On a problem for the loaded degenerating mixed type equation involving integral-differential operators

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This work aims to study the existence and uniqueness of a solution for a problem of the loaded degenerating mixed type equation. We consider the loaded parabolic-hyperbolic equation involving the Caputo fractional derivative and Riemann-Liouville integrals. The uniqueness of solution is proved by using the method of integral energy applying an extremum principle. Based on the statement of equivalence between "The existence and uniqueness of solution" and "Solvability of the respectively Fredholm type integral equations", the existence of a solution was proved.

Keywords: loaded degenerating equation, parabolic-hyperbolic type, integral operators, Caputo fractional derivative, existence and uniqueness of solution, integral equations.

Received: 18 May 2017

Revised: 30 May 2017

1. Introduction

The first, most general definitions of loaded equations and various loaded equations were classified in detail by A. M. Nakhushev [1].

Definition 1. An equation $Au(x) = f(x)$ is called a loaded equation in n dimensional Euclidean domain Ω if the operator A depends of the restriction of the unknown function to a closed subset of $\overline{\Omega}$, of measure strictly less than n.

To that end, some local and non-local problems were investigated for the loaded mixed type equation involving fractional order integral-differential operators (see [2, 3]). In the works [4–7] we can see significant development in the field of fractional differential equations in recent years. Various phenomena in physics, like diffusion in a disordered or fractal medium, or in image analysis, or in risk management have been modeled by means of fractional partial differential equations. In general, there exists no method that yields an exact solution for these equations. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, etc., (for details, see [8–11]). See Ref. [12] for deterministic fractional models in bioengineering and nanotechnology. Fractional calculus is widely applied to the investigation of partial differential equations of mixed type and hyperbolic type with degenerations (see [7,13,14]). In a series of papers (see [15–17]), the authors considered some classes of boundary value problems for mixed type non degenerating and degenerating differential equations involving Caputo and Riemann-Liouville fractional derivatives of order $0 < \alpha \leq 1$.

2. Preliminaries

2.1. Riemann-Liouville and Caputo integral-differential operators

Definition 2. Let $f(x)$ be an absolutely continuous function over (a,b). Then the left and right Riemann–Liouville fractional integrals order α ($\alpha \in \mathbb{R}^+$) (respectively) are (see [4], p. 69)

$$\left(I_{a+}^{\alpha}f\right) x = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}dt, \quad x > a \tag{1}$$

$$\left(I_{-b}^{\alpha}f\right) x = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}dt, \quad x < b. \tag{2}$$

Definition 3. The Riemann–Liouville fractional derivatives $D_{ax}^{\alpha}f$ and $D_{xb}^{\alpha}f$ of order $\alpha(\alpha \in \mathbb{R}^+)$, are defined by (see [4], p. 70):

$$(D_{ax}^{\alpha}f)\, x = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt,\ n=[\alpha]+1,\quad x>a; \tag{3}$$

$$(D_{xb}^{\alpha}f)\, x = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}}dt,\ \ n=[\alpha]+1,\quad x<b; \tag{4}$$

respectively, where $[\alpha]$ is the integer part of α. In particular, for $\alpha = N \cup \{0\}$ we have $\left(D_{ax}^0 f\right)x = f(x)$, $\left(D_{xb}^0 f\right)x = f(x)$, $(D_{ax}^n f)\, x = f^{(n)}(x)$; $(D_{xb}^n f)\, x = (-1)^n f^{(n)}(x)$, $n \in \mathbb{N}$, where $f^{(n)}(x)$ is the usual derivative of $f(x)$ of order n.

Definition 4. Caputo fractional derivatives ${}_C D_{ax}^{\alpha}f$ and ${}_C D_{xb}^{\alpha}f$ of order $\alpha > 0$ $(\alpha \notin \mathbb{N} \cup \{0\})$ are defined by (see [4], p. 92):

$$({}_C D_{ax}^{\alpha}f)\, x = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,\ \ n=[\alpha]+1,\quad x>a; \tag{5}$$

$$({}_C D_{xb}^{\alpha}f)\, x = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt,\ \ n=[\alpha]+1,\quad x<b; \tag{6}$$

respectively. From (3)–(6), as a conclusion we will have: $k-1 < \alpha \leq k$, $k \in \mathbb{N}$; consequently, while for $\alpha \in \mathbb{N} \cup \{0\}$ we have

$$\left({}_C D_{ax}^0 f\right)x = f(x),\quad \left({}_C D_{xb}^0 f\right)x = f(x),$$

$$({}_C D_{ax}^n f)\, x = f^{(n)}(x),\quad ({}_C D_{xb}^n f)\, x = (-1)^n f^{(n)}(x), n \in \mathbb{N}.$$

2.2. Gauss hypergeometric function

Gauss hypergeometric function $F(a, b, c, z)$ is defined in the unit desk as the sum of the hypergeometric series (see [4], p. 27):

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!}, \tag{7}$$

where $|z| < 1$, a, $b \in \mathbb{C}$, $c \in \mathbb{C}\backslash Z_0^-$ and $(a)_0 = 1$, $(a)_n = a(a+1)...(a+n-1) = \dfrac{\Gamma(a)}{\Gamma(a+n)}$, $(n = 1, 2, ...)$.

One such analytic continuation is given by Euler integral representation:

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx, \tag{8}$$

$0 < \operatorname{Re} b < \operatorname{Re} c$, $|\arg(1-z)| < \pi$. The Gauss hypergeometric function $F(a, b, c, z)$ allows the following estimation:

$$F(a,b,c;z) \leq \begin{cases} c_1, & \text{if } c-a-b>0,\ 0 \leq z \leq 1; \\ c_2(1-z)^{c-a-b}, & \text{if } c-a-b<0,\ 0 < z < 1; \\ c_3\left(1+|\ln(1-z)|\right), & \text{if } c-a-b=0. \end{cases} \tag{9}$$

2.3. Wright type functions

The elementary definition of the Wright type function at $\alpha > \beta$, $\alpha > 0$ and for all $z \in \mathbb{C}$, is [18]

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}. \tag{10}$$

If $\alpha = \mu = 1$, then from (10) we have:

$$e_{1,\beta}^{1,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{n!\Gamma(\delta - \beta n)}. \tag{11}$$

3. Problem formulation and main functional relation

We consider equation:

$$0 = \begin{cases} u_{xx} - {}_C D_{oy}^{\alpha} u + \sum_{k=1}^{n} p_k I_{0x}^{\beta_k} u(x,0), & \text{at } y > 0; \\ (-y)^m u_{xx} - x^m u_{yy} + \sum_{k=1}^{n} q_k I_{0\eta}^{\gamma_k} u\left(\eta^{\frac{1}{m+2}}, 0\right), & \text{at } y < 0 \end{cases} \tag{12}$$

with operators (see (1) and (5)):

$$_C D_{oy}^{\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} u_t(x,t) dt, \quad \left(I_{ax}^{\beta} u\right)(x,0) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} u(t,0) dt, \tag{13}$$

where $0 < \alpha$, β_k, $\gamma_k < 1$, $\xi = x^{\frac{m+2}{2}} - (-y)^{\frac{m+2}{2}}$, m, p_k, $q_k = const$, $m > 0$.

We set the Ω domain, bounded with segments: $A_1 A_2 = \{(x,y) : x = 1, 0 < y < h\}$, $B_1 B_2 = \{(x,y) : x = 0, 0 < y < h\}$, $B_2 A_2 = \{(x,y) : y = h, 0 < x < 1\}$ at the $y > 0$, and by the characteristics: $A_1 C : x^{\frac{m+2}{2}} + (-y)^{\frac{m+2}{2}} = 1$; $B_1 C : x^{\frac{m+2}{2}} - (-y)^{\frac{m+2}{2}} = 0$ of equation (12) at $y < 0$, where $A_1(1;0)$, $A_2(1;h)$, $B_1(0;0)$, $B_2(0;h)$, $C\left(2^{-\frac{2}{m+2}}; -2^{-\frac{2}{m+2}}\right)$. Introduce designations: $\Omega^+ = \Omega \cap (y > 0)$, $\Omega^- = \Omega \cap (y < 0)$.

For equation (12), we consider the following problem: find a solution $u(x,y)$ of equation (12) from the following class of functions:

$$W = \left\{ u(x,y) : u(x,y) \in C(\bar{\Omega}) \cap C^2(\Omega^-), \quad u_{xx} \in C\left(\Omega^+\right), \quad _C D_{oy}^{\alpha} u \in C\left(\Omega^+\right) \right\}$$

satisfies boundary conditions:

$$u(x,y)\big|_{A_1 A_2} = \varphi(y), 0 \le y \le h, \tag{14}$$

$$u(x,y)\big|_{B_1 B_2} = \psi(y), 0 \le y \le h, \tag{15}$$

$$u(x,y)|_{B_1 C} = h(x), \quad 0 \le x \le 2^{-\frac{2}{m+2}}. \tag{16}$$

and gluing conditions:

$$\lim_{y \to +0} y^{1-\alpha} u_y(x,y) = \lambda u_y(x,-0), \quad (x,0) \in A_1 B_1, \tag{17}$$

where $\varphi(y)$, $\psi(y)$, $h(x)$ are given functions and $\lambda = const$ $\left(\lambda \in \mathbb{R}^+\right)$. In order to solve the above problem we need the following Riemann function.

Its known [19] that the Riemann function for equation (12) at $y < 0$ on the characteristics coordinate $\xi = x^{\frac{m+2}{2}} - (-y)^{\frac{m+2}{2}}$ and $\eta = x^{\frac{m+2}{2}} + (-y)^{\frac{m+2}{2}}$ defined with the Gauss hypergeometric function

$$R(\xi_0, \eta_0; \xi, \eta) = \frac{\left(\eta^2 - \xi^2\right)^{2\delta}}{\left(\eta^2 - \xi_0^2\right)^{\delta} \left(\eta_0^2 - \xi^2\right)^{\delta}} F\left(\delta, \delta, 1; \frac{\left(\xi_0^2 - \xi^2\right)\left(\eta^2 - \eta_0^2\right)}{\left(\eta^2 - \xi_0^2\right)\left(\eta_0^2 - \xi^2\right)}\right), \tag{18}$$

where $\delta = \dfrac{m}{2(m+2)}$. In fact, that a solution of the Cauchy problem for equation (12) in the domain Ω^- with initial dates

$$u(x,0) = \tau(x), \quad 0 \le x \le 1; \\ u_y(x,-0) = \nu^-(x), \quad 0 < x < 1, \tag{19}$$

will be given by formula:

$$u(x,y) = k_1 \int_{\xi}^{\eta} \left(t^2 - \xi^2\right)^{-\delta} \left(\eta^2 - t^2\right)^{-\delta} \nu^-\left(t^{1-2\delta}\right) dt -$$

$$k_2 \int_{\xi}^{\eta} \left(\eta^2 - \xi^2\right)^{1-2\delta} \left(t^2 - \xi^2\right)^{\delta-1} \left(\eta^2 - t^2\right)^{\delta-1} t \tau^-\left(t^{1-2\delta}\right) dt +$$

$$\sum_{k=1}^{n} \int_{\xi}^{\eta} q_k I_{0t^2}^{\gamma_k} \tau(t^{1-2\delta}) dt \int_t^{\eta} \frac{\left(\eta^2 - \xi^2\right)^{2\delta}}{\left(\eta^2 - t^2\right)^{\delta}(z^2 - \xi^2)^{\delta}} F\left(\delta, \delta, 1; \frac{\left(t^2 - \xi^2\right)\left(\eta^2 - z^2\right)}{\left(\eta^2 - t^2\right)\left(z^2 - \xi^2\right)}\right) dz, \tag{20}$$

where

$$k_1 = \frac{\Gamma(2\delta)}{\Gamma^2(\delta)}, \quad k_2 = \frac{\Gamma(1-2\delta)}{2^{1-2\delta}\Gamma^2(1-\delta)}\left(\frac{m+2}{2}\right)^{-2\delta}.$$

Let us find the relation between $\tau^-(x)$ and $\nu^-(x)$ transferred from the hyperbolic part Ω^- to the line $y = 0$. By using condition (16) and taking (13) into account, from (20) we obtain:

$$\tilde{\nu}^-\left(\eta^{1-2\delta}\right) = \frac{k_2\Gamma(1-\delta)}{k_1\Gamma(\delta)}\eta D_{0\,\eta^2}^{1-2\delta}\tilde{\tau}\left(\eta^{1-2\delta}\right) - \frac{2\Gamma(1-\delta)}{k_1}\eta^{2\delta}D_{0\,\eta^2}^{1-\delta}\int_0^\eta \sum_{k=1}^n \frac{q_k}{\Gamma(\gamma_k)}dt\int_0^{t^2}(t^2-s)^{\gamma_k-1}\tilde{\tau}(s^{1/2-\delta})ds\times$$

$$\int_t^\eta \frac{(\eta^2)^{2\delta}}{(\eta^2-t^2)^\delta(z^2)^\delta}F\left(\delta,\delta,1;\frac{t^2(\eta^2-z^2)}{z^2(\eta^2-t^2)}\right)dz + \frac{2\Gamma(1-\delta)}{k_1}\eta^{2\delta}D_{0\,\eta^2}^{1-\delta}h\left[\left(\frac{\eta}{4}\right)^{\frac{2}{m+2}}\right].$$

Moreover, replacing $\eta^2 \sim \eta_1$, $t^2 \sim t$, $z^2 \sim z$ we have:

$$\tilde{\nu}^-(\eta) = \frac{k_2\Gamma(1-\delta)}{k_1\Gamma(\delta)}\eta_1^{1/2}D_{0\,\eta_1}^{1-2\delta}\tilde{\tau}(\eta) - \frac{\Gamma(1-\delta)}{2k_1}\eta_1^\delta D_{0\,\eta_1}^{1-\delta}\int_0^{\eta_1}\sum_{k=1}^n\frac{q_k t^{1/2}}{\Gamma(\gamma_k)}dt\int_0^t(t-s)^{\gamma_k-1}\tilde{\tau}(s)ds\times$$

$$\int_t^{\eta_1}\frac{\eta_1^{2\delta}}{(\eta_1-t)^\delta z^{\delta+1/2}}F\left(\delta,\delta,1;\frac{t(\eta_1-z)}{z(\eta_1-t)}\right)dz + \frac{2\Gamma(1-\delta)}{k_1}\eta_1^\delta D_{0\,\eta_1}^{1-\delta}h\left[\left(\frac{\eta_1}{4}\right)^{\frac{1}{m+2}}\right] \tag{21}$$

were

$$\tilde{\nu}^-(\eta) = \nu^-\left(\eta^{1-2\delta}\right) = \nu^-\left(\eta_1^{1/2-\delta}\right),$$

$$\tilde{\tau}(\eta) = \tau\left(\eta^{1-2\delta}\right) = \tau\left(\eta_1^{1/2-\delta}\right).$$

Considering designations (19) and $\lim_{y\to+0}y^{1-\alpha}u_y(x,y) = \nu^+(x)$, $0 < x < 1$ from gluing condition (17), we have

$$\nu^+(x) = \lambda\nu^-(x). \tag{22}$$

For further consideration, from Eq. (12) at $y \to +0$ considering (13), (22) and

$$\lim_{y\to0}D_{0y}^{\alpha-1}f(y) = \Gamma(\alpha)\lim_{y\to0}y^{1-\alpha}f(y)$$

we obtain:

$$\tau''(x) - \Gamma(\alpha)\nu^+(x) + \sum_{k=1}^n p_k I_{0x}^{-\beta_k}\tau(x) = 0. \tag{23}$$

4. Uniqueness of the solution

Theorem 1. If satisfy conditions $\lambda > 0$ and

$$0 < \alpha,\ \beta_k,\ \gamma_k < 1, \quad p_k < 0, \quad q_k < 0, \quad k = 1,2,...,n \tag{24}$$

are satisfied, then, the solution is unique.

Proof. Now, we consider the corresponding homogeneous problem $[\varphi(y) \equiv \psi(y) \equiv 0]$. Let us estimate the integral $J = \int_0^1 \tau(x)\nu^+(x)dx$.

We multiply to $\tau(x)$ equation (23) and integrate from 0 to 1:

$$\Gamma(\alpha)\int_0^1 \tau(x)\nu^+(x)dx = \int_0^1 \tau''(x)\tau(x)dx + \int_0^1 \tau(x)\sum_{k=1}^n p_k I_{0x}^{-\beta_k}\tau(x)dx. \tag{25}$$

Integrating by parts and using the relations $\tau(0) = \tau(1) = 0$, we obtain

$$J \equiv \int_0^1 \tau(x)\nu^+(x)dx = -\int_0^1 (\tau'(x))^2 dx + \sum_{k=1}^n \frac{p_k}{\Gamma(\beta_k)}\int_0^1 \tau(x)dx\int_0^x (x-t)^{\beta_k-1}\tau(t)dt.$$

By using the formula (see [8], p. 188):

$$|x - t|^{-\gamma} = \frac{1}{\Gamma(\gamma) \cos \frac{\pi \gamma}{2}} \int\limits_0^\infty z^{\gamma-1} \cos\left[z\left(x - t\right)\right] dz, \quad 0 < \gamma < 1,$$

after some simplifications, we will obtain (see [3]):

$$\sum_{k=1}^n \frac{1}{\Gamma(\beta_k)} \int\limits_0^1 \tau(x)dx \int\limits_0^x (x - t)^{\beta_k - 1}\tau(t)dt \geq 0,$$

consequently by virtue (24), we will conclude that

$$\int\limits_0^1 \tau(x)\nu^+(x)dx \leq 0. \tag{26}$$

Let us show that $\int\limits_0^1 \tau(x)\nu^-(x)dx \geq 0$ for the hyperbolic domain Ω^-. For this aim we investigate the integral (see (21)):

$$A(\eta) \equiv \eta^\delta D_{0\eta}^{1-\delta} \int\limits_0^\eta \sum_{k=1}^n \frac{q_k t^{1/2}}{\Gamma(\gamma_k)} dt \int\limits_0^t \frac{\tilde\tau(s)ds}{(t - s)^{1-\gamma_k}} \times \int\limits_t^\eta \frac{\eta^{2\delta}}{(\eta - t)^\delta z^{\delta+1/2}} F\left(\delta, \delta, 1; \frac{t(\eta - z)}{z(\eta - t)}\right) dz.$$

Entering replacement $\dfrac{t(\eta - z)}{z(\eta - t)} = \sigma$ and after some simplifications we have:

$$A(\eta) \equiv \sum_{k=1}^n \frac{q_k}{\Gamma(\gamma_k)\Gamma(\delta)} \eta^\delta \frac{d}{d\eta} \int\limits_0^\eta (\eta - \mu)^{\delta-1} d\mu \int\limits_0^\mu t^{1/2} dt \int\limits_0^t (t - s)^{\gamma_k-1}\tilde\tau(s)ds \times$$

$$\int\limits_0^1 \frac{\mu^{\delta+1/2}[\sigma(\mu - t) + t]^{\delta-3/2}}{(\mu - t)^{\delta-1} t^{\delta-1/2}} F(\delta, \delta, 1; \sigma)\, d\sigma =$$

$$\sum_{k=1}^n \frac{q_k}{\Gamma(\gamma_k)\Gamma(\delta)} \eta^\delta \frac{d}{d\eta} \eta^{2+\delta} \int\limits_0^1 \frac{\tilde\mu^{\delta+1/2} d\tilde\mu}{(1 - \tilde\mu)^{1-\delta}} \int\limits_0^{\tilde\mu} \frac{\tilde t^{1-\delta}}{(\tilde\mu - \tilde t)^{\delta-1}} d\tilde t \int\limits_0^{\tilde t\eta} \frac{\tilde\tau(s)ds}{(\tilde t\eta - s)^{\gamma_k-1}} \times$$

$$\int\limits_0^1 \left[\sigma(\tilde\mu - \tilde t) + \tilde t\right]^{\delta-3/2} F(\delta, \delta, 1; \sigma)\, d\sigma,$$

where

$$\tilde\mu = \mu/\eta, \quad \tilde t = t/\eta.$$

Further, taking $\tau(0) = \varphi(0)$ into account and integrating by parts on third integral we will obtain:

$$A(\eta) \equiv \varphi(0) \sum_{k=1}^n \frac{q_k(2 + \delta + \gamma_k)}{\Gamma(\gamma_k + 1)\Gamma(\delta)} \eta^{1+2\delta+\gamma_k} \int\limits_0^1 \frac{\tilde\mu^{\delta+1/2} d\tilde\mu}{(1 - \tilde\mu)^{1-\delta}} \int\limits_0^{\tilde\mu} \frac{\tilde t^{1+\gamma_k-\delta}}{(\tilde\mu - \tilde t)^{\delta-1}} d\tilde t \times \int\limits_0^1 \left[\sigma(\tilde\mu - \tilde t) + \tilde t\right]^{\delta-3/2} F(\delta, \delta, 1; \sigma)\, d\sigma +$$

$$\sum_{k=1}^n \frac{q_k(2 + \delta)}{\Gamma(\gamma_k + 1)\Gamma(\delta)} \eta^{1+2\delta} \int\limits_0^1 \frac{\tilde\mu^{\delta+1/2} d\tilde\mu}{(1 - \tilde\mu)^{1-\delta}} \int\limits_0^{\tilde\mu} \frac{\tilde t^{1-\delta}}{(\tilde\mu - \tilde t)^{\delta-1}} d\tilde t \int\limits_0^{\tilde t\eta} (\tilde t\eta - s)^{\gamma_k}\tilde\tau'(s)ds \times \int\limits_0^1 \left[\sigma(\tilde\mu - \tilde t) + \tilde t\right]^{\delta-3/2} F(\delta, \delta, 1; \sigma)\, d\sigma +$$

$$\sum_{k=1}^n \frac{q_k}{\Gamma(\gamma_k)\Gamma(\delta)} \eta^{2+\delta} \int\limits_0^1 \frac{\tilde\mu^{\delta+1/2} d\tilde\mu}{(1 - \tilde\mu)^{1-\delta}} \int\limits_0^{\tilde\mu} \tilde t^{2-\delta}(\tilde\mu - \tilde t)^{1-\delta} d\tilde t \int\limits_0^{\tilde t\eta} (\tilde t\eta - s)^{\gamma_k-1}\tilde\tau'(s)ds \times \int\limits_0^1 \left[\sigma(\tilde\mu - \tilde t) + \tilde t\right]^{\delta-3/2} F(\delta, \delta, 1; \sigma)\, d\sigma.$$

$$\tag{27}$$

By changing the order of integration in the last two terms in (27), we obtain:

$$A(\eta) \equiv \varphi(0) \sum_{k=1}^{n} \frac{q_k(2+\delta+\gamma_k)}{\Gamma(\gamma_k+1)\Gamma(\delta)} \eta^{1+2\delta+\gamma_k} \int_0^1 \frac{\tilde{\mu}^{\delta+1/2}d\tilde{\mu}}{(1-\tilde{\mu})^{1-\delta}} \int_0^{\tilde{\mu}} \frac{\tilde{t}^{1+\gamma_k-\delta}d\tilde{t}}{(\tilde{\mu}-\tilde{t})^{\delta-1}} \times \int_0^1 \left[\sigma(\tilde{\mu}-\tilde{t})+\tilde{t}\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma +$$

$$\sum_{k=1}^{n} \frac{q_k(2+\delta)}{\Gamma(\gamma_k+1)\Gamma(\delta)} \eta^{1+2\delta} \int_0^{\eta} \tilde{\tau}'(s)ds \int_{s/\eta}^1 \frac{\tilde{\mu}^{\delta+1/2}d\tilde{\mu}}{(1-\tilde{\mu})^{1-\delta}} \int_{s/\eta}^{\tilde{\mu}} \frac{\tilde{t}^{1-\delta}(\tilde{t}\eta-s)^{\gamma_k}d\tilde{t}}{(\tilde{\mu}-\tilde{t})^{\delta-1}} \times \int_0^1 \left[\sigma(\tilde{\mu}-\tilde{t})+\tilde{t}\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma +$$

$$\sum_{k=1}^{n} \frac{q_k}{\Gamma(\gamma_k)\Gamma(\delta)} \eta^{2+2\delta} \int_0^{\eta} \tilde{\tau}'(s)ds \int_{s/\eta}^1 \frac{\tilde{\mu}^{\delta+1/2}d\tilde{\mu}}{(1-\tilde{\mu})^{1-\delta}} \int_{s/\eta}^{\tilde{\mu}} \tilde{t}^{2-\delta}(\tilde{\mu}-\tilde{t})^{1-\delta}(\tilde{t}\eta-s)^{\gamma_k-1}d\tilde{t} \times$$

$$\int_0^1 \left[\sigma(\tilde{\mu}-\tilde{t})+\tilde{t}\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma.$$

Consequently, using inverse replacements $\mu = \tilde{\mu}\eta$ and $t = \tilde{t}\eta$, we obtain:

$$A(\eta) \equiv \varphi(0) \sum_{k=1}^{n} \frac{q_k(2+\delta+\gamma_k)}{\Gamma(\gamma_k+1)\Gamma(\delta)} \eta^{\delta-1} \int_0^{\eta} \frac{\mu^{\delta+1/2}d\mu}{(\eta-\mu)^{1-\delta}} \int_0^{\mu} \frac{t^{1+\gamma_k-\delta}dt}{(\mu-t)^{\delta-1}} \times \int_0^1 \left[\sigma(\mu-t)+t\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma +$$

$$\sum_{k=1}^{n} \frac{q_k(2+\delta)}{\Gamma(\gamma_k+1)\Gamma(\delta)} \eta^{\delta-1} \int_0^{\eta} \tilde{\tau}'(s)ds \int_s^{\eta} \frac{\mu^{\delta+1/2}d\mu}{(\eta-\mu)^{1-\delta}} \int_s^{\mu} \frac{t^{1-\delta}(t-s)^{\gamma_k}dt}{(\mu-t)^{\delta-1}} \times \int_0^1 \left[\sigma(\mu-t)+t\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma +$$

$$\sum_{k=1}^{n} \frac{q_k}{\Gamma(\gamma_k)\Gamma(\delta)} \eta^{\delta-1} \int_0^{\eta} \tilde{\tau}'(s)ds \int_s^{\eta} \frac{\mu^{\delta+1/2}d\mu}{(\eta-\mu)^{1-\delta}} \int_s^{\mu} \frac{t^{2-\delta}(t-s)^{\gamma_k-1}}{(\mu-t)^{\delta-1}}dt \times \int_0^1 \left[\sigma(\mu-t)+t\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma. \quad (28)$$

There, the following preliminary assertion holds.

Lemma 1. If a function $\tau(x)$ has a positive maximum (respectively a negative minimum) at the point $x = x_0 \in (0,1)$, then $\nu^-(x_0) > 0$ (respectively $\nu^-(x_0) < 0$) at $q_k < 0$, $(k = \overline{1,n})$.

Proof. Let function $\tau(x)$ have a positive maximum at the point $x = x_0 \in (0,1)$ and $\varphi(y) \equiv 0$, then, from (28), we have:

$$A(x_0) = \sum_{k=1}^{n} \frac{q_k(2+\delta)}{\Gamma(\gamma_k+1)\Gamma(\delta)} x_0^{\delta-1} \int_0^{x_0} \tilde{\tau}'(s)ds \int_s^{x_0} \frac{\mu^{\delta+1/2}d\mu}{(x_0-\mu)^{1-\delta}} \int_s^{\mu} \frac{t^{1-\delta}(t-s)^{\gamma_k}}{(\mu-t)^{\delta-1}}dt \times$$

$$\int_0^1 \left[\sigma(\mu-t)+t\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma +$$

$$\sum_{k=1}^{n} \frac{q_k}{\Gamma(\gamma_k)\Gamma(\delta)} x_0^{\delta-1} \int_0^{x_0} \tilde{\tau}'(s)ds \int_s^{x_0} \frac{\mu^{\delta+1/2}d\mu}{(x_0-\mu)^{1-\delta}} \int_s^{\mu} \frac{t^{2-\delta}(t-s)^{\gamma_k-1}dt}{(\mu-t)^{\delta-1}} \times$$

$$\int_0^1 \left[\sigma(\mu-t)+t\right]^{\delta-3/2} F(\delta,\delta,1;\sigma)\, d\sigma.$$

From here, due to $\Gamma(\gamma_k) > 0$, $\Gamma(\delta) > 0$, $q_k < 0$, $(k = \overline{1,n})$, $F(\delta,\delta,1;\sigma) > 0$ and

$$\int_0^{x_0} \tilde{\tau}'(s)ds = \int_0^{x_0} \lim_{x_0 \to s} \frac{\tilde{\tau}(x_0)-\tilde{\tau}(s)}{x_0-s}ds > 0$$

we deduce that $A(x_0) < 0$.

In fact, if $\tau(x)$ has a positive maximum at the point $x = x_0$ then $D_{0\,x_0}^{1-2\delta}\tilde{\tau}(x_0) > 0$ (see [19]). Thus, owing to $k_1, k_2 > 0$ from (21) at $h(x) \equiv 0$ on the point of positive maximum, we will also deduce:

$$\tilde{\nu}^-(x_0) = \frac{k_2\Gamma(1-\delta)}{k_1\Gamma(\delta)}x_0^{1/2}D_{0\,x_0}^{1-2\delta}\tilde{\tau}(x_0) - \frac{\Gamma(1-\delta)}{2k_1}A(x_0) > 0.$$

Similarly, we can prove that on the point of negative minimum $\tilde{\nu}^-(x_0) < 0$. The lemma is proved.

Hence, based on the Lemma, we will deduce that

$$\int_0^1 \tau(x)\nu^-(x)dx \geq 0. \tag{29}$$

Thus, due to conditions of the Theorem 1 from (29) and (26) we infer that $\tau(x) \equiv 0$. Consequently, from (21), we will obtain $\nu^-(x) \equiv 0$.

As a conclusion, based on the solution of the first boundary problem for the Eq.(12), [15], [20] owing to account (14) and (15) we will get $u(x, y) \equiv 0$ in $\overline{\Omega}^+$, similarly, based on the solution (20) we obtain $u(x, y) \equiv 0$ in closed domain $\overline{\Omega}^-$.

5. The existence of solution

Theorem 2. If all conditions of Theorem 1 are satisfied and

$$\varphi(y),\ \psi(y) \in C\left(\overline{I_2}\right) \cap C^1(I_2)\,; \quad h(x) \in C^1\left(\overline{I_1}\right) \cap C^2(I_1) \tag{30}$$

then the solution of the investigating problem is exist.

Proof. Taking (22) into account from Eq.(23) we will obtain

$$\tau''(x) = f(x) \tag{31}$$

where

$$f(x) = \lambda\Gamma(\alpha)\nu^-(x) - \varphi(0)\sum_{k=1}^n \frac{x^{\beta_k}p_k}{\Gamma(\beta_k)} - \sum_{k=1}^n \frac{p_k}{\Gamma(\beta_k)}\int_0^x (x-t)^{\beta_k}\tau'(t)dt. \tag{32}$$

Solution of equation (31) together with conditions $\tau(0) = \psi(0)$, $\tau(1) = \varphi(0)$ has the form:

$$\tau(x) = \int_0^x (x-t)f(t)dt - x\int_0^1 (1-t)f(t)dt + \varphi(0)(1-x) + x\psi(0),$$

consequently, we can find:

$$\tau'(x) = \int_0^x f(t)dt - \int_0^1 (1-t)f(t)dt + \psi(0) - \varphi(0). \tag{33}$$

Taking (13) and (28) into account from (21) we will obtain:

$$\nu^-(x) = \tilde{k} \cdot x^{\omega/2}\int_0^x (x^\omega - t^\omega)^{2\delta-1}\tau'(t)\,dt -$$

$$\sum_{k=1}^n q_k^1 x^{\frac{\delta-1}{1-2\delta}}\int_0^x \tau'(s)ds \int_s^x (x^\omega - \mu^\omega)^{\delta-1}\mu^{\frac{6\delta+1}{2(1-2\delta)}}d\mu \int_s^\mu t^{\frac{1+\delta}{1-2\delta}}(\mu^\omega - t^\omega)^{1-\delta}\times$$

$$(t^\omega - s^\omega)^{\gamma_k}dt \int_0^1 [\sigma(\mu^\omega - t^\omega) + t^\omega]^{\delta-3/2}F(\delta, \delta, 1; \sigma)\,d\sigma -$$

$$\sum_{k=1}^n q_k^2 x^{\frac{\delta-1}{1-2\delta}}\int_0^x \tau'(s)ds \int_s^x \left(x^{\frac{1}{1-2\delta}} - \mu^{\frac{1}{1-2\delta}}\right)^{\delta-1}\mu^{\frac{6\delta+1}{2(1-2\delta)}}d\mu \times$$

$$\int\limits_{s}^{\mu} t^{\frac{2+\delta}{1-2\delta}}(\mu^{\omega}-t^{\omega})^{1-\delta}(t^{\omega}-s^{\omega})^{\gamma_k-1}dt\int\limits_{0}^{1}[\sigma(\mu^{\omega}-t^{\omega})+t^{\omega}]^{\delta-3/2}F(\delta,\delta,1;\sigma)\,d\sigma-$$

$$\sum_{k=1}^{n}q_k^3 x^{\frac{\delta-1}{1-2\delta}}\int\limits_{0}^{x}(x^{\omega}-\mu^{\omega})^{\delta-1}\mu^{\frac{6\delta+1}{2(1-2\delta)}}d\mu\int\limits_{0}^{\mu}t^{\frac{1+\gamma_k+\delta}{1-2\delta}}(\mu^{\omega}-t^{\omega})^{\delta-1}dt\times$$

$$\int\limits_{0}^{1}[\sigma(\mu^{\omega}-t^{\omega})+t^{\omega}]^{\delta-3/2}F(\delta,\delta,1;\sigma)\,d\sigma+\frac{k_2\Gamma(1-\delta)}{k_1\Gamma(\delta)\Gamma(2\delta)}x^{\frac{4\delta-1}{2(1-2\delta)}}\varphi(0),\tag{34}$$

where

$$\omega=\frac{1}{1-2\delta},\quad \tilde{k}=\frac{\omega k_2\Gamma(1-\delta)}{k_1\Gamma(\delta)\Gamma(2\delta)},\quad q_k^1=\frac{(2+\delta)\Gamma(1-\delta)q_k\omega^3}{2k_1\Gamma(\gamma_k+1)\Gamma(\delta)},$$

$$q_k^2=\frac{\Gamma(1-\delta)q_k\omega^3}{2k_1\Gamma(\gamma_k)\Gamma(\delta)},\quad q_k^3=\frac{\Gamma(1-\delta)(2+\delta+\gamma_k)q_k\omega^2}{2k_1\Gamma(\delta)\Gamma(\gamma_k+1)}\varphi(0).$$

Further, considering (32) from (33), after some simplifications we will obtain:

$$\tau'(x)=\lambda\Gamma(\alpha)\int\limits_{0}^{x}\nu^{-}(t)dt-\lambda\Gamma(\alpha)\int\limits_{0}^{1}(1-t)\nu^{-}(t)dt-\sum_{k=1}^{n}\frac{p_k}{\Gamma(\beta_k)}\int\limits_{0}^{x}dt\int\limits_{0}^{t}(t-s)^{\beta_k}\tau'(s)ds+$$

$$\sum_{k=1}^{n}\frac{p_k}{\Gamma(\beta_k)}\int\limits_{0}^{1}(1-t)dt\int\limits_{0}^{t}(t-s)^{\beta_k}\tau'(s)ds-\varphi(0)\sum_{k=1}^{n}\frac{x^{\beta_k+1}p_k}{(\beta_k+1)\Gamma(\beta_k)}+$$

$$\varphi(0)\sum_{k=1}^{n}\frac{p_k}{(\beta_k+1)(\beta_k+2)\Gamma(\beta_k)}+\psi(0)-\varphi(0).\tag{35}$$

Substituting (34) into (35), we have:

$$\tau'(x)=\tilde{k}\lambda\Gamma(\alpha)\cdot\int\limits_{0}^{x}t^{\frac{3-4\delta}{2(1-2\delta)}}dt\int\limits_{0}^{t}(t^{\omega}-s^{\omega})^{2\delta-1}\tau'(s)\,ds-\tilde{k}\lambda\Gamma(\alpha)\cdot\int\limits_{0}^{1}(1-t)t^{\frac{1}{2(1-2\delta)}}dt\int\limits_{0}^{t}(t^{\omega}-s^{\omega})^{2\delta-1}\tau'(s)\,ds-$$

$$\lambda\Gamma(\alpha)\sum_{k=1}^{n}q_k^1\int\limits_{0}^{x}t^{\frac{-\delta}{1-2\delta}}dt\int\limits_{0}^{t}\tau'(s)A(s,t)ds+\lambda\Gamma(\alpha)\sum_{k=1}^{n}q_k^1\int\limits_{0}^{1}(1-t)t^{\frac{\delta-1}{1-2\delta}}dt\int\limits_{0}^{t}\tau'(s)A(s,t)ds-$$

$$\lambda\Gamma(\alpha)\sum_{k=1}^{n}q_k^2\int\limits_{0}^{x}t^{\frac{-\delta}{1-2\delta}}dt\int\limits_{0}^{t}\tau'(s)B(s,t)ds+\lambda\Gamma(\alpha)\sum_{k=1}^{n}q_k^2\int\limits_{0}^{1}(1-t)t^{\frac{\delta-1}{1-2\delta}}dt\int\limits_{0}^{t}\tau'(s)B(s,t)ds-$$

$$\sum_{k=1}^{n}\frac{p_k}{\Gamma(\beta_k)}\int\limits_{0}^{x}dt\int\limits_{0}^{t}(t-s)^{\beta_k}\tau'(s)ds+\sum_{k=1}^{n}\frac{p_k}{\Gamma(\beta_k)}\int\limits_{0}^{1}(1-t)dt\int\limits_{0}^{t}(t-s)^{\beta_k}\tau'(s)ds+F(x),\tag{36}$$

where

$$F(x)=\varphi(0)\sum_{k=1}^{n}\left(\frac{p_k}{(\beta_k+1)(\beta_k+2)\Gamma(\beta_k)}-\frac{x^{\beta_k+1}p_k}{(\beta_k+1)\Gamma(\beta_k)}\right)-\lambda\Gamma(\alpha)\sum_{k=1}^{n}q_k^3\int\limits_{0}^{x}t^{\frac{\delta-1}{1-2\delta}}C(t)dt+$$

$$\lambda\Gamma(\alpha)\sum_{k=1}^{n}q_k^3\int\limits_{0}^{1}(1-t)t^{\frac{\delta-1}{1-2\delta}}C(t)dt-\frac{k_2\lambda\Gamma(\alpha)\Gamma(1-\delta)}{k_1\Gamma(\delta)\Gamma(2\delta)}\varphi(0)\int\limits_{0}^{1}(1-t)t^{\frac{4\delta-1}{2(1-2\delta)}}dt+$$

$$\frac{2(1-2\delta)k_2\lambda\Gamma(\alpha)\Gamma(1-\delta)}{k_1\Gamma(\delta)\Gamma(2\delta)}\varphi(0)x^{\frac{\omega}{2}}-\psi(0)-\varphi(0),\tag{37}$$

$$A(s,t)=\int\limits_{s}^{t}(t^{\omega}-\mu^{\omega})^{\delta-1}\mu^{\frac{6\delta+1}{2(1-2\delta)}}d\mu\int\limits_{s}^{\mu}z^{\frac{1+\delta}{1-2\delta}}(\mu^{\omega}-z^{\omega})^{1-\delta}\times$$

$$(z^\omega - s^\omega)^{\gamma_k} dz \int\limits_0^1 [\sigma(\mu^\omega - z^\omega) + z^\omega]^{\delta - 3/2} F(\delta, \delta, 1; \sigma) \, d\sigma, \tag{38}$$

$$B(s,t) = \int\limits_s^t (t^\omega - \mu^\omega)^{\delta - 1} \mu^{\frac{6\delta + 1}{2(1 - 2\delta)}} d\mu \int\limits_s^\mu z^{\frac{2+\delta}{1-2\delta}} (\mu^\omega - z^\omega)^{1-\delta} (z^\omega - s^\omega)^{\gamma_k - 1} dz \times$$

$$\int\limits_0^1 [\sigma(\mu^\omega - z^\omega) + z^\omega]^{\delta - 3/2} F(\delta, \delta, 1; \sigma) \, d\sigma, \tag{39}$$

$$C(t) = \int\limits_0^t (t^\omega - \mu^\omega)^{\delta - 1} \mu^{\frac{6\delta + 1}{2(1 - 2\delta)}} d\mu \int\limits_0^\mu z^{\frac{1 + \gamma_k + \delta}{1 - 2\delta}} (\mu^\omega - z^\omega)^{\delta - 1} dz \times$$

$$\int\limits_0^1 [\sigma(\mu^\omega - z^\omega) + z^\omega]^{\delta - 3/2} F(\delta, \delta, 1; \sigma) \, d\sigma. \tag{40}$$

Changing the order of integration in (36), in total, we have integral equation:

$$\tau'(x) = \int\limits_0^1 K(x,s)\tau'(s) ds + F(x). \tag{41}$$

Here

$$K(x,s) = \begin{cases} K_1(x,s); & 0 \le s \le x; \\ K_2(x,s); & x \le s \le 1, \end{cases} \tag{42}$$

$$K_1(x,s) = \lambda \Gamma(\alpha) \int\limits_s^1 (1-t) t^{\frac{\delta-1}{1-2\delta}} \left[\sum_{k=1}^n q_k^1 A(s,t) + \sum_{k=1}^n q_k^2 B(s,t) \right] dt +$$

$$\lambda \tilde{k} \Gamma(\alpha) \left[\int\limits_s^x t^{\frac{3-4\delta}{2-4\delta}} (t^\omega - s^\omega)^{2\delta - 1} dt - \int\limits_x^1 (1-t) t^{\frac{1}{2-4\delta}} (t^\omega - s^\omega)^{2\delta - 1} dt \right] -$$

$$\lambda \Gamma(\alpha) \int\limits_s^x t^{\frac{-\delta}{1-2\delta}} \left[\sum_{k=1}^n q_k^1 A(s,t) + \sum_{k=1}^n q_k^2 B(s,t) \right] dt +$$

$$\sum_{k=1}^n \frac{p_k}{\Gamma(\beta_k)} \left[\int\limits_s^1 (1-t)(t-s)^{\beta_k} dt + \frac{(x-s)^{\beta_k + 1}}{\beta_k + 1} \right], \tag{43}$$

$$K_2(x,s) = \lambda \Gamma(\alpha) \int\limits_s^1 (1-t) t^{\frac{\delta-1}{1-2\delta}} \left[\sum_{k=1}^n q_k^1 A(s,t) + \sum_{k=1}^n q_k^2 B(s,t) \right] dt -$$

$$\lambda \tilde{k} \Gamma(\alpha) \int\limits_s^1 (1-t) t^{\frac{1}{2-4\delta}} (t^\omega - s^\omega)^{2\delta - 1} dt + \sum_{k=1}^n \frac{p_k}{\Gamma(\beta_k)} \int\limits_s^1 (1-t)(t-s)^{\beta_k} dt. \tag{44}$$

In fact, the estimates of functions $A(s,t)$, $B(s,t)$ and $C(t)$ play an important roles for estimating $K(x,s)$. Now, we estimate $|A(s,t)|$. Due to properties of the hypergeometric function (9) from (38), we obtain:

$$|A(s,t)| \le \left| \int\limits_s^t (t^\rho - \mu^\rho)^{\delta - 1} \mu^{\frac{4\delta}{2(1-2\delta)}} d\mu \int\limits_s^\mu z^{\frac{1+\delta}{1-2\delta}} \frac{(z^\rho - s^\rho)^{\gamma_k}}{(\mu^\rho - z^\rho)^\delta} \left| z^{1/2} - \mu^{1/2} \right| dz \right|.$$

Introducing the change of variables $z^\rho = s^\rho + (\mu^\rho - s^\rho) z_1$ and due to (8), we get

$$|A(s,t)| \le \left| t^{1/2} - s^{1/2} \right| s^{\frac{1}{2-4\delta}} \left| \int\limits_s^t \frac{(\mu^\rho - s^\rho)^{1-\delta+\gamma_k}}{(t^\rho - \mu^\rho)^{1-\delta}} \mu^{\frac{4\delta}{1-2\delta}} F\left(-\frac{1}{2}, 1 + \gamma_k, 2 + \gamma_k - \delta; \frac{s-\mu}{s} \right) d\mu \right|.$$

In the sequel, due to

$$2 + \gamma_k - \delta - 1 - \gamma_k + \frac{1}{2} = \frac{3}{2} - \delta > 0,$$

$$\left| \frac{s - \mu}{s} \right| \le 1,$$

we have (see (9))

$$F\left(-\frac{1}{2}, 1 + \gamma_k, 2 + \gamma_k - \delta; \frac{s - \mu}{s}\right) \le const$$

consequently, changing of variables $\mu^\rho = s^\rho + (t^\rho - s^\rho)\mu_1$ and by virtue formula (8), after some evaluation finally, we get

$$|A(s,t)| \le \left(t^{1/2} - s^{1/2}\right) s^{\frac{1+4\delta}{2-4\delta}} (t - s)^{1+\gamma_k} \le const. \tag{45}$$

Similarly, by estimating $B(s,t)$ and $C(t)$ (see (39) and (40)), we can get

$$|B(s,t)| \le \left(t^{1/2} - s^{1/2}\right) s^{\frac{3+4\delta}{2-4\delta}} (t - s)^{\gamma_k} \le const, \quad C(t) \le const. \tag{46}$$

Hence, due to class of given functions (see (30)) and (45), (46) considering (43) and (44) from (37) and (42) respectively we deduce $|K(x,s)| \le const$, for all $0 \le x,\ s \le 1$, $|F(x)| \le const$, $0 \le x \le 1$. Since kernel $K(x,s)$ is continuous and function in right-side $F(x)$ is continuously differentiable, for the solution of integral equation (41) we can write via resolvent-kernel:

$$\tau'(x) = F(x) - \int\limits_0^1 \Re(x,s)F(s)ds, \tag{47}$$

where $\Re(x,s)$ is the resolvent-kernel of $K(x,s)$. Unknown functions $\nu^-(x)$ we will find from (34). Solution of the Problem I in the domain Ω^+ we write as follows [15,20]:

$$u(x,y) = \int\limits_0^y G_\xi(x,y,0,\eta)\psi(\eta)d\eta - \int\limits_0^y G_\xi(x,y,1,\eta)\varphi(\eta)d\eta + \int\limits_0^1 G_0(x - \xi, y)\tau(\xi)d\xi - $$

$$\int\limits_0^y \int\limits_0^1 G(x,y,0,\eta) \sum_{k=1}^n p_k I_{0\xi}^{\beta_k} \tau(\xi)d\xi d\eta.$$

Here

$$G_0(x - \xi, y) = \frac{1}{\Gamma(1 - \alpha)} \int\limits_0^y \eta^{-\alpha} G(x,y,\xi,\eta)d\eta,$$

$$G(x,y,\xi,\eta) = \frac{(y - \eta)^{\alpha/2 - 1}}{2} \sum_{n=-\infty}^{\infty} \left[e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x - \xi + 2n|}{(y - \eta)^{\alpha/2}}\right) - e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x + \xi + 2n|}{(y - \eta)^{\alpha/2}}\right) \right]$$

is the Green's function of the first boundary problem Eq. (12) in the domain Ω^+ with the Riemann–Liouville fractional differential operator instead of the Caputo ones [20],

$$e_{1,\delta}^{1,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\delta - \delta n)}$$

is the Wright type function [18]. Solution of the Problem I in the domain Ω^- will be found by the formula (20). Hence, the Theorem 2 is proved.

Acknowledgements

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