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Lyapunov operator $\mathcal{L}$ with degenerate kernel and Gibbs measures

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In this paper, we studied the fixed points of the Lyapunov operator with degenerate kernel, in which each fixed point of the operator is corresponds to a *translation-invariant* Gibbs measure with four competing interactions of models with uncountable set of spin values on the Cayley tree of order two. Also, it was proved that Lyapunov operator with degenerate kernel has at most three positive fixed points.

Keywords: Cayley tree, Gibbs measure, translation-invariant Gibbs measure, Lyupanov operator, degenerate kernel, fixed point.

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1. Introduction

The existence of Gibbs measures for a wide class of Hamiltonians was established in the groundbreaking work of Dobrushin. A complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is a difficult problem. Also, Spin systems on lattices are a large class of systems considered in statistical mechanics. Some of them have a real physical meaning, others are studied as suitably simplified models of more complicated systems [1, 2].

The Ising model is an important model in statistical mechanics. The various partial cases of Ising model have been investigated in numerous works. For example, In [3] and [4], the exact solutions of an Ising model with competing restricted interactions with zero external field was presented. Also, it was proved that there are two translation-invariant and uncountable number of distinct non-translation-invariant extreme Gibbs measures and considered Ising model with four competing interactions on the Cayley tree of order two (see [5–7]). In [8], other important results are given on a Cayley tree. Mainly, these papers are devoted to models with a finite set of spin values. In [9], the Potts model, with a countable set of spin values on a Cayley tree is considered and it was shown that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independent of the parameters for the Potts model with a countable set of spin values on the Cayley tree.

Gibbs measures have been considered for models with uncountable sets of spin values for the last five years. Until now, models with nearest-neighbor interactions have been considered (i.e., $J_3 = J = \alpha = 0,\ J_1 \neq 0$) and with the set $[0, 1]$ of spin values on a Cayley tree, we obtained the following results: "Splitting Gibbs measures" of the model on a Cayley tree of order k is described by the solutions of a nonlinear integral equation. For $k = 1$ it has been shown that the integral equation has a unique solution (i.e., there is a unique Gibbs measure). For periodic splitting Gibbs measures, a sufficient condition was found under which the measure is unique and proved the existence of phase transitions on a Cayley tree of order $k \geq 2$ (see [10–13]).

In [14] it was described splitting Gibbs measures on Γ_2 were described by solutions to a nonlinear integral equation for the case $J_3^2 + J_1^2 + J^2 + \alpha^2 \neq 0$ which is a generalization of the case $J_3 = J = \alpha = 0,\ J_1 \neq 0$. Also, it was proved that periodic Gibbs measure for the Hamiltonian with four competing interactions is either *translation-invariant* or $G_k^{(2)}$*- periodic*, and given examples of non-uniqueness for Hamiltonian (2.1) in the case $J_3 \neq 0,\ J = J_1 = \alpha = 0$. Gibbs measures for the Hamiltonian which corresponds to the degenerate kernel was not considered in the paper.

In this paper, we provide a connection between Gibbs measures for the model which is defined in [14] and positive solutions of the Lyupanov integral equations. Also we study the fixed points of the Lyapunov operator with degenerate kernel. Using each fixed point for the operator, the *translation-invariant* Gibbs measure for the Hamiltonian can be founded which corresponds to the degenerate kernel.

2. Preliminaries

A Cayley tree $\Gamma^k = (V, L)$ of order $k \in \mathbb{N}$ is an infinitely homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertex. Here, V is the set of vertices and L that of edges (arcs). Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y\rangle$. The distance $d(x, y)$, x, $y \in V$ on the Cayley tree is defined by the formula

$$d(x,y) = \min\{d |\ x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \quad \text{such that the pairs}$$
$$\langle x_0, x_1\rangle, ..., \langle x_{d-1}, x_d\rangle \quad \text{are neighboring vertices}\}.$$

Let $x^0 \in V$ be a fixed and we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \le n\},$$
$$L_n = \{l =< x, y >\in L \mid x, y \in V_n\},$$

The set of the direct successors of x is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1} |\ d(x, y) = 1\},\ x \in W_n.$$

We observe that for any vertex $x \neq x^0$, x has k direct successors and x^0 has $k+1$. The vertices x and y are called second neighbor which is denoted by $\rangle x, y\langle$, if there exist a vertex $z \in V$ such that x, z and y, z are nearest neighbors. We will consider only second neighbors $\rangle x, y\langle$, for which there exist n such that $x, y \in W_n$. Three vertices x, y and z are called a triple of neighbors and they are denoted by $\langle x, y, z\rangle$, if $\langle x, y\rangle$, $\langle y, z\rangle$ are nearest neighbors and $x,\ z \in W_n$, $y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Now we consider models with four competing interactions where the spin takes values in the set $[0, 1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \to [0, 1]$ is called a configuration and the set of all configurations on A we denote by $\Omega_A = [0, 1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi_1 : (t, u, v) \in [0, 1]^3 \to \xi_1(t, u, v) \in R$, $\xi_i : (u, v) \in [0, 1]^2 \to \xi_i(u, v) \in R$, $i \in \{2, 3\}$ are given bounded, measurable functions. Next, we consider the model with four competing interactions on the Cayley tree which is defined by following Hamiltonian

$$H(\sigma) = \\ - J_3 \sum_{<x,y,z>} \xi_1(\sigma(x), \sigma(y), \sigma(z)) - J \sum_{>x,y<} \xi_2(\sigma(x), \sigma(z)) - J_1 \sum_{<x,y>} \xi_3(\sigma(x), \sigma(y)) - \alpha \sum_{x\in V} \sigma(x), \quad (2.1)$$

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors and $J, J_1, J_3, \alpha \in R \setminus \{0\}$. Let $h : [0, 1] \times V \setminus \{x^0\} \to \mathbb{R}$ and $|h(t, x)| = |h_{t,x}| < C$ where x_0 is a root of Cayley tree and C is a constant which does not depend on t. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right), \quad (2.2)$$

$$Z_n = \int\limits_{\Omega^{(p)}_{V_{n-1}}}\!\!...\!\int \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda^{(p)}_{V_{n-1}}(d\widetilde{\sigma}_n), \quad (2.3)$$

where

$$\underbrace{\Omega_{W_n} \times \Omega_{W_n} \times ... \times \Omega_{W_n}}_{3\cdot 2^{p-1}} = \Omega^{(p)}_{W_n}, \quad \underbrace{\lambda_{W_n} \times \lambda_{W_n} \times ... \times \lambda_{W_n}}_{3\cdot 2^{p-1}} = \lambda^{(p)}_{W_n},\ n, p \in \mathbb{N},$$

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$\iint\limits_{\Omega_{W_n}\times\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)(\lambda_{W_n} \times \lambda_{W_n})(d\omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \quad (2.4)$$

By Kolmogorov's extension theorem there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$. The measure μ is called *splitting Gibbs measure* corresponding to Hamiltonian (2.1)

and function $x \mapsto h_x$, $x \neq x^0$.
We denote

$$K(u,t,v) = \exp\left\{J_3\beta\xi_1\left(t,u,v\right) + J\beta\xi_2\left(u,v\right) + J_1\beta\left(\xi_3\left(t,u\right) + \xi_3\left(t,v\right)\right) + \alpha\beta(u+v)\right\}, \tag{2.5}$$

and

$$f(t,x) = \exp(h_{t,x} - h_{0,x}), \quad (t,u,v) \in [0,1]^3, \ x \in V \setminus \{x^0\}.$$

The following statement describes conditions on h_x guaranteeing the compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

Theorem 2.1. *The measure $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ satisfies the consistency condition (2.4) iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$f(t,x) = \prod_{\rangle y,z\langle \in S(x)} \frac{\int_0^1 \int_0^1 K(t,u,v) f(u,y) f(v,z) du dv}{\int_0^1 \int_0^1 K(0,u,v) f(u,y) f(v,z) du dv}, \tag{2.6}$$

where $S(x) = \{y,z\}$, $\langle y,x,z\rangle$ is a ternary neighbor and $du = \lambda(du)$ is the Lebesgue measure.

3. Lyapunov's operator $\mathcal{L}$ with degenerate kernel

Now we consider the case $J_3 \neq 0$, $J = J_1 = \alpha = 0$ for the model (2.1) in the class of translational-invariant functions $f(t,x)$ i.e $f(t,x) = f(t)$, for any $x \in V$. For such functions, equation (2.1) can be written as

$$f(t) = \frac{\int_0^1\int_0^1 K(t,u,v) f(u) f(v) du dv}{\int_0^1\int_0^1 K(0,u,v) f(u) f(v) du dv}, \tag{3.1}$$

where $K(t,u,v) = \exp\left\{J_3\beta\xi_1\left(t,u,v\right) + J\beta\xi_2\left(u,v\right) + J_1\beta\left(\xi_3\left(t,u\right) + \xi_3\left(t,v\right)\right) + \alpha\beta(u+v)\right\}$, $f(t) > 0$, $t, u \in [0,1]$.

We shall find positive continuous solutions to (3.1) i.e. such that $f \in C^+[0,1] = \{f \in C[0,1] : f(x) \geq 0\}$.

We define a nonlinear operator H on the cone of positive continuous functions on $[0,1]$:

$$(Hf)(t) = \frac{\int_0^1 \int_0^1 K(t,s,u) f(s) f(u) ds du}{\int_0^1 \int_0^1 K(0,s,u) f(s) f(u) ds du}.$$

We'll study the existence of positive fixed points for the nonlinear operator H (i.e., solutions of the equation (3.1)). Put $C_0^+[0,1] = C^+[0,1] \setminus \{\theta \equiv 0\}$. Then the set $C^+[0,1]$ is the cone of positive continuous functions on $[0,1]$.

We define the Lyapunov integral operator $\mathcal{L}$ on $C[0,1]$ by the equality (see [15])

$$\mathcal{L}f(t) = \int_0^1 K(t,s,u) f(s) f(u) ds du.$$

We put

$$\mathcal{M}_0 = \left\{f \in C^+[0,1] : f(0) = 1\right\}.$$

We denote by $N_{fix.p}(H)$ and $N_{fix.p}(\mathcal{L})$ are the set of positive numbers of nontrivial positive fixed points of the operators $N_{fix.p}(H)$ and $N_{fix.p}(\mathcal{L})$, respectively.

Theorem 3.1. *[14]*
i) The equation

$$Hf = f, \quad f \in C_0^+[0,1] \tag{3.2}$$

has a positive solution iff the Lyapunov equation

$$\mathcal{L}g = \lambda g, \quad g \in C^+[0,1] \tag{3.3}$$

has a positive solution in $\mathcal{M}_0$ for some $\lambda > 0$.
ii) The equation $Hf = f$ has a nontrivial positive solution iff the Lyapunov equation $\mathcal{L}g = g$ has a nontrivial positive solution.
iii) The equation

$$\mathcal{L}f = \lambda f, \quad \lambda > 0 \tag{3.4}$$

has at least one solution in $C_0^+[0,1]$.
iv) The equation (3.2) has at least one solution in $C_0^+[0,1]$.

v) The equality $N_{fix.p}(H) = N_{fix.p}(\mathcal{L})$ *holds.*

Let $\varphi_1(t),\ \ \varphi_2(t)$ and $\psi_1(t),\ \ \psi_2(t)$ are positive functions from $C_0^+[0,1]$. We consider Lyapunov's operator $\mathcal{L}$

$$(\mathcal{L}f)(t) = \int_0^1 (\psi_1(t)\varphi_1(u) + \psi_2(t)\varphi_2(v))f(u)f(v)dudv, \tag{3.5}$$

and quadratic operator P on $\mathbb{R}^2$ by the rule

$$P(x,y) = (\alpha_{11}x^2 + \alpha_{12}xy + \alpha_{22}y^2, \quad \beta_{11}x^2 + \beta_{12}xy + \beta_{22}y^2).$$

$$\alpha_{11} = \int_0^1\int_0^1 \psi_1(u)\psi_1(v)\varphi_2(v)dudv, \qquad \alpha_{12} = \int_0^1\int_0^1 (\psi_1(v)\psi_2(u) + \psi_1(u)\psi_2(v))\varphi_2(v)dudv,$$

$$\alpha_{22} = \int_0^1\int_0^1 \psi_2(u)\psi_2(v)\varphi_2(v)dudv, \qquad \beta_{11} = \int_0^1\int_0^1 \psi_1(u)\psi_1(v)\varphi_1(u)dudv,$$

$$\beta_{12} = \int_0^1\int_0^1 (\psi_1(u)\psi_2(v) + \psi_1(v)\psi_2(u))\varphi_1(u)dudv, \quad \beta_{22} = \int_0^1\int_0^1 \psi_2(u)\psi_2(v)\varphi_1(u)dudv.$$

Lemma 3.2. *Lyapunov's operator* $\mathcal{L}$ *has a nontrivial positive fixed point iff the quadratic operator* P *has a nontrivial positive fixed point, moreover* $N^+_{fix}(H_k) = N^+_{fix}(P)$.

Proof. a) Put

$$\mathbb{R}_2^+ = \{(x,y) \in \mathbb{R}^2 :\ \ x \geq 0, y \geq 0\},\ \mathbb{R}_2^> = \{(x,y) \in \mathbb{R}^2 :\ \ x > 0, y > 0\}.$$

Let $f(t) \in C_0^+[0,1]$ be a nontrivial positive fixed point of $\mathcal{L}$. Let

$$c_1 = \int_0^1 \varphi_1(u)f(u)f(v)dudv, \quad c_2 = \int_0^1 \varphi_2(u)f(u)f(v)dudv$$

Clearly, $c_1 > 0$, $c_2 > 0$ and $f(t) = c_1\psi_1(t) + c_2\psi_2(t)$. If we put $f(t) = c_1\psi_1(t) + c_2\psi_2(t)$ to the equation (3.5) we'll get

$$c_1 = \alpha_{11}c_1^2 + \alpha_{12}c_1c_2 + \alpha_{22}c_2^2, \quad c_2 = \beta_{11}c_1^2 + \beta_{12}c_1c_2 + \beta_{22}c_2^2.$$

Therefore, the point (c_1, c_2) is fixed point of the quadratic operator P.

b) Assume, that the point (x_0, y_0) is a nontrivial positive fixed point of the quadratic operator P, i.e. $(x_0, y_0) \in \mathbb{R}_2^+ \setminus \{\theta\}$ and numbers x_0, y_0 satisfies following equalities

$$\alpha_{11}x_0^2 + \alpha_{12}x_0y_0 + \alpha_{22}y_0^2 = x_0, \quad \beta_{11}x_0^2 + \beta_{12}x_0y_0 + \beta_{22}y_0^2 = y_0.$$

Similarly, we can prove that the function $f_0(t) = x_0\psi_1(t) + y_0\psi_2(t)$ is a fixed point of the operator $\mathcal{L}$ and $f_0(t) \in C_0^+[0,1]$. This completes the proof. □

We define positive quadratic operator $\mathcal{Q}$:

$$\mathcal{Q}(x,y) = (a_{11}x^2 + a_{12}xy + a_{22}y^2,\ \ b_{11}x^2 + b_{12}xy + b_{22}y^2).$$

Proposition 3.3.

i) If $\omega = (x_0, y_0) \in \mathbb{R}_2^+$ *is a positive fixed point of* $\mathcal{Q}$*, then* $\lambda_0 = \dfrac{x_0}{y_0}$ *is a root of the following equation*

$$a_{11}\lambda^3 + (a_{12} - b_{11})\lambda^2 + (a_{22} - b_{12})\lambda - b_{22} = 0. \tag{3.6}$$

ii) If the positive number λ_0 *is a positive root of the equation (3.6), then the point* $\omega_0 = (\lambda_0 y_0, y_0)$ *is a positive fixed point of* $\mathcal{Q}$*, where* $y_0^{-1} = a_{11} + a_{12}\lambda_0 + a_{22}\lambda_0^2$.

Proof. i) Let the point $\omega = (y_0, x_0) \in \mathbb{R}_2^+$ be a fixed point of $\mathcal{Q}$. Then

$$a_{11}x_0^2 + a_{12}x_0y_0 + a_{22}y_0^2 = x_0, \quad b_{11}x_0^2 + b_{12}x_0y_0 + b_{22}y_0^2 = y_0$$

Using the equality $\dfrac{x_0}{y_0} = \lambda_0$, we obtain

$$a_{11}\lambda_0^2y_0^2 + a_{12}\lambda_0y_0^2 + a_{22}y_0^2 = \lambda_0y_0, \quad b_{11}\lambda_0^2y_0^2 + b_{12}\lambda_0^2y_0^2 + b_{22}y_0^2 = y_0.$$

Thus we get

$$\frac{a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22}}{b_{11}\lambda_0^2 + b_{12}\lambda_0 + b_{22}} = \lambda_0.$$

Consequently,

$$a_{22} + (a_{12} - b_{22})\lambda_0 + (a_{11} - b_{12})\lambda_0^2 - b_{11}\lambda_0^3 = 0.$$

ii) Let $\lambda_0 > 0$ be a root of the cubic equation (3.6). We set $x_0 = \lambda_0 y_0$, where

$$x_0 = \frac{\lambda_0}{a_{11}\lambda_0^2 + 2a_{12}\lambda_0 + a_{22}}.$$

Since

$$a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 = \frac{1}{a_{11}\lambda_0^2 + 2a_{12}\lambda_0 + a_{22}},$$

we get

$$a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 = y_0.$$

Alternatively, we get:

$$a_{22} + (a_{12} - b_{22})\lambda_0 + (a_{11} - b_{12})\lambda_0^2 - b_{11}\lambda_0^3 = 0.$$

Then, we get

$$b_{11}\lambda_0^2 + b_{12}\lambda_0 + b_{22} = \lambda_0(a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22}).$$

From the last equality we get

$$\frac{\lambda_0}{a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22}} = \frac{b_{11}\lambda_0^2 + b_{12}\lambda_0 + b_{22}}{(a_{11}\lambda_0^2 + a_{12}\lambda_0 + a_{22})^2} = b_{11}x_0^2 + 2b_{12}x_0y_0 + b_{22}y_0^2 = y_0.$$

This completes the proof. □

We denote

$$\begin{gathered} P(\lambda) = \alpha_{11}\lambda^3 + (\alpha_{12} - \beta_{11})\lambda^2 + (\alpha_{22} - \beta_{12})\lambda - \beta_{22} = 0, \\ \mu_0 = \alpha_{11}, \quad \mu_1 = \alpha_{12} - \beta_{11}, \quad \mu_2 = \alpha_{22} - \beta_{12}, \quad \mu_3 = \beta_{22}, \\ P_3(\xi) = \mu_0\xi^3 + \mu_1\xi^2 + \mu_2\xi - \mu_3, \\ D = \mu_1^2 - 3\mu_0\mu_2,\ \alpha = -\frac{\mu_1 + \sqrt{D}}{3\mu_0}, \quad \beta = -\frac{\mu_1 - \sqrt{D}}{3\mu_0}. \end{gathered} \tag{3.7}$$

Theorem 3.4. *Let $\mathcal{Q}$ satisfy one of the following conditions*
i) $D \le 0$;
ii) $D > 0, \beta \le 0$;
iii) $D > 0, \alpha \le 0, \beta > 0$;
iv) $D > 0, \alpha > 0, P_3(\alpha) < 0$;
v) $D > 0, \alpha > 0, P_3(\alpha) > 0, P_3(\beta) > 0$, *then $\mathcal{Q}$ has a unique nontrivial positive fixed point.*

Proof. The proof of Theorem 3.4 is basis on monotonous property of the function $P_3(\xi)$. Clearly,

$$(P_3(\xi))' = 3\mu_0\xi^2 + 2\mu_1\xi + \mu_2. \tag{3.8}$$

and $P_3'(\alpha) = P_3'(\beta) = 0$. Moreover,

$i)$ In the case $D \le 0$, by the equality (3.8) the function $P_3(\xi)$ is an increasing function on $\mathbb{R}$ and $P_3(0) = -b_{11} < 0$. Therefore, the polynomial $P_3(\xi)$ has a unique positive root.

$ii)$ Let $D > 0$ and $\beta \le 0$. For the case $D > 0$ the function $P_3(\xi)$ is an increasing function on $(-\infty, \alpha) \cup (\beta, \infty)$ and it is a decreasing function on (α, β). Hence, from the inequality $P_3(0) < 0$ the polynomial $P_3(\xi)$ has a unique positive root.

$iii)$ Let $D > 0, \alpha \le 0$ and $\beta > 0$. Since the function $P_3(\xi)$ is decreasing on (α, β) and increasing on (β, ∞), the polynomial $P_3(\xi)$ has a unique positive root as $P_3(0) < 0$.

$iv)$ Let $D > 0, \alpha > 0$ and $P_3(\alpha) < 0$. Then $\max\limits_{\xi \in (-\infty, \beta)} P_3(\xi) = P_3(\alpha) < 0$. Consequently, by the function $P_3(\xi)$ is increasing on (β, ∞) the polynomial $P_3(\xi)$ has a unique positive root $\xi_0 \in (\beta, \infty)$.

$v)$ Let $D > 0, \alpha > 0, P_3(\alpha) > 0$ and $P_3(\beta) > 0$. Then $\min\limits_{\xi \in (\alpha, \infty)} P_3(\xi) = P_3(\beta) > 0$. From the function $P_3(\xi)$ on $(-\infty, \alpha)$, $P_3(\xi)$ $P_3(\xi)$ has a unique positive root $\xi_0 \in (0, \alpha)$, as $P_3(0) < 0$ and $P_3(\alpha) > 0$.

From the upper analysis and by Lemmas 3.3, it follows that the Theorem 3.4. □

Theorem 3.5. *Let be $D > 0$. If $\mathcal{Q}$ satisfies one of the following conditions*
i) $\alpha > 0,\ P_3(\alpha) = 0,\ P_3(\beta) < 0$;
ii) $\alpha > 0,\ P_3(\alpha) > 0,\ P_3(\beta) = 0$, *then QO $\mathcal{Q}$ has two nontrivial positive fixed points and* $N^{+}_{fix}(\mathcal{Q}) = N^{>}_{fix}(\mathcal{Q}) = 2$.

Proof. $i)$ Let $\alpha > 0,\ P_3(\alpha) = 0$ and $P_3(\beta) < 0$. Then $\max\limits_{\xi\in(-\infty,\beta)} P_3(\xi) = P_3(\alpha) = 0$ and $\xi_1 = \alpha$ is the root of the polynomial $P_3(\xi)$. By the increase property on (β, ∞) of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_2 \in (\beta, \infty)$, as $\beta > 0$ and $P_3(\beta) < 0$. There are no other positive roots of the polynomial $P_3(\xi)$.
$ii)$ Let $\alpha > 0, P_3(\alpha) > 0$ and $P_3(\beta) = 0$. Then by the increase property on $(-\infty, \alpha)$ of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_1 \in (0, \alpha)$. Alternatively, $\min\limits_{\xi\in(\alpha,\infty)} P_3(\xi) = P_3(\beta) = 0$ and the number $\xi_2 = \alpha$ is the second positive root of the polynomial $P_3(\xi)$. The polynomial $P_3(\xi)$ has no other roots. From above, and by Lemmas 3.3, we get Theorem 3.6. □

Theorem 3.6. *Let be $D > 0$. If $\mathcal{Q}$ satisfies one of the following conditions*
i) $\alpha > 0,\ P_3(\alpha) = 0,\ P_3(\beta) < 0$;
ii) $\alpha > 0,\ P_3(\alpha) > 0,\ P_3(\beta) = 0$,
then $\mathcal{Q}$ has two nontrivial positive fixed points and $N^{+}_{fix}(\mathcal{Q}) = N^{>}_{fix}(\mathcal{Q}) = 2$.

Proof. $i)$ Let $\alpha > 0, P_3(\alpha) = 0$ and $P_3(\beta) < 0$. Then $\max\limits_{\xi\in(-\infty,\beta)} P_3(\xi) = P_3(\alpha) = 0$ and $\xi_1 = \alpha$ is the root of the polynomial $P_3(\xi)$. By the increase property on (β, ∞) of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_2 \in (\beta, \infty)$, as $\beta > 0$ and $P_3(\beta) < 0$. There are no other positive roots of the polynomial $P_3(\xi)$.
$ii)$ Let $\alpha > 0, P_3(\alpha) > 0$ and $P_3(\beta) = 0$. Then by the increase property on $(-\infty, \alpha)$ of the function $P_3(\xi)$ the polynomial $P_3(\xi)$ has a root $\xi_1 \in (0, \alpha)$. Alternatively, $\min\limits_{\xi\in(\alpha,\infty)} P_3(\xi) = P_3(\beta) = 0$ and the number $\xi_2 = \alpha$ is the second positive root of the polynomial $P_3(\xi)$. The polynomial $P_3(\xi)$ has no other roots. From above, and by Lemmas 3.3, we get Theorem 3.6. □

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