, *where* V_{ij} *is an integral operator with kernel* $v_{ij}(x, y), x, y \in \mathbb{T}^d$, $d \geq 2$. *Then the discrete spectrum of* $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ *is finite set.*

3. Proof of the Main results

Resolvent of $\mathbf{H}_0$. Proof of the Lemma 2.1. The operator $\mathbf{H}_0 - \lambda\mathbf{I}$ has a matrix form

$$\begin{pmatrix} -\lambda I & E \\ \overline{E} & -\lambda I \end{pmatrix},$$

where I is an identity operator and E is operator multiplication by function $E(x)$.

Therefore, the inverse of this matrix has the form

$$\begin{pmatrix} -\lambda I & E \\ \overline{E} & -\lambda I \end{pmatrix}^{-1} = \begin{pmatrix} L^{-1}(\lambda) & 0 \\ 0 & L^{-1}(\lambda) \end{pmatrix} \begin{pmatrix} -\lambda I & -\overline{E} \\ -E & -\lambda I \end{pmatrix},$$

where $L(\lambda)$ is operator multiplication by function $L(\lambda, x)$, $L(\lambda, x) = \lambda^2 - |E(x)|^2$.

Let us denote by $\mathbf{L}_\lambda$ and $\mathbf{A}_\lambda$ 2×2 matrix operators

$$\mathbf{L}_\lambda = \begin{pmatrix} L(\lambda) & 0 \\ 0 & L(\lambda) \end{pmatrix}, \quad \mathbf{A}_\lambda = \begin{pmatrix} -\lambda I & -\overline{E} \\ -E & -\lambda I \end{pmatrix}.$$

Then the resolvent $\mathbf{R}_0(\lambda) = (\mathbf{H}_0 - \lambda\mathbf{I})^{-1}$ of $\mathbf{H}_0$ has the form

$$\mathbf{R_0}(\lambda) = \mathbf{L}_\lambda^{-1}\mathbf{A}_\lambda.$$

It follows from this that the operator $\mathbf{R}_0(\lambda)$ exists if and only if $L(\lambda, x) \neq 0$ for all $x \in \mathbb{T}^d$, i.e iff $\lambda \notin \{y = |E(x)| : x \in \mathbb{T}^d\} = [-1, 1]$. Hence, we have $\sigma(\mathbf{H}_0) = [-1, 1]$.

The lemma is thus proven.

Remember that $L(\lambda, x) > 0$ as $|\lambda| > 1$ for all $x \in \mathbb{T}^d$. Therefore $L(\lambda)$ is a positive operator for all real λ with $|\lambda| > 1$. Hence $\mathbf{L}_\lambda$ is a positive operator for all real λ with $|\lambda| > 1$. A positive root $\mathbf{L}_\lambda^{-1/2}$ of $\mathbf{L}_\lambda^{-1}$ has the form

$$\mathbf{L}_\lambda^{-1/2} = \begin{pmatrix} L^{-1/2}(\lambda) & 0 \\ 0 & L^{-1/2}(\lambda) \end{pmatrix},$$

where $L^{-1/2}(\lambda)$ is an operator multiplication by function $1/\sqrt{L(\lambda, \cdot)}$, $|\lambda| > 1$.

Let

$$\mathbf{A}_\lambda(x) = \begin{pmatrix} -\lambda & -\overline{E(x)} \\ -E(x) & -\lambda \end{pmatrix}.$$

For any fixing $x \in \mathbb{T}^d$ the eigenvalues of the matrices $\mathbf{A}_\lambda(x)$ are $\xi_-(\lambda, x) = -\lambda - |E(x)|$ and $\xi_+(\lambda, x) = -\lambda + |E(x)|$. The numbers $-\lambda \pm |E(x)|$ are positive as $\lambda < -1$ and negative as $\lambda > 1$. Then, $\mathbf{A}_\lambda \geq 0$ as $\lambda < -1$ and $-\mathbf{A}_\lambda \geq 0$ as $\lambda > 1$. Since the operator $\mathbf{L}_\lambda^{-1}$ is commutative with $\mathbf{A}_\lambda$, the operator $\mathbf{L}_\lambda^{-1}\mathbf{A}_\lambda$ is self-adjoint and $\mathbf{L}_\lambda^{-1}\mathbf{A}_\lambda \geq 0$ as $\lambda < -1$, $-\mathbf{L}_\lambda^{-1}\mathbf{A}_\lambda \geq 0$ as $\lambda > 1$.

The positive roots $[\mathbf{R}_0(\lambda)]^{1/2}, \lambda < -1$ and $[-\mathbf{R}_0(\lambda)]^{1/2}$, $\lambda > 1$ of the operators $\mathbf{R}_0(\lambda)$, $\lambda > 1$ and $-\mathbf{R}_0(\lambda)$, $\lambda < 1$ have the following forms, respectively:

$$\mathbf{R}_0(\lambda)^{1/2} = \mathbf{L}_\lambda^{-1/2}\mathbf{A}_\lambda^{-1/2} \quad \text{as} \quad \lambda > 1 \tag{3.1}$$

and

$$[-\mathbf{R}_0(\lambda)]^{1/2} = \mathbf{L}_\lambda^{-1/2}[-\mathbf{A}_\lambda]^{-1/2} \quad \text{as} \quad \lambda < -1. \tag{3.2}$$

Lemma 3.1.*The positive roots $[\mathbf{A}_\lambda(x)]^{-1/2}$ and $[-\mathbf{A}_\lambda(x)]^{-1/2}$ of the matrix $\mathbf{A}_\lambda(x)$, $\lambda < -1$ and $-\mathbf{A}_\lambda(x)$, $\lambda > 1$ are given, respectively, by*

$$\mathbf{A}_\lambda^{-1/2} = \frac{1}{2}\begin{pmatrix} \sqrt{\xi_-(\lambda,x)} + \sqrt{\xi_+(\lambda,x)} & \left[\sqrt{\xi_-(\lambda,x)} - \sqrt{\xi_+(\lambda,x)}\right]\frac{\overline{E(x)}}{|E(x)|} \\ \left(\sqrt{\xi_-(\lambda,x)} - \sqrt{\xi_+(\lambda,x)}\right)\frac{E(x)}{|E(x)|} & \sqrt{\xi_-(\lambda,x)} + \sqrt{\xi_+(\lambda,x)} \end{pmatrix}$$

and

$$[-\mathbf{A}_\lambda]^{1/2}(x) = \frac{1}{2}\begin{pmatrix} \sqrt{-\xi_-(\lambda,x)} + \sqrt{-\xi_+(\lambda,x)} & \left[\sqrt{-\xi_-(\lambda,x)} - \sqrt{-\xi_+(\lambda,x)}\right]\frac{\overline{E(x)}}{|E(x)|} \\ \left(\sqrt{-\xi_-(\lambda,x)} - \sqrt{-\xi_+(\lambda,x)}\right)\frac{E(x)}{|E(x)|} & \sqrt{-\xi_-(\lambda,x)} + \sqrt{-\xi_+(\lambda,x)} \end{pmatrix}.$$

Proof. The eigenvectors of the matrix $\mathbf{A}_\lambda(x)$ corresponding to the eigenvalues $\xi_-(\lambda,x)$ and $\xi_+(\lambda,x)$ are $\varphi_- = 1/\sqrt{2}\ (1,\ E(x)/|E(x)|)$ and $\varphi_+ = 1/\sqrt{2}\ \left(\overline{E(x)}/|E(x)|,\ -1\right)$, respectively, with $||\varphi_\pm|| = 1$. Therefore the matrix $\mathbf{A}_\lambda(x)$ in a sense operator can be represented as

$$\mathbf{A}_\lambda(x) = \xi_-(\lambda,x)(\cdot,\varphi_-)_{\mathbb{C}^2}\varphi_- + \xi_+(\lambda,x)(\cdot,\varphi_+)_{\mathbb{C}^2}\varphi_+,$$

where $(\cdot,\cdot)_{\mathbb{C}^2}$ is a usual scalar product of $\mathbb{C}^2$. Therefore the positive roots $[\mathbf{A}_\lambda(x)]^{1/2}$, $\lambda < -1$ and $[-\mathbf{A}_\lambda]^{1/2}(x)$, $\lambda > 1$ of the matrices $\mathbf{A}_\lambda(x)$, $\lambda < -1$ and $-\mathbf{A}_\lambda(x)$, $\lambda > 1$ have the forms

$$[\mathbf{A}_\lambda(x)]^{1/2} = \sqrt{\xi_-(\lambda,x)}(\cdot,\varphi_-)_{\mathbb{C}^2}\varphi_- + \sqrt{\xi_+(\lambda,x)}(\cdot,\varphi_+)_{\mathbb{C}^2}\varphi_+, \quad \lambda < 1,$$

$$[-\mathbf{A}_\lambda(x)]^{1/2} = \sqrt{-\xi_-(\lambda,x)}(\cdot,\varphi_-)_{\mathbb{C}^2}\varphi_- + \sqrt{-\xi_+(\lambda,x)}(\cdot,\varphi_+)_{\mathbb{C}^2}\varphi_+, \quad \lambda > 1.$$

These equalities prove the desired results of the lemma.

Note that the Lemma 3.1 shows that the matrix valued function $[\mathbf{A}_\lambda(\cdot)]^{1/2}$, $\lambda \le -1$ $\left([-\mathbf{A}_\lambda(\cdot)]^{1/2},\ \lambda \ge 1\right)$ is bounded for all $x \in \mathbb{T}^d$ and $\lambda \le -1$ $\left(\lambda \ge 1\right)$.

The Birman-Schwinger principle. We define the self-adjoint compact operators $\mathbf{T}_\mp(z)$, acting in the Hilbert space $L_2^{(2)}(\mathbb{T}^d)$ determined by

$$\mathbf{T}_-(z) = \mathbf{R}_0^{1/2}(z)\mathbf{Q}\mathbf{R}_0^{1/2}(z) \quad \text{for} \quad z < -1$$

and

$$\mathbf{T}_+(z) = -\left[-\mathbf{R}_0(z)\right]^{1/2}\mathbf{Q}\left[-\mathbf{R}_0(z)\right]^{1/2} \quad \text{for} \quad 1 < z.$$

By $N_-(z)$ and $N_+(z)$, we denote the number of eigenvalues of the operator $\mathbf{H}$ lying to the left from $z < -1$ and lying to the right from $z > 1$, respectively.

Let A be a self-adjoint operator acting in a Hilbert space $\mathcal{H}$, and let $\mathcal{H}_A(\lambda)$, $\lambda > \sup\sigma_{ess}(A)$, be the subspace consisting of the vectors $f \in \mathcal{H}$ satisfying the condition $(Af,f) > \lambda(f,f)$. We set

$$n(\lambda, A) = \sup_{\mathcal{H}_A(\lambda)} \dim\mathcal{H}_A(\lambda).$$

By definition, we have $N_-(z) = n(-z,-\mathbf{H})$, $-z > 1$ and $N_+(z) = n(z,\mathbf{H})$, $z > 1$.

The following lemma is a modification of the well-known Birman–Schwinger principle for the operator H (see [17, 18]).

Lemma 3.2. *For the numbers $N_-(z)$ and $N_+(z)$ of eigenvalues (counted with multiplicities) of the operator $\mathbf{H}$ we have the equalities, respectively,*

$$N_-(z) = n(1,\mathbf{T}_-(z)), \quad z < -1, \tag{3.3}$$

and

$$N_+(z) = n(1,\mathbf{T}_+(z)), \quad z > 1. \tag{3.4}$$

Proof. We suppose that $u \in \mathcal{H}_{\mathbf{H}}(-z)$, i.e.,

$$(\mathbf{H}u, u) < z(u, u)$$

or

$$((\mathbf{H}_0 - z\mathbf{I})u, u) < -(\mathbf{Q}u, u).$$

Then we have

$$(y, y) < -\left(\mathbf{R}_0^{1/2}(z)\mathbf{Q}\mathbf{R}_0^{1/2}(z)y, y\right), \quad y = (\mathbf{R}_0 - z\mathbf{I})^{1/2}u.$$

Therefore, $N_-(z) \leq n\left(1, \mathbf{R}_0^{1/2}(z)\mathbf{Q}\mathbf{R}_0^{1/2}(z)\right)$.

By analogous arguments, we obtain the converse statement:

$$N_-(z) \geq n\left(1, \mathbf{R}_0^{1/2}(z)\mathbf{Q}\mathbf{R}_0^{1/2}(z)\right).$$

Hence inequality (3.3) follows. The equality (3.4) can be proven similarly.

Proof of Theorem 2.1. To prove the theorem, we use the technique proposed in [15], i.e., we show that the operator-valued function $T_\pm(\cdot)$ is well defined at the limits of the essential spectrum and we also use Lemma 3.2 and apply the Weyl inequality [19]

$$n(a + b, A + B) \leq n(a, A) + n(b, B),$$

which holds for compact operators A and B.

Let us first show that the operator-valued functions $T_-(\cdot)$ and $T_+(\cdot)$ are continuous in the norm, $(-\infty, 0]$ and $[1, \infty)$ respectively.

Note that since **0** is a unite maximum point of $|E(\cdot)|^2$, we have

$$C_1x^2 \leq |E(x)|^2 \leq C_2x^2 \quad \text{for all} \quad x \in \mathbb{T}^d.$$

From this, we get the estimation

$$\frac{1}{\sqrt{L(z,x)}} = \frac{1}{\sqrt{z^2 - |E(x)|^2}} \leq \frac{C}{|x|}, \quad \forall |z| \geq 1. \tag{3.5}$$

Using (3.1) and (3.2) we rewrite $\mathbf{T}_-(z)$ and $\mathbf{T}_+(z)$ as

$$\mathbf{T}_-(z) = \mathbf{L}_z^{-1/2}\mathbf{A}_z^{-1/2}\mathbf{Q}\mathbf{A}_z^{-1/2}\mathbf{L}_z^{-1/2}, \quad z < -1$$

and

$$\mathbf{T}_+(z) = \mathbf{L}_z^{-1/2}[-\mathbf{A}_z]^{-1/2}\mathbf{Q}[-\mathbf{A}_z]^{-1/2}\mathbf{L}_z^{-1/2}, \quad z > 1.$$

We denote by $q_{ij}^\pm(z)$ the entries of the matrix operator

$$[\pm\mathbf{A}_\lambda]^{-1/2}\mathbf{Q}[\pm\mathbf{A}_\lambda]^{-1/2}.$$

Then $q_{ij}^\pm(z)$ are integral operators. Since $Q(\cdot,\cdot)$ and $\xi_\pm(z,\cdot)$ are continuous functions, respectively on $(\mathbb{T}^d)^2$ and $\mathbb{T}^d$ as $|z| \geq 1$, the kernel $q_{ij}^\pm(z;\cdot,\cdot)$ of the integral operator $q_{ij}^\pm(z)$ are bounded functions on $(\mathbb{T}^d)^2$.

It follows from (3.5) that the kernel $t_{ij}^\pm(z;x,y) = \frac{1}{\sqrt{L(z,x)}}q_{ij}^\pm(z;x,y)\frac{1}{\sqrt{L(z,y)}}$ of $\mathbf{T}_\pm(z)$ estimated by

$$|t_{ij}^\pm(z;x,y)| \leq \frac{C}{|x||y|} \quad \text{as} \quad x, y \in \mathbb{T}^d,$$

where the constant C does not depend of $z, |z| \geq 1$. This implies that the functions $t_{ij}^-(z;\cdot,\cdot), z < -1$ and $t_{ij}^+(z;\cdot,\cdot), z > 1$ are square-integrable on $(\mathbb{T}^d)^2$ and $t_{ij}^\mp(z;\cdot,\cdot)$ converges almost everywhere to $t_{ij}^\mp(\mp 1;\cdot,\cdot)$ as $z \to \mp 1 \mp 0$. By the Lebesgue theorem, the operator $\mathbf{T}_\mp(z)$ then converges in the norm to $\mathbf{T}_\mp(\mp 1)$ as $z \to \mp 1 \mp 0$. Further, using the Weyl inequality, from (3.3) and (3.4) we obtain

$$N_-(z) \leq n\left(\frac{1}{2}, \mathbf{T}_-(z) - \mathbf{T}_-(-1)\right) + n\left(\frac{1}{2}, \mathbf{T}_-(-1)\right), \quad z < -1,$$

$$N_+(z) \leq n\left(\frac{1}{2}, \mathbf{T}_+(z) - \mathbf{T}_+(1)\right) + n\left(\frac{1}{2}, \mathbf{T}_+(1)\right), \quad z > 1.$$

Since the operator $\mathbf{T}(\pm 1)$ is compact,

$$n\left(\frac{1}{2}, \mathbf{T}(\pm 1)\right) < \infty.$$

For small $|z+1|$ and $|z-1|$ we have the equalities, respectively,

$$n\left(\frac{1}{2}, \mathbf{T}_-(z) - \mathbf{T}_-(-1)\right) = 0$$

and

$$n\left(\frac{1}{2}, \mathbf{T}_+(z) - \mathbf{T}_+(1)\right) = 0.$$

Hence, by Lemma 3.2, the number of eigenvalues of $\mathbf{H}$ in $(-\infty,-1)\cup(1,\infty)$ must be finite.

Acknowledgements

We thank the referee for valuable comments and gratefully acknowledge the support of the Malaysian Ministry of Education (MOE) through the Research Management Center (RMC), Universiti Teknologi Malaysia (Vote: QJ130000.2726.01K82)

References

[1] Faddeev L.D., Merkuriev S.P. *Quantum scattering theory for several particle systems*. Kluwer Academic Publishers, 1993.
[2] Mattis D.C. The few-body problem on lattice. *Rew. mod. Phys.*, 1986, **58**, P. 361—379.
[3] Reed M.,Simon B. *Methods of modern Mathematical Physics*. V. 4. Analysis of Operators, Academic Press, London, 1980.
[4] Minlos R.A., Mogilner A.I. Some problems concerning spectra of lattice models, in: Schrödinger Operators, Standard and Nonstandard (P. Exner and P. Seba, eds.), World Scientific, Teaneck N.J., 1989, P. 243–257.
[5] Abdullaev J.I., Ikromov I.A. Finiteness of the number of eigenvalues of the two-particle Schredinger operator on a lattice. *Theoretical and Mathematical Physics*, 2007, **152**(3), P. 1299–1312.
[6] Remling, Christian. Discrete and embedded eigenvalues for one-dimensional Schrdinger operators. *Comm. Math. Phys.*, 2007, **271**, P. 275–287.
[7] Muminov M.I., Khurramov A.M. Spectral properties of two-particle hamiltonian on a d-dimensional lattice. *Nanosystems: Physics, Chemistry, Mathematics*, 2016, **7**(5), P. 880–887.
[8] Kostrykin V., Schrader R. Kirchhoffs rule for quan tum wires. II: The inverse problem with possible applications to quantum c omputers. *Fortschr. Phys.*, 2000, **48**, P. 703–716.
[9] Melnikov Yu.B., Pavlov B.S. Two-body scattering on a graph and application to simple nanoelectronic devices. *J. Math. Phys.*, 1995, **36**, P. 2813–2825.
[10] Melikhov I.F., Popov I.Yu. The discrete spectrum of the multiparticle Hamiltonian in the framework of the Hartree-Fock approximation. *Journal of Physics: Conference Series*, 2014, **541**, P. 012099.
[11] Melikhov I.F., Popov I.Yu. Multi-Particle Bound States in Window-Coupled 2D Quantum Waveguides. *Chinese Journal of Physics*, 2015, **53**(3), P. 1–12.
[12] Kazunori Ando, Hiroshi Isozaki, Hisashi Morioka, Spectral Properties of Schrödinger Operators on Perturbed Lattices Ann. Henri Poincaré, 2016, **17**(8), P. 2103–2171.
[13] Isozaki H., Korotyaev E. Inverse Problems, Trace Formulae for Discrete Schrdinger Operators. *Annales Henri Poincare*, 2012, **13**(4), P. 751–788
[14] Ando K. Inverse scattering theory for discrete Schrödinger operators on the hexagonal lattice. *Ann. Henri Poincaré*, 2013, **14**, P. 347–383.
[15] Lakaev S.N.,Muminov M.I. Essential and Discrete Spectra of the Three-Particle Schrdinger Operator on a Lattice. *Teoret. Mat. Fiz.*,2003, **135**(3),P. 478–503.
[16] Kotani M., Shirai T. and Sunada T. Asymptotic Behavior of the Transition Probability of a Random Walk on an Infinite Graph. *J. Func. Anal.*, 1998, **159**, P. 664–689.
[17] Sobolev A.V., The Efimov effect. Discrete spectrum asymptotics. *Commun. Math. Phys.*, 1993, **156**(1), P. 101–126.
[18] Muminov M.E. The infiniteness of the number of eigenvalues in the gap in the essential spectrum for the three-particle Schrödinger operator on a lattice. *Theor. Math. Phys.*, 2009, **159**(2), P. 299–317.
[19] Birman M.Sh., Solomyak M.Z. *Spectral theory of selfadjoint operators in Hubert space*. Dordrecht, D. Reidel P.O., 1987.