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# To the qualitative properties of solution of system equations not in divergence form of polytrophic filtration in variable density

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In this paper, the properties of solutions for the nonlinear system equations not in divergence form:

$$|x|^n \frac{\partial u}{\partial t} = u^{\gamma_1} \nabla \left( |\nabla u|^{p-2} \nabla u \right) + |x|^n u^{q_1} v^{q_2} ,$$
$$|x|^n \frac{\partial v}{\partial t} = v^{\gamma_2} \nabla \left( |\nabla v|^{p-2} \nabla v \right) + |x|^n v^{q_4} u^{q_3},$$

are studied. In this work, we used method of nonlinear splitting, known previously for nonlinear parabolic equations, and systems of equations in divergence form, asymptotic theory and asymptotic methods based on different transformations. Asymptotic representation of self-similar solutions for the nonlinear parabolic system of equations not in divergence form is constructed. The property of finite speed propagation of distributions (FSPD) and the asymptotic behavior of the weak solutions were studied for the slow diffusive case.

**Keywords:** nonlinear system of equations, not in divergence form, global solutions, self-similar solutions, asymptotic representation of solution.

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## 1. Introduction

Consider in $Q = \{(t, x) : \quad t > 0, \ x \in \mathbb{R}^N\}$ parabolic system of nonlinear equations not in divergence form:

$$|x|^n \frac{\partial u}{\partial t} = u^{\gamma_1} \nabla \left( |\nabla u|^{p-2} \nabla u \right) + |x|^n u^{q_1} v^{q_2} ,$$
$$|x|^n \frac{\partial v}{\partial t} = v^{\gamma_2} \nabla \left( |\nabla v|^{p-2} \nabla v \right) + |x|^n v^{q_4} u^{q_3}, \tag{1}$$

$$u|_{t=0} = u_0 (x) \geq 0, \ \ v|_{t=0} = v_0 (x) \geq 0, \ \ \forall x \in \mathbb{R}^N \tag{2}$$

where $n$, $p$, $\gamma_i$ $(i = 1, 2)$, $q_i$ $(i = 1, 2, 3, 4)$ the numerical parameters, $\nabla(\cdot) = \mathrm{grad}_x(\cdot)$, $t$ and $x \in \mathbb{R}^N$ – respectively, the temporal and spatial coordinates, $u = u(t, x) \geq 0$, $v = v(t, x) \geq 0$ are the solutions.

Such systems arise in various applications, such as the spatial segregation of interacting species [1], chemotactic cell migration in tissues [2], and ion transport through biological and synthetic channels (nanopores) [3].

In [4], the Cauchy problem (1)–(2) was studied for $p = 2$, $n = 0$ and the absence of absorption, proved the existence of a single viscous solutions, and in [5] investigated the existence and uniqueness of a classical solution of the Cauchy problem for $p = 2$, $n = 0$.

In previous research [6], a degenerate nonlinear parabolic system with localized source was considered $u_t = u^{\alpha} (\Delta u + u^p (x, t) v^q (x_0, t))$, $v_t = v^{\beta} (\Delta v + v^m (x, t) u^n (x_0, t))$. In that work [6], the authors investigated blow-up properties for a degenerate parabolic system with nonlinear localized sources subject to homogeneous Dirichlet boundary conditions. The main aim of [6] was to study the blow-up rate estimate and the uniform blow-up profile for the blow-up solution. At the end, the blow-up set and blow up rate with respect to the radial variable was considered when the domain $Q$ is a ball.

In [7], the nonlinear degenerate parabolic system $u_t = v^{\gamma_1} (u_{xx} + au)$, $v_t = u^{\gamma_2} (v_{xx} + bv)$ with Dirichlet boundary conditions was studied. The regularization method and upper-lower solutions technique were employed to show the local existence of a solution for the nonlinear degenerate parabolic system. The global existence of a solution was discussed. The finite time blow-up results, together with an estimate of the blow-up time, were found. The blow-up set with positive measure was analyzed in some detail.

In [8] Chunhua and Jingxue were concerned with the self-similar solutions of the form:

$$u(t, x) = (t + 1)^{-\alpha} f \left( (t + 1)^{\beta} |x|^2 \right),$$

for the following degenerate and singular parabolic equation in non-divergence form:

$$\frac{\partial u}{\partial t} = u^m \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad m \geq 1,\, p > 1.$$

They first established the existence and uniqueness of solutions $f$ with compact supports, which implies that the self-similar solution shrinks. On that basis, the convergent rates of these solutions on the boundary of the supports were also established. Conversely, the convergent speeds of solutions were also considered and compared with the Dirac function as t tends to infinity.

In [9], Raimbekov studied some properties of solutions for the Cauchy problem for nonlinear parabolic equations in non-divergence form with variable density $|x|^n \frac{\partial u}{\partial t} = u^m \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, $0 \leq m < \frac{(p-2)(N+n)+p+n}{p-N}$ where they obtained a self-similar solution of the Barenblatt–Zeldovich–Kompaneets type and compared solution methods that proved the asymptotic behavior of solutions in the fast and slow diffusion cases. In this article, some comparative numerical results were also given for the case $m = 0$, $m = 1$ and $m = 1.5$. Using this result, the author discussed the properties for the finite speed of heating propagation for divergent equations and localization for non-divergent case.

In [10] and [11], the authors studied the asymptotic behavior of self-similar solutions of a parabolic system:

$$|x|^n \frac{\partial u}{\partial t} = v^{\alpha_1} \nabla\left(|x|^k u^{m_1-1}\nabla u\right) + |x|^n u^{\beta_1},$$

$$|x|^n \frac{\partial v}{\partial t} = u^{\alpha_2} \nabla\left(|x|^k v^{m_1-1}\nabla v\right) + |x|^n v^{\beta_2}.$$

The Zeldovich–Barenblatt type solution of the Cauchy problem was obtained for a cross-diffusion parabolic system not in divergence form with a source and a variable density. Based on the comparison method, the properties of finite speed perturbation of distribution is considered.

This paper is devoted to constructing a Zeldovich–Barenblatt type solution for the system equation (1). Based on comparing solution methods the properties of FSPD of the Cauchy problem for a parabolic system not in divergence form is established. The asymptotic behavior of a self-similar solution for a nonlinear parabolic system of equations in non-divergence form for slow diffusion case (depending on value of the numerical parameters) is discussed.

## 2. The self-similar system of equations

Below, a method of nonlinear splitting [13] is provided to construct a self-similar system of equations. For construction of the self-similar solutions of the system (1) in the form:

$$\begin{aligned} u(x,t) &= (t+T)^{-\alpha_1} f(\xi), \\ v(x,t) &= (t+T)^{-\alpha_2} \varphi(\xi), \\ \xi &= (t+T)^{-\gamma}|x|, \end{aligned} \tag{3}$$

where $\alpha_1 = -\frac{1+q_2-q_4}{(1-q_1)(1-q_4)-q_2q_3}$, $\alpha_2 = -\frac{1+q_3-q_1}{(1-q_1)(1-q_4)-q_2q_3}$, $\gamma = \frac{1-\alpha_1(p+\gamma_1-2)}{p+n}$, $T > 0$, $\alpha_1(p+\gamma_1-2) = \alpha_2(p+\gamma_2-2)$, it can be a self-similar system of equations:

$$\begin{aligned} & f^{\gamma_1}\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \alpha_1\xi^n f + \gamma\xi^{n+1}\frac{df}{d\xi} + \xi^n f^{q_1}\varphi^{q_2} = 0, \\ & \varphi^{\gamma_2}\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\right) + \alpha_2\xi^n \varphi + \gamma\xi^{n+1}\frac{d\varphi}{d\xi} + \xi^n f^{q_3}\varphi^{q_4} = 0. \end{aligned} \tag{4}$$

In [12], the qualitative properties of solutions for system (4) in divergence form were studied based on the self-similar and approximately self-similar approaches.

## 3. Slow diffusion (case $p+\gamma_i-2>0,\ i=1,2$). A global solvability of solutions

The properties of a global solvability for weak solutions of the system (1) were proved using a comparison principle [14]. For this goal, a new system of equations was constructed using the standard equation method as in [13]:

$$u_+(t,x)=(t+T)^{-\alpha_1}\overline{f}(\xi),\qquad v_+(t,x)=(t+T)^{-\alpha_2}\overline{\varphi}(\xi), \tag{5}$$

where $\alpha_1=-\dfrac{1+q_2-q_4}{(1-q_1)(1-q_4)-q_2q_3}$, $\alpha_2=-\dfrac{1+q_3-q_1}{(1-q_1)(1-q_4)-q_2q_3}$, $\gamma=\dfrac{1-\alpha_1(p+\gamma_1-2)}{p+n}$, $T>0$, $\xi=(t+T)^{-\gamma}|x|$.

In case, $\alpha_1(p+\gamma_1-2)=\alpha_2(p+\gamma_2-2)$,

$$\overline{f}(\xi)=A_1\Big(a-\xi^{\frac{p+n}{p-1}}\Big)_+^{\frac{p-1}{p+\gamma_1-2}},\quad \overline{\varphi}(\xi)=A_2\Big(a-\xi^{\frac{p+n}{p-1}}\Big)_+^{\frac{p-1}{p+\gamma_2-2}}, \tag{6}$$

where $a>0$, $A_i=\left(\dfrac{\gamma(p+\gamma_i-2)}{(1-\gamma_i)(p+n)}\right)^{\frac{1}{p+\gamma_i-2}}\left|\dfrac{p+n}{p+\gamma_i-2}\right|^{\frac{2-p}{p+\gamma_i-2}}$, $(i=1,2)$, $b_+=\max(0,b)$.

The following notations can be introduced:

$$k_i=\frac{(p-1)\,q_{2i-1}}{p+\gamma_1-2}+\frac{(p-1)\,q_{2i}}{p+\gamma_2-2}-\frac{p-1}{p+\gamma_i-2},\quad h_i=\frac{n(p-1)+N(p+\gamma_i-2)}{(n+p)(\gamma_i-1)}-\frac{p}{n+p},\ i=1,2,$$

$$m_1=A_1^{q_1-1}A_2^{q_2},\quad m_2=A_1^{q_3}A_2^{q_4-1}.$$

**Theorem 1.** *(A global solvability). Let the conditions of* $p+\gamma_i-2>0$, $k_i\ge 0$,

$$-\frac{N+n}{(n+p)(1-\gamma_i)}-h_i\alpha_i+m_ia^{k_i}\le 0,\quad i=1,2,$$

$$u_+(0,x)\ge u_0(x),\ \ v_+(0,x)\ge v_0(x),\ \ x\in\mathbb{R}^N.$$

*Then, for sufficiently small* $u_0(x)$, $v_0(x)$, *the followings holds:*

$$u(t,x)\le u_+(t,x),\quad v(t,x)\le v_+(t,x)\quad \text{in } Q, \tag{7}$$

*where the functions* $u_+(t,x)$, $v_+(t,x)$ *defined as above.*

**Proof.** Theorem 1 is proved by the comparing solution method [14]. Hence, comparing solution methods it is taken the functions $u_+(t,x)$, $v_+(t,x)$. Substituting (5) in (1) the following inequality can be obtained:

$$\begin{aligned}
&\overline{f}^{\gamma_1}\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\left|\frac{d\overline{f}}{d\xi}\right|^{p-2}\frac{d\overline{f}}{d\xi}\right)+\alpha_1\xi^n\overline{f}+\gamma\xi^{n+1}\frac{d\overline{f}}{d\xi}+\xi^n\overline{f}^{q_1}\overline{\varphi}^{q_2}\le 0,\\
&\overline{\varphi}^{\gamma_2}\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\left|\frac{d\overline{\varphi}}{d\xi}\right|^{p-2}\frac{d\overline{\varphi}}{d\xi}\right)+\alpha_2\xi^n\overline{\varphi}+\gamma\xi^{n+1}\frac{d\overline{\varphi}}{d\xi}+\xi^n\overline{f}^{q_3}\overline{\varphi}^{q_4}\le 0.
\end{aligned} \tag{8}$$

If the specific form (6) is given for the functions $\overline{f}(\xi)$, $\overline{\varphi}(\xi)$, inequality (8) can be rewritten as follows:

$$-\frac{N+n}{(n+p)(1-\gamma_1)}-h_1\alpha_1+m_1\Big(a-\xi^{\frac{p+n}{p-1}}\Big)^{k_1}\le 0,$$

$$-\frac{N+n}{(n+p)(1-\gamma_2)}-h_2\alpha_2+m_2\Big(a-\xi^{\frac{p+n}{p-1}}\Big)^{k_2}\le 0.$$

It is easy to check that $m_1\Big(a-\xi^{\frac{p+n}{p-1}}\Big)^{k_1}\le m_1a^{k_1}$, $m_2\Big(a-\xi^{\frac{p+n}{p-1}}\Big)^{k_2}\le m_2a^{k_2}$.

Then, according to the hypotheses of Theorem 1 and comparison principle, it will be: $u(t,x)\le u_+(t,x)$, $v(t,x)\le v_+(t,x)$ in $Q$, if $u_+(0,x)\ge u_0(x)$, $v_+(0,x)\ge v_0(x)$, $x\in\mathbb{R}^N$.

The proof of the theorem is completed.

## 4. Asymptotic of the self-similar solutions

Next, the asymptotic behavior of the self-similar solutions of the system (4) is studied. Self-similar solution of system equations (4) will be searched for in the form:

$$f(\xi) = \overline{f}(\xi)y(\eta), \quad \varphi(\xi) = \overline{\varphi}(\xi)z(\eta), \quad \eta = -\ln\left(a - \xi^{\frac{p+n}{p-1}}\right), \tag{9}$$

where $\overline{f}(\xi) = \left(a - \xi^{\frac{p+n}{p-1}}\right)^{\frac{p-1}{p+\gamma_1-2}}$, $\overline{\varphi}(\xi) = \left(a - \xi^{\frac{p+n}{p-1}}\right)^{\frac{p-1}{p+\gamma_2-2}}$, $a > 0$.

Then, substituting (9) into (4) for the function $y(\eta) > 0$, $z(\eta) > 0$, the following system of nonlinear equations is obtained:

$$\begin{aligned} &y^{\gamma_1}\frac{d}{d\eta}(L_1 y) + a_{11}(\eta)\, y^{\gamma_1}(L_1 y) + a_{12}(\eta)\left(\frac{dy}{d\eta} + a_{10}(\eta)\, y\right) + a_{13}(\eta)\, y^{q_1} z^{q_2} + a_{14}(\eta)\, y = 0\,, \\ &z^{\gamma_2}\frac{d}{d\eta}(L_2 z) + a_{21}(\eta)\, z^{\gamma_2}(L_2 z) + a_{22}(\eta)\left(\frac{dz}{d\eta} + a_{20}(\eta)\, z\right) + a_{23}(\eta)\, y^{q_3} z^{q_4} + a_{24}(\eta)\, z = 0. \end{aligned} \tag{10}$$

Here, $a_{i0}(\eta) = -\dfrac{p-1}{p+\gamma_i-2}$, $a_{i1}(\eta) = \dfrac{(N+n)(p-1)}{p+n}\dfrac{e^{-\eta}}{a-e^{-\eta}} - \dfrac{(p-1)(1-\gamma_i)}{p+\gamma_i-2}$, $a_{i2}(\eta) = \gamma\left(\dfrac{p-1}{p+n}\right)^{p-1}$, $a_{i4}(\eta) = \alpha_i\left(\dfrac{p-1}{p+n}\right)^{p}$, $a_{i3}(\eta) = \left(\dfrac{p-1}{p+n}\right)^{p}\dfrac{e^{-s_i\eta}}{a-e^{-\eta}}$, $s_i = 1 + \dfrac{(p-1)\,q_{2i-1}}{p+\gamma_1-2} + \dfrac{(p-1)\,q_{2i}}{p+\gamma_2-2} - \dfrac{p-1}{p+\gamma_i-2}$ $(i = 1, 2)$, $L_1 y = \left|\dfrac{dy}{d\eta} + a_{10}(\eta)\, y\right|^{p-2}\left(\dfrac{dy}{d\eta} + a_{10}(\eta)\, y\right)$, $L_2 z = \left|\dfrac{dz}{d\eta} + a_{20}(\eta)\, z\right|^{p-2}\left(\dfrac{dz}{d\eta} + a_{20}(\eta)\, z\right)$.

There, it is assumed that $\xi \in [\xi_0,\ \xi_1)$, $0 < \xi_0 < \xi_1$, $\xi_1 = a^{\frac{p-1}{p+n}}$.

Therefore, the function $\eta(\xi)$ has the properties: $\eta'(\xi) > 0$ at $\xi \in [\xi_0, \xi_1)$, $\eta_0 = \eta(\xi_0) > 0$, $\lim\limits_{\xi_0\to\xi_1} \eta(\xi) = +\infty$.

Further, the self-similar system of equations (10) is investigated in the following limitations: $\lim\limits_{\eta\to+\infty} a_{ij}(\eta) = a_{ij}^0$, $0 < \left|a_{ij}^0\right| < +\infty$, $(i = 1, 2; j = 0, 1, 2, 3, 4)$.

Through the introduction of transformations (3), (9) and properties $\eta \to +\infty$, study of the solutions of (1) is reduced to the study of the solutions of (10), each of which is in the vicinity $+\infty$ and satisfies the inequalities:

$$y(\eta) > 0, \quad y' + a_{10}(\eta)y \neq 0,$$

$$z(\eta) > 0, \quad z' + a_{20}(\eta)z \neq 0.$$

Now, the asymptotic behavior of the positive solutions of (10), having a nonzero a finite limit as $\eta \to +\infty$ is studied.

## 5. The main results

Here, we introduce the notations:

$$c_{i1} = \frac{1-\gamma_i}{(p+\gamma_i-2)^p}, \quad c_{i2} = \frac{1}{(p+n)^{p-1}}\left(\frac{\alpha_i}{p+n} - \frac{\gamma}{p+\gamma_i-2}\right), \quad c_{i3} = \frac{1}{(p+n)^p a} \quad (i = 1, 2)\,.$$

Let $y(\eta) = y^0 + o(1)$, $z(\eta) = z^0 + o(1)$ at $\eta \to +\infty$ and the equality $(1+q_1)(\gamma_1+p-2) = (1+q_2)(\gamma_2+p-2)$ is performed.

Then, this is validated by the following theorems:

**Theorem 2.** *Let $s_1 = 0$, $s_2 = 0$. Then, the self-similar solution of system (1) has the asymptotic at $|x| \to a^{\frac{p-1}{p+n}}(t+T)^{\gamma}$*

$$\begin{aligned} u_A(t,x) &= (T+t)^{-\alpha_1}\left(a - \left(\frac{|x|}{(t+T)^{\gamma}}\right)^{\frac{p+n}{p-1}}\right)_+^{\frac{p-1}{p+\gamma_1-2}}\left(y^0 + o(1)\right), \\ v_A(t,x) &= (T+t)^{-\alpha_2}\left(a - \left(\frac{|x|}{(t+T)^{\gamma}}\right)^{\frac{p+n}{p-1}}\right)_+^{\frac{p-1}{p+\gamma_2-2}}\left(z^0 + o(1)\right), \end{aligned} \tag{11}$$

*where $0 < y^0 < +\infty$, $0 < z^0 < +\infty$ and $y^0$, $z^0$ are the solutions $w_1$, $w_2$ for the system of nonlinear algebraic equations:*

$$c_{i1} w_i^{p+\gamma_i-1} + c_{i2} w_i + c_{i3} w_1^{q_{2i-1}} w_2^{q_{2i}} = 0 \quad (i = 1, 2). \tag{12}$$

**Theorem 3.** *Let $s_1 = 0$, $s_2 > 0$. Then, the self-similar solution of system (1) has the asymptotic at $|x| \to a^{\frac{p-1}{p+n}}(t+T)^{\gamma}$ form (11), where $0 < y^0 < +\infty$, $0 < z^0 < +\infty$ and $y^0$, $z^0$ are the solutions $w_1$, $w_2$ for the system of nonlinear algebraic equations:*

$$c_{11}w_1^{p+\gamma_1-1} + c_{12}w_1 + c_{13}w_1^{q_1}w_2^{q_2} = 0, \quad c_{21}w_2^{p+\gamma_2-1} + c_{22}w_2 = 0.$$

**Theorem 4.** *Let $s_1 > 0$, $s_2 = 0$. Then, the self-similar solution of equation (1) has an asymptotic at $|x| \to a^{\frac{p-1}{p+n}}(t+T)^{\gamma}$ form (11), where $0 < y^0 < +\infty$, $0 < z^0 < +\infty$ and $y^0$, $z^0$ are the solutions $w_1$, $w_2$ the system of nonlinear algebraic equations*

$$c_{11}w_1^{p+\gamma_1-2} + c_{12}w_1 = 0, \quad c_{21}w_2^{p+\gamma_2-2} + c_{22}w_2 + c_{23}w_1^{q_3}w_2^{q_4} = 0.$$

**Theorem 5.** *Let $s_1 > 0$, $s_2 > 0$. Then, the self-similar solution of equation (1) has the asymptotic at $|x| \to a^{\frac{p-1}{p+n}}(t+T)^{\gamma}$ form (11), where $0 < y^0 < +\infty$, $0 < z^0 < +\infty$ and $y^0$, $z^0$ are the solutions $w_1$, $w_2$ for the system of nonlinear algebraic equations:*

$$c_{11}w_1^{p+\gamma_1-2} + c_{12}w_1 = 0, \quad c_{21}w_2^{p+\gamma_2-2} + c_{22}w_2 = 0.$$

**The proof.** Assuming that the system (10) as:

$$\vartheta_1(\eta) = L_1 y, \quad \vartheta_2(\eta) = L_2 z, \tag{13}$$

the following identity is obtained:

$$\begin{aligned} \vartheta_1'(\eta) &\equiv -a_{11}(\eta)\vartheta_1(\eta) - a_{12}(\eta)y^{-\gamma_1}\vartheta_1^{\frac{1}{p-1}}(\eta) - a_{13}(\eta)y^{q_1-\gamma_1}z^{q_2} - a_{14}(\eta)y^{1-\gamma_1}, \\ \vartheta_2'(\eta) &\equiv -a_{21}(\eta)\vartheta_2(\eta) - a_{22}(\eta)z^{-\gamma_2}\vartheta_2^{\frac{1}{p-1}}(\eta) - a_{23}(\eta)z^{q_4-\gamma_2}y^{q_3} - a_{24}(\eta)z^{1-\gamma_2}. \end{aligned} \tag{14}$$

Now, we consider the function as:

$$\begin{aligned} g_1(\lambda_1,\eta) &\equiv -a_{11}(\eta)\lambda_1 - a_{12}(\eta)y^{-\gamma_1}\lambda_1^{\frac{1}{p-1}} - a_{13}(\eta)y^{q_1-\gamma_1}z^{q_2} - a_{14}(\eta)y^{1-\gamma_1}, \\ g_2(\lambda_2,\eta) &\equiv -a_{21}(\eta)\lambda_2 - a_{22}(\eta)z^{-\gamma_2}\lambda_2^{\frac{1}{p-1}} - a_{23}(\eta)z^{q_4-\gamma_2}y^{q_3} - a_{24}(\eta)z^{1-\gamma_2}, \end{aligned} \tag{15}$$

where $\lambda_i \in \mathbb{R}$ $(i = 1,2)$.

Suppose first that $s_i = 0$ $(i = 1,2)$. Then, the functions $g_i(\lambda_i,\eta)$ $(i = 1,2)$ preserves sign on some interval $[\eta_1, +\infty) \subset [\eta_0, +\infty)$ for every fixed value $\lambda_i$ $(i = 1,2)$, different from the values satisfying system:

$$\begin{aligned} -a_{11}^0\lambda_1 - a_{12}^0\left(y^0\right)^{-\gamma_1}\lambda_1^{\frac{1}{p-1}} - a_{13}^0\left(y^0\right)^{q_1-\gamma_1}\left(z^0\right)^{q_2} - a_{14}^0\left(y^0\right)^{1-\gamma_1} = 0, \\ -a_{21}^0\lambda_2 - a_{22}^0\left(z^0\right)^{-\gamma_2}\lambda_2^{\frac{1}{p-1}} - a_{23}^0\left(z^0\right)^{q_4-\gamma_2}\left(y^0\right)^{q_3} - a_{24}^0\left(z^0\right)^{1-\gamma_2} = 0. \end{aligned}$$

Now, we let $s_i > 0$ $(i = 1,2)$. It is easy to see that the functions $g_i(\lambda_i,\eta)$ $(i = 1,2)$, for every fixed value $\lambda_i$ $(i = 1,2)$, are different from the values satisfying the system:

$$\begin{aligned} -a_{11}^0\lambda_1 - a_{12}^0\left(y^0\right)^{-\gamma_1}\lambda_1^{\frac{1}{p-1}} - a_{14}^0\left(y^0\right)^{1-\gamma_1} = 0, \\ -a_{21}^0\lambda_2 - a_{22}^0\left(z^0\right)^{-\gamma_2}\lambda_2^{\frac{1}{p-1}} - a_{24}^0\left(z^0\right)^{1-\gamma_2} = 0, \end{aligned}$$

which preserves the sign on some interval $[\eta_2, +\infty) \subset [\eta_0, +\infty)$.

And in the case $s_i < 0$ $(i = 1,2)$, the functions $g_i(\lambda_i,\eta)$ $(i = 1,2)$ can be rewritten in the following form:

$$\begin{aligned} g_1(\lambda_1,\eta) &\equiv -a_{11}(\eta)\lambda_1 - a_{12}(\eta)y^{-\gamma_1}\lambda_1^{\frac{1}{p-1}} - a_{13}(\eta)y^{1-\gamma_1}\left(y^{q_1-1}z^{q_2} - a_{14}(\eta)a_{13}^{-1}(\eta)\right), \\ g_2(\lambda_2,\eta) &\equiv -a_{21}(\eta)\lambda_2 - a_{22}(\eta)z^{-\gamma_2}\lambda_2^{\frac{1}{p-1}} - a_{23}(\eta)z^{1-\gamma_2}\left(y^{q_3}z^{q_4-1} - a_{24}(\eta)a_{23}^{-1}(\eta)\right). \end{aligned}$$

From here:

$$\begin{aligned} &\lim_{\eta\to+\infty} a_{i1}(\eta) = -\frac{(p-1)(1-\gamma_i)}{p+\gamma_i-2}, \quad \lim_{\eta\to+\infty} a_{i2}(\eta) = \gamma\left(\frac{p-1}{p+n}\right)^{p-1}, \\ &\lim_{\eta\to+\infty} a_{i3}(\eta) = \infty, \quad \lim_{\eta\to+\infty} a_{i4}(\eta) = \alpha_i\left(\frac{p-1}{p+n}\right)^{p} \quad (i = 1,2) \end{aligned}$$

implies that the functions $g_i(\lambda_i,\eta)$ $(i = 1,2)$ preserve sign on the interval $[\eta_2, +\infty) \subset [\eta_0, +\infty)$, where $\lambda_i \neq 0$ $(i = 1,2)$. Thus, the functions $g_i(\lambda_i,\eta)$ $(i = 1,2)$ for all $\eta \in [\eta_i, +\infty)$ $(i = 1,2)$ satisfy one of the inequalities:

$$g_i(\lambda_i,\eta) > 0 \quad \text{or} \quad g_i(\lambda_i,\eta) < 0 \quad (i = 1,2). \tag{16}$$

Suppose now that for the functions $\vartheta_i(\eta)$ $(i = 1,2)$ limit as $\eta \to +\infty$ does not exist. Consider that case when one of the inequalities (16) is satisfied. As $\vartheta_i(\eta)$ $(i = 1,2)$ are oscillating functions around straight line

$\overline{\vartheta}_i = \lambda_i\ (i = 1, 2)$ its graph intersects this straight line infinitely many times in $[\eta_i, +\infty)\ (i = 1, 2)$. However, this is impossible, since on the interval $[\eta_i, +\infty)\ (i = 1, 2)$ just one of the inequalities (16) is valid, and therefore, from (15), it follows that the graph of the functions $\vartheta_i(\eta)\ (i = 1, 2)$ intersects the straight lines $\overline{\vartheta}_i = \lambda_i\ (i = 1, 2)$ only once in the interval $[\eta_i, +\infty)\ (i = 1, 2)$. Accordingly, the functions $\vartheta_i(\eta)\ (i = 1, 2)$ has a limit at $\eta \to +\infty$.

By assumption, $y(\eta) = y^0 + o(1)$, $z(\eta) = z^0 + o(1)$ at $\eta \to +\infty$, and the functions $\vartheta_i(\eta)\ (i = 1, 2)$ defined in (13), has a limit at $\eta \to +\infty$. Then $y'(\eta)$ and $z'(\eta)$ have a limit at $\eta \to +\infty$, and this limit is zero. Then,

$$\vartheta_1(\eta) = \left|\frac{dy}{d\eta} + a_{10}(\eta)\, y\right|^{p-2} \left(\frac{dy}{d\eta} + a_{10}(\eta)\, y\right) = \left|a_{10}^0 y^0\right|^{p-2} a_{10}^0 y^0 + o(1),$$

$$\vartheta_2(\eta) = \left|\frac{dz}{d\eta} + a_{20}(\eta)\, z\right|^{p-2} \left(\frac{dz}{d\eta} + a_{20}(\eta)\, z\right) = \left|a_{20}^0 z^0\right|^{p-2} a_{20}^0 z^0 + o(1)$$

at $\eta \to +\infty$ and by (14), the derivative of functions $\vartheta_i(\eta)\ (i = 1, 2)$ has a limit at $\eta \to +\infty$, which are obviously equal to zero.

Consequently, the following is necessary:

$$\lim_{\eta \to +\infty} \left(a_{11}(\eta)\vartheta_1(\eta) + a_{12}(\eta) y^{-\gamma_1} \vartheta_1^{\frac{1}{p-1}}(\eta) + a_{13}(\eta) y^{q_1-\gamma_1} z^{q_2} + a_{14}(\eta) y^{1-\gamma_1}\right) = 0,$$

$$\lim_{\eta \to +\infty} \left(a_{21}(\eta)\vartheta_2(\eta) + a_{22}(\eta) z^{-\gamma_2} \vartheta_2^{\frac{1}{p-1}}(\eta) + a_{23}(\eta) z^{q_4-\gamma_2} y^{q_3} + a_{24}(\eta) z^{1-\gamma_2}\right) = 0.$$

From this expression, it is easy to see that the system (13) has a solution $(y(\eta), z(\eta))$ with a finite non-zero limit, at $\eta \to +\infty$, necessary, for compliance with one of the conditions of Theorems 2, 3, 4, 5.

Consequently, by the transformations introduced by (3) and (9), self-similar solution of the system equation (1) has an asymptotic at $|x| \to a^{\frac{p-1}{p+n}} (t+T)^{\gamma}$ of the following form:

$$u_A(t,x) = (T+t)^{-\alpha_1} \left(a - \left(\frac{|x|}{(t+T)^{\gamma}}\right)^{\frac{p+n}{p-1}}\right)_+^{\frac{p-1}{p+\gamma_1-2}} \left(y^0 + o(1)\right),$$

$$v_A(t,x) = (T+t)^{-\alpha_2} \left(a - \left(\frac{|x|}{(t+T)^{\gamma}}\right)^{\frac{p+n}{p-1}}\right)_+^{\frac{p-1}{p+\gamma_2-2}} \left(z^0 + o(1)\right).$$

The theorems are thus proved.

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