NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2018, 9 (2), P. 215–224

Inverse dynamic problems for canonical systems and de Branges spaces

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DOI 10.17586/2220-8054-2018-9-2-215-224

We show the equivalence of inverse problems for different dynamical systems and corresponding canonical systems. For canonical system with general Hamiltonian we outline the strategy of studying the dynamic inverse problem and procedure of construction of corresponding de Branges space.

Keywords: inverse problem, Boundary Control method, de Branges spaces, Schrödinger operator, Dirac system, Jacobi matrices, canonical systems.

Received: 8 January 2018

Revised: 19 January 2018

1. Introduction

This is an accompanying paper to [1], in which the authors have shown the relationship between the de Branges method and the Boundary Control (BC) method on a basis of three dynamical systems: wave equation with a potential on a half-line, Dirac system on a half-line and dynamical system with discrete time for semi-infinite discrete Schrödinger operator. For each system, they constructed the related de Branges space using natural dynamic objects and operators, used in the BC method. In the present note, we will show the equivalence of dynamic inverse problems (IP) for different dynamical systems (wave equation, Dirac system, Jacobi matrices), and IPs for equivalent canonical systems. We note that every original system will be equivalent to canonical system with different dynamics (the dependence on t is given by one of the following operators: $\frac{d^2}{dt^2}$, $i\frac{d}{dt}$, ∂_t, where ∂_t is a difference operator).

Let $H \in L_{1,loc}(0, L; \mathbb{R}^{2\times 2})$ be a locally summable on $(0, L)$, $L \leq \infty$ matrix-valued function $H \geq 0$, called Hamiltonian, $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, vector $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$. We choose the "proper" dynamics and fix the general dynamical canonical system, the initial boundary value problem (IBVP) of which will be the subject of our interest:

$$iH\frac{dY}{dt} - J\frac{dY}{dx} = 0, \quad x \geq 0,\, t \geq 0.$$

For such a system we set up an IP and outline the strategy of solving it by the BC method, provided the Hamiltonian is smooth and strictly positive. We also provide a method of construction of the de Branges space for such a Hamiltonian in natural dynamic terms following [1].

In the second section, we expose all necessary information on de Branges spaces and canonical systems following [2] and [3]. In the third section, we deal with dynamical systems for Schrödinger operator on a half-line, wave equation on a half-line, Dirac operator on a half-line and a semi-infinite Jacobi matrices. We formulate dynamic IP for each system, then we transform IBVP for each system to the IBVP for certain canonical system, formulate IP for canonical system, and show that it is equivalent the original ones.

In the fourth section, we will show that one specific choice of dynamics give a finite speed of wave propagation in a canonical system, provided the Hamiltonian is smooth and strictly positive. We note that the finiteness of the wave propagation is important: initially the BC method was developed and applied in the case of multidimensional wave equation [4, 5] on a bounded manifold, but later on the BC method was successfully applied to parabolic and Schrödinger equations (where the speed is infinite) as well [6–8]. We provide algorithms of solving dynamic IP and construction of de Branges space for such a Hamiltonian. Based on these results, we formulate the hypothesis for constructing the de Branges space for general Hamiltonian by the dynamic method.

2. de Branges spaces

Here, we provide the information on de Branges spaces in accordance with [2, 3]. The entire function $E : \mathbb{C} \mapsto \mathbb{C}$ is called a *Hermite–Biehler function* if $|E(z)| > |E(\overline{z})|$ for $z \in \mathbb{C}_+$. We use the notation $F^{\#}(z) = \overline{F(\overline{z})}$. The *Hardy space* H_2 is defined by: $f \in H_2$ if f is holomorphic in $\mathbb{C}^+$ and $\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx < \infty$. Then the *de Branges space* $B(E)$ consists of entire functions such that:

$$B(E) := \left\{ F : \mathbb{C} \mapsto \mathbb{C},\, F \text{ entire}, \int_{\mathbb{R}} \left| \frac{F(\lambda)}{E(\lambda)} \right|^2 d\lambda < \infty,\, \frac{F}{E}, \frac{F^{\#}}{E} \in H_2 \right\}.$$

The space $B(E)$ with the scalar product:

$$[F, G]_{B(E)} = \frac{1}{\pi} \int_{\mathbb{R}} \overline{F(\lambda)} G(\lambda) \frac{d\lambda}{|E(\lambda)|^2},$$

is a Hilbert space. For any $z \in \mathbb{C}$ the *reproducing kernel* is introduced by the relation

$$J_z(\xi) := \frac{\overline{E(z)} E(\xi) - E(\overline{z}) \overline{E(\overline{\xi})}}{2i(\overline{z} - \xi)}. \tag{1}$$

Then

$$F(z) = [J_z, F]_{B(E)} = \frac{1}{\pi} \int_{\mathbb{R}} \overline{J_z(\lambda)} F(\lambda) \frac{d\lambda}{|E(\lambda)|^2}.$$

We observe that a Hermite–Biehler function $E(\lambda)$ defines J_z by (1). The converse is also true [9, 10]: a Hilbert space of analytic functions with reproducing kernel is a de Branges space (provided some nonrestrictive conditions on the set of function and on the norm hold true).

Let $H \in L_{1,loc}(0, L; R^{2\times 2})$ be a Hamiltonian and the vector $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ be solution to the following Cauchy problem:

$$\begin{gathered} -J\frac{dY}{dx} = \lambda H Y, \\ Y(0) = C, \end{gathered} \tag{2}$$

for $C \in \mathbb{R}^2$, $C \neq 0$. Without loss of generality, it is assumed that $\operatorname{tr} H(x) = 1$. Then, the function $E_x(\lambda) = Y_1(x, \lambda) + iY_2(x, \lambda)$ is a Hermite–Biehler function ($E_L(\lambda)$ makes sense if $L < \infty$), it is called de Branges function of the system (2) since one can construct de Branges space based on this function. On the other hand, E_L serves as an inverse spectral data for the canonical system (2). The solution to (2) and $Y(0) = (1, 0)^T$ is denoted by $\Theta(x, \lambda)$. The main result of the theory [3, 9] says that the opposite is also true: every Hermite–Biehler function satisfying some condition comes from some canonical system.

3. Dynamical canonical systems for wave equation, Dirac system and Jacobi system with discrete time

In this section, we use some ideas from [3] to rewrite IBVPs for different dynamical systems as IBVPs for canonical dynamical systems. Everywhere below, $T > 0$ is fixed.

3.1. Wave equation with a potential on a half-line

For a potential $q \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$, we consider the IBVP for the 1d wave equation on a half-line:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0, & x \geq 0,\ t \geq 0, \\ u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t). \end{cases} \tag{3}$$

Here, f is an arbitrary $L^2_{loc}(\mathbb{R}_+)$ function referred to as a *boundary control*. The *response operator* $R_q^T : L_2(0,T) \mapsto L_2(0,T)$ with the domain $\mathcal{D} = C_0^\infty(0,T)$ is introduced by $\left(R_q^T f\right)(t) := u_x^f(0,t)$, it plays a role of a dynamic inverse data [11–13]. The IP is to recover q on $(0,T)$ from R_q^{2T}.

We consider the solutions $y_{1,2}$ to following Cauchy problems:

$$\begin{cases} -y_{1,2}''(x) + q(x)y_{1,2}(x) = 0, & x \geq 0, \\ y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0, y_2'(0) = 1, \end{cases} \tag{4}$$

and look for the solution to (3) in the form:

$$u^f(x,t) = c^1(x,t)y_1(x) + c^2(x,t)y_2(x). \tag{5}$$

Plugging this representation to (3) yields:

$$\begin{aligned} c^1_{tt}y_1 + c^2_{tt}y_2 &= -qc^1y_1 - qc^2y_2 + c^1_{xx}y_1 + 2c^1_xy_1' + c_1y_1'' + c^2_{xx}y_2 + 2c^2_xy_2' + c_2y'' \\ &= \left(c^1_xy_1 + c^2_xy_2\right)_x + c^1_xy_1' + c^2_xy_2'. \end{aligned}$$

If we demand the equality $c^1_xy_1 + c^2_xy_2 = 0$, then unknown $c^{1,2}$ satisfies the following system:

$$\begin{cases} c^1_{tt}y_1 + c^2_{tt}y_2 = c^1_xy_1' + c^2_xy_2', \\ c^1_xy_1 + c^2_xy_2 = 0. \end{cases} \tag{6}$$

We note that due to the boundary conditions in (4) and (6), we have that:

$$u^f_x(0,t) = c^1_x(0,t)y_1(0) + c^1(0,t)y_1'(0) + c^2_x(0,t)y_2(0) + c^2(0,t)y_2'(0) = c^2(0,t).$$

On expressing $c^{1,2}_x$ from (6), and bearing in mind the equality $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = 1$, we obtain that:

$$\begin{cases} c^1_x = -c^1_{tt}y_1y_2 - c^2_{tt}y_2^2, \\ c^2_x = c^1_{tt}y_1^2 + c^2_{tt}y_1y_2. \end{cases}$$

On introducing the notations $C = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} y_1^2 & y_1y_2 \\ y_1y_2 & y_2^2 \end{pmatrix}$ and counting the initial and boundary conditions on u^f at $t = 0$ and at $x = 0$, we obtain that C satisfies the following IBVP:

$$\begin{cases} HC_{tt} - JC_x = 0, & x \geq 0,\, t \geq 0, \\ C(x,0) = 0, C_t(x,0) = 0, & x \geq 0, \\ c^1(0,t) = f(t), & t \geq 0. \end{cases} \tag{7}$$

The response operator $\widetilde{R}^T_q : L_2(0,T) \mapsto L_2(0,T)$ for (7) is introduced by the equality $\left(\widetilde{R}^T_s f\right)(t) := c^2(0,t)$. On the other hand, using (5) and second line in (6), we have that:

$$\left(R^T_q f\right)(t) := u^f_x(0,t) = c^1_x(0,t)y_1(0) + c^2(0,t)y_2'(0) = c^2(0,t) = \left(\widetilde{R}^T_s f\right)(t).$$

So we can see that IPs for (3) and for (7) are equivalent.

3.2. Wave equation on a half-line

For a smooth positive density $\rho \in C^2(\mathbb{R}_+)$, $\rho(x) \geq \delta > 0$, we consider the IBVP for a wave equation on a half-line:

$$\begin{cases} \rho(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \geq 0,\ t \geq 0, \\ u(x,0) = u_t(x,0) = 0,\ u(0,t) = f(t). \end{cases} \tag{8}$$

Where the function $f \in L^2_{loc}(\mathbb{R}_+, \mathbb{C})$ is interpreted as a *boundary control*. The *response operator* $R^T_\rho : L_2(0,T) \mapsto L_2(0,T)$ with the domain $\mathcal{D} = C_0^\infty(0,T)$ is defined by $R^T_\rho f := u^f_x(0,t)$. We introduce the *eikonal* $\tau(x) := \int_0^x \rho^{\frac{1}{2}}(s)\, ds$, from physical point of view, it is a time at which a wave initiated at $x = 0$ fills the segment $(0,x)$, let $\Omega^l = \{x > 0 \,|\, \tau(x) < l\}$. Then, the natural set up of IP is to recover $\rho(x)|_{\Omega^T}$ from R^{2T}_ρ, see [14].

We introduce the new function:

$$C(x,t) = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} := \begin{pmatrix} u_t \\ iu_x \end{pmatrix},$$

and a Hamiltonian $H := \begin{pmatrix} \rho(x) & 0 \\ 0 & 1 \end{pmatrix}$. Then it is easy to see that Y satisfies the canonical system:

$$\begin{cases} iHC_t - JC_x = 0, & x \geq 0,\, t \geq 0, \\ C(x,0) = 0, & x \geq 0, \\ c^1(0,t) = g(t) := f'(t), & t \geq 0. \end{cases} \tag{9}$$

The response operator $\widetilde{R}^T_\rho : L_2(0,T) \mapsto L_2(0,T)$ for (9) with the domain $\mathcal{D} = C_0^\infty(0,T)$ is introduced by $\left(\widetilde{R}^T_s g\right)(t) := c^2(0,t)$. We can see that IPs for (8) and for (9) are equivalent.

3.3. Dirac system on a half-line

With a matrix potential $V = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$, $p, q \in C^1_{loc}(\mathbb{R}_+)$, vector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ we associate the IBVP for a Dirac system:

$$\begin{cases} iu_t + Ju_x + Vu = 0, & x \geq 0,\, t \geq 0, \\ u\big|_{t=0} = 0, & x \geq 0, \\ u_1\big|_{x=0} = f, & t \geq 0, \end{cases} \tag{10}$$

Here f is an arbitrary $L^2_{loc}(\mathbb{R}_+, \mathbb{C})$ function referred to as a *boundary control*. The *response operator* $R^T_D : L_2(0,T) \mapsto L_2(0,T)$ with the domain $\mathcal{D} = C^\infty_0(0,T)$ is introduced by $\left(R^T_D f\right)(t) := u_2(0,t)$, it plays a role of a dynamic inverse data. The IP is to recover V on $(0,T)$ from R^{2T}_D, see [15].

Let $Y^{1,2}$ be solutions to the following Cauchy problems:

$$\begin{cases} JY^{1,2}_x + VY^{1,2} = 0, \\ Y^1_1(0) = 1, \quad Y^1_2(0) = 0, \quad Y^2_1(0) = 0, \quad Y^2_2(0) = 1. \end{cases}$$

We will look for the solution to (10) in the form:

$$u(x,t) = c^1(x,t)Y^1(x) + c^2(x,t)Y^2(x). \tag{11}$$

Plugging this representation in (10) yields:

$$\begin{aligned} i\left(c^1_tY^1 + c^2_tY^2\right) + c^1_xJY^1 + c^2_xJY^2 + c_1JY^1_x + c_2JY^2_x + c_1VY^1 + c_2VY^2 \\ = i\left(c^1_tY^1 + c^2_tY^2\right) + J\left(c^1_xY^1 + c^2_xY^2\right) = 0, \end{aligned}$$

on introducing $C = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}$, we see that the above equality is equivalent to:

$$i\begin{pmatrix} Y^1_1 & Y^2_1 \\ Y^1_2 & Y^2_2 \end{pmatrix} C_t + J\begin{pmatrix} Y^1_1 & Y^2_1 \\ Y^1_2 & Y^2_2 \end{pmatrix} C_x = 0.$$

We introduce the notation: $A = \begin{pmatrix} Y^1_1 & Y^2_1 \\ Y^1_2 & Y^2_2 \end{pmatrix}$, $B = JAJ$. Then the above system is equivalent to:

$$iAC_t - BJC_x = 0,$$

on multiplying it by B^{-1} and introducing the Hamiltonian by $H = B^{-1}A$, we obtain:

$$iHC_t - JC_x = 0.$$

Counting that $\det B = \det A = 1$, we evaluate:

$$H = B^{-1}A = \begin{pmatrix} Y^1Y^1 & Y^1Y^2 \\ Y^1Y^2 & Y^2Y^2 \end{pmatrix},$$

Bearing in mind the initial and boundary conditions in (10), we see that C satisfies the following IBVP:

$$\begin{cases} iHC_t - JC_x = 0, & x \geq 0,\, t \geq 0, \\ C(x,0) = 0, & x \geq 0, \\ c^1(0,t) = f(t), & t \geq 0. \end{cases} \tag{12}$$

The response operator $\widetilde{R}^T_D : L_2(0,T) \mapsto L_2(0,T)$ for (12) is introduced by $\left(\widetilde{R}^T_D f\right)(t) := c^2(0,t)$. The representation (11) implies that IPs for (10) and for (12) are equivalent.

3.4. Semi-infinite Jacobi matrices

Let $0 = b_0 < b_1 < b_2 < \ldots < b_n < \ldots$ be a partition of $[0, +\infty)$. We introduce the notations: $\Delta_j := (b_{j-1}, b_j)$, $l_j = |\Delta_j| = b_j - b_{j-1}$. Let for each j we define $e_j \in \mathbb{R}^2$, $|e_j| = 1$, $e_j \neq \pm e_{j\pm1}$, and $e_j(x) = e_j$, $x \in \Delta_j$. We define a Hamiltonian H:

$$H(x)f(x) = (f(x), e_j(x))\, e_j(x) = \begin{pmatrix} e^2_{1j}(x) & e_{1j}(x)e_{2j}(x) \\ e_{1j}(x)e_{2j}(x) & e^2_{2j}(x) \end{pmatrix} \begin{pmatrix} f^1(x) \\ f^2(x) \end{pmatrix}.$$

Consider functions of the type (i.e. functions from the domain of operator, corresponding to such a Hamiltonian, see [RR]):

$$f(x) = \begin{pmatrix} f^1(x) \\ f^2(x) \end{pmatrix} = f_je_j(x) + \xi_j(x)e^\perp_j(x), \quad x \in \Delta_j, f_j \in \mathbb{R}, \quad e^\perp_j = Je_j, \tag{13}$$

and note that $(f, e_j) = f_j$. For such a Hamiltonian H we study the equation:

$$Jf' = Hg, \tag{14}$$

where the function g has a form (13), $g = g_j e_j(x) + \eta_j(x) e_j^{\perp}(x)$, $x \in \Delta_j$. The equality in (14) implies that

$$\xi_j'(x) J e_j^{\perp}(x) = g_j e_j(x), \quad x \in \Delta_j,$$

which yields the following expression for $\xi_j(x)$ for some s_j:

$$\xi_j(x) = s_j + g_j(b_j - x), \quad x \in \Delta_j. \tag{15}$$

We use the continuity condition at $x = b_{j-1}$ to obtain:

$$f_{j-1} e_{j-1} + s_{j-1} e_{j-1}^{\perp} = f_j e_j + (s_j + g_j l_j)\, e_j^{\perp}.$$

Multiplying the above equality by e_j we get:

$$s_{j-1} = \frac{1}{\left(e_j, e_{j-1}^{\perp}\right)} \left(f_j - f_{j-1}\left(e_j, e_{j-1}^{\perp}\right)\right), \tag{16}$$

and multiplying by e_{j-1} we obtain:

$$f_{j-1} = f_j\left(e_j, e_{j-1}\right) + \left(s_j + g_j l_j\right)\left(e_j^{\perp}, e_{j-1}\right). \tag{17}$$

Using (16), (17) we can express g_j via f_{j-1}, f_j, f_{j+1}:

$$g_j l_j = \frac{1}{\left(e_j, e_{j-1}^{\perp}\right)} f_{j-1} + \left(\frac{(e_{j+1}, e_j)}{\left(e_{j+1}, e_j^{\perp}\right)} - \frac{(e_j, e_{j-1})}{\left(e_j^{\perp}, e_{j-1}\right)}\right) f_j - \frac{1}{\left(e_j^{\perp}, e_{j+1}\right)} f_{j+1}. \tag{18}$$

Making the substitution:

$$u_j = g_j \sqrt{l_j}, \quad v_j = f_j \sqrt{l_j}, \tag{19}$$

from (18) we obtain the relation:

$$u_j = \frac{1}{\left(e_j, e_{j-1}^{\perp}\right)\sqrt{l_{j-1} l_j}} v_{j-1} + \frac{1}{l_j}\left(\frac{(e_{j+1}, e_j)}{\left(e_{j+1}, e_j^{\perp}\right)} - \frac{(e_j, e_{j-1})}{\left(e_j^{\perp}, e_{j-1}\right)}\right) v_j - \frac{1}{\left(e_j^{\perp}, e_{j+1}\right)\sqrt{l_j l_{j+1}}} v_{j+1}. \tag{20}$$

On introducing the notations:

$$\rho_j = \frac{-1}{\left(e_{j+1}, e_j^{\perp}\right)\sqrt{l_j l_{j+1}}}, \quad j \geq 1,$$

$$q_j = \frac{1}{l_j}\left(\frac{(e_j, e_{j+1})}{\left(e_j^{\perp}, e_{j+1}\right)} - \frac{(e_j, e_{j-1})}{\left(e_j^{\perp}, e_{j-1}\right)}\right), \quad j \geq 2,$$

we can rewrite (20) in a form:

$$u_j = \rho_{j-1} v_{j-1} + q_j v_j + \rho_j v_{j+1}, \quad j \geq 2,$$

and q_1 is found from the condition at zero. So finally we obtain the following result: if f and g having representation (13) are connected by (14), then u and v defined by (19) satisfy:

$$Av = u, \quad A = \begin{pmatrix} q_1 & \rho_1 & 0 & 0 & 0 \\ \rho_1 & q_2 & \rho_2 & 0 & 0 \\ 0 & \rho_2 & q_3 & \rho_3 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \end{pmatrix}.$$

We can introduce the dependence on (continuous) time t: let $f(x,t)$, $g(x,t)$ have form:

$$f(x,t) = f_j(t) e_j(x) + \xi(x,t) e_j^{\perp}(x), \quad x \in \Delta_j,$$
$$g(x,t) = g_j(t) e_j(x) + \eta(x,t) e_j^{\perp}(x), \quad x \in \Delta_j,$$

then if $g(x,t) = i f_t(x,t)$, then f solves:

$$J f_x = i H f_t.$$

On the other hand (19) implies the relationship $u_j(t) = i {v_j}_t(t)$, which yields that v solves $i v_t - Av = 0$. Adding initial and boundary conditions gives well-posed IBVP for dynamical system with continuous time governed by Jacobi matrix:

$$\begin{cases} i v_t - Av = 0, & x \geq 0, t \geq 0, \\ v_n(0) = 0, & n \geq 1, \\ v_1(t) = h(t), & t \geq 0. \end{cases} \tag{21}$$

The response operator $R_J^T : L_2(0,T) \mapsto L_2(0,T)$ with the domain $D = C_0^\infty(0,T)$ for this system is introduced by the rule $\left(R_J^T h\right)(t) := v_2(t)$. On the other hand, IBVP (21) is equivalent to (we assume that $e_1 = (1,0)^T$):

$$\begin{cases} iHf_t - Jf_x = 0, & x \geq 0,\, t \geq 0, \\ f(x,0) = 0, & x \geq 0, \\ f^1(0,t) = j(t) := \dfrac{h(t)}{\sqrt{l_1}}, & t \geq 0. \end{cases} \tag{22}$$

For the system (22), the response operator $\widetilde{R}_J^T : L_2(0,T) \mapsto L_2(0,T)$ with the domain $D = C_0^\infty(0,T)$ is introduced by the rule $\left(R_J^T h\right)(t) := f^2(0,t)$. Note that by (13), $f^2(0,t) = \xi_1(0,t)$. From (15), the relationship $g(x,t) = if_t(x,t)$ and (16), we have that:

$$\left(R_J^T h\right)(t) = s_1(t) + g_1(t) l_1 = \frac{f_2(t)}{\left(e_2, e_1^\perp\right)} - f_1(t) + if^1(0,t) l_1$$
$$= \frac{f_2(t)}{\left(e_2, e_1^\perp\right)} - \frac{h(t)}{\sqrt{l_1}} + ih(t)\sqrt{l_1} = -\rho_1 v_2(t)\sqrt{l_1} - h(t)\left(\frac{1}{\sqrt{l_1}} - i\sqrt{l_1}\right).$$

So IP for (21) and (22) from corresponding response operators are equivalent. We note that we can introduce the different type of continuous dynamics for Jacobi matrices (for example the dynamics of the type $\dfrac{d}{dt^2}$ was considered in [16]).

We can also introduce the dependence on the discrete time $t \in \mathbb{N}$ by letting $f_t(x)$, $g_t(x)$ have form:

$$f_t(x) = f_{j,t} e_j(x) + \xi_t(x) e_j^\perp(x), \quad x \in \Delta_j,\, t \in \mathbb{N},$$
$$g_t(x) = g_{j,t} e_j(x) + \eta_t(x) e_j^\perp(x), \quad x \in \Delta_j,\, t \in \mathbb{N}.$$

If f, g are related by $g_t(x) = f_t(x) + f_{t-1}(x) =: \partial_t f(x)$, then counting (14), f solves:

$$Jf_x = H\partial_t f.$$

The equality (19) implies $u_j = \partial_t v_j$, which yields that v satisfies $\partial_t v_{\cdot,t} - Av_{\cdot,t} = 0$. Adding initial and boundary conditions gives the following IBVP:

$$\begin{cases} \partial_t v_{\cdot,t} - Av_{\cdot,t} = 0, & t \in \mathbb{N} \\ v_{n,1} = v_{n,0}(0) = 0, & n \geq 1, \\ v_{1,t} = h_t, & t \in \mathbb{N}. \end{cases} \tag{23}$$

where $h_t \in l_2$ is referred to as a *boundary control*. The response operator $R_{J,d}^T$ with the domain $D = \mathrm{R}^T$ for this system is introduced by $R_{J,d}^T : \mathbb{R}^T \mapsto \mathbb{R}^T$, $\left(R_{J,d}^T h\right)_t = v_{2,t}$, $t = 1\ldots, T$. The forward and inverse problem was studied in [17, 18]. The IBVP (23) is equivalent to, which is equivalent to the following IBVP for a canonical system:

$$\begin{cases} H\partial_t f - Jf_x = 0, & x \geq 0,\, t \in \mathbb{N}, \\ f_0(x) = 0, & x \geq 0, \\ f_t^1(0) = j_t := \dfrac{h_t}{\sqrt{l_1}}, & t \in \mathbb{N}. \end{cases} \tag{24}$$

For the system (24) the response operator $\widetilde{R}_{J,d}^T : l_2 \mapsto l_2$ is introduced by the rule $\left(R_J^T j\right)(t) := f_t^2(0)$. By (13), $f_t^2(0) = \xi_{1t}(0)$, from (15), the relationship $g_t(x) = \partial_t f(x)$ and (16), we have that:

$$\left(R_{J,d}^T h\right)_t = s_{1t} + g_{1t} l_1 = \frac{f_{2t}}{\left(e_2, e_1^\perp\right)} - f_{1t} + if_t^1(0) l_1$$
$$= \frac{f_{2t}}{\left(e_2, e_1^\perp\right)} - \frac{h_t}{\sqrt{l_1}} + ih_t\sqrt{l_1} = -\rho_1 v_{2,t}\sqrt{l_1} - h_t\left(\frac{1}{\sqrt{l_1}} - i\sqrt{l_1}\right).$$

So, IP for (23) and (24) from corresponding response operators are equivalent.

We see that different dynamic systems after transformations come to dynamical canonical systems with different dynamics ($i\dfrac{d}{dt}$, $\dfrac{d}{dt^2}$, and even discrete one ∂_t).

We will investigate the dynamics given by $i\dfrac{d}{dt}$, the canonical system with this dynamics possess property of finite speed of wave propagation.

4. Canonical systems with smooth strictly positive Hamiltonian

We consider the IBVP for a canonical system. Assuming that the Hamiltonian satisfies conditions: $H = H^* \in C^2(0,T;\mathbb{R}^{2\times 2})$, $H \geq \delta > 0$, $\operatorname{tr} H = 1$, we set $Y^f = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$ to be a solution to:

$$\begin{cases} iH\dfrac{d}{dt}Y - J\dfrac{d}{dx}Y = 0, & x \geq 0,\ t \geq 0, \\ Y(x,0) = 0, & x \geq 0, \\ y^1(0,t) = f(t), & t \geq 0. \end{cases} \tag{25}$$

Where the *boundary control* $f \in \mathcal{F}^T := L_2(0,T;\mathbb{C})$. The *response operator* $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is introduced as $\left(R^T f\right)(t) := y_2^f(0,t)$. The inverse problem we will be dealing with consists in a recovering $H(x)$, on an interval $(0,l)$ for some $l > 0$ from given R^{2T}.

4.1. One-velocity wave system

We rewrite (25): differentiate the first line in (25) w.r.t. t and use equation to get:

$$HY_{tt} + JH^{-1}JY_{xx} + JH_x^{-1}JY_x = 0,$$

which is equivalent to the equation:

$$HY_{tt} - \frac{1}{\det H}HY_{xx} + JH_x^{-1}JY_x = 0.$$

Counting the initial and boundary condition, we obtain that Y satisfies the following IBVP for one-velocity system:

$$\begin{cases} \det HY_{tt} - Y_{xx} + \det HH^{-1}JH_x^{-1}JY_x = 0, & x \geq 0,\ t \geq 0, \\ Y(x,0) = Y_t(x,0) = 0, & x \geq 0, \\ \begin{pmatrix} y^1(0,t) \\ y^2(0,t) \end{pmatrix} = G(t) := \begin{pmatrix} f(t) \\ (Rf)(t) \end{pmatrix}, & t \geq 0. \end{cases} \tag{26}$$

Here, the velocity is given by $c(x) = \dfrac{1}{\sqrt{\det H(x)}}$. The response operator $R_w^T : L_2(0,T;\mathbb{C}) \mapsto L_2(0,T;\mathbb{C})$ with the domain $\mathcal{D} = C_0^\infty(0,T,\mathbb{C})$ for (26) is introduced as $\left(R_w^T G\right)(t) := Y_x^G(0,t)$. The *eikonal* function is introduced by $\tau(x) := \displaystyle\int_0^x \sqrt{\det H(s)}\,ds$, and $\Omega^l = \{x > 0 \,|\, \tau(x) < l\}$. Then the natural setup of IP is to recover $H(x)|_{\Omega^T}$ from R_w^{2T}.

We see that the IP for the system (26), is equivalent to IP for (25). But there is one important disadvantage – in studying IP for (26) which comes from (25), we need to use the specific set of controls of the type $\begin{pmatrix} f \\ Rf \end{pmatrix}$, which makes application of the BC method problematic. Instead, we will reduce (25) to Dirac-type system, and follow the scheme offered in [15].

4.2. Dirac-type dynamical system

We introduce the following transformation: let

$$U = \begin{pmatrix} \cos\phi(x) & \sin\phi(x) \\ -\sin\phi(x) & \cos\phi(x) \end{pmatrix}$$

be a unitary matrix such that $U^*HU = D := \begin{pmatrix} d_1(x) & 0 \\ 0 & d_2(x) \end{pmatrix}$, where $d_1, d_2 \geq \delta > 0$, $d_1 + d_2 = 1$. If $Y = U\widetilde{Y}$, then $\widetilde{Y}$ satisfies the following IBVP for Dirac-type dynamical system:

$$\begin{cases} iD\dfrac{d}{dt}\widetilde{Y} + J\dfrac{d}{dx}\widetilde{Y} - \phi'(x)\widetilde{Y} = 0, & x \geq 0,\ t \geq 0, \\ \widetilde{Y}(x,0) = 0, & x \geq 0, \\ \widetilde{y}^1(0,t) = g(t) := \cos\phi(0)f(t) + \sin\phi(0)(Rf)(t), & t \geq 0. \end{cases} \tag{27}$$

The response operator $R_{CD}^T : L_2(0,T) \mapsto L_2(0,T)$ is introduced by $\left(R_{CD}^T g\right)(t) := \widetilde{y}^2(0,t)$. We can see that $\widetilde{y}^2(0,t) = -\sin\phi(0)f(t) + \cos\phi(0)(Rf)(t)$, so IP for (25) and for (27) are equivalent.

Thus our first goal will be to study the dynamic IP for the following Dirac-type system:

$$\begin{cases} iD\dfrac{d}{dt}V + J\dfrac{d}{dx}V + \psi(x)V = 0, & x \geq 0,\, t \geq 0, \\ V(x,0) = 0, & x \geq 0, \\ v^1(0,t) = f(t), & t \geq 0, \end{cases} \tag{28}$$

where D as above is a diagonal matrix with twice differentiable entries and unit trace, $\psi \in C^2(\mathbb{R}_+)$. The function $f \in \widetilde{\mathcal{F}}^T := L_2(0,T;\mathbb{C})$ is a *boundary control*. The response $R_D^T : \widetilde{\mathcal{F}}^T \mapsto \widetilde{\mathcal{F}}^T$ is introduced by $\left(R_D^T f\right)(t) := v^2(0,t)$. The IP consists in recovering $D|_{\Omega^T}$, $\psi|_{\Omega^T}$ from R^{2T}. We outline the scheme offered in [1,15]:

Proposition 1. *The solution to (28) admits the following representation:*

$$V(x,t) = A(x)f(t-\tau(x)) + \int_0^{x(t)} w(x,s)f(t-\tau(s))\,ds,$$

where $\tau(s) = \int_0^s \sqrt{d_1(\alpha)d_2(\alpha)}\,d\alpha$ *is eikonal,* $x(t)$ *is a function inverse to* $\tau(x)$*, the kernel* $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ *is twice differentiable in* $\{(x,s)\,|\,0 \leq \tau(x) \leq s \leq T\}$, $A = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}$*, where* $a^{1,2}$ *are solutions to the following system:*

$$i\sqrt{d_1}a_x^1 = \sqrt{d_2}a_x^2,$$
$$\sqrt{d_2}\left(\psi a^1 + a_x^2\right) = i\sqrt{d_1}\left(\psi a^2 - a_x^1\right).$$

We introduce the *outer space*, the space o states of (28): $\mathcal{H}^T := L_2(0,\tau(T);\mathbb{C})$ and a *control operator* $\widetilde{W}^T : \widetilde{\mathcal{F}}^T \mapsto \mathcal{H}^T$ acting by the rule:

$$\left(\widetilde{W}^T f\right)(x) := V^f(x,T).$$

The Proposition 1 implies that $\widetilde{W}^T$ is not an isomorphism, and the system (28) is not boundary controllable. To restore the controllability, we introduce the auxiliary system:

$$\begin{cases} iD\dfrac{d}{dt}U - J\dfrac{d}{dx}U - \psi(x)U = 0, & x \geq 0,\, t \geq 0, \\ U(x,0) = 0, & x \geq 0, \\ u^1(0,t) = g(t), & t \geq 0, \end{cases} \tag{29}$$

and note that solutions to (28) and (29) are connected by the formula $V^f = \overline{U^{\overline{f}}}$. The extended outer space is defined by $\mathcal{F}^T := L_2(0,T;\mathbb{C}^2)$, and the *extended control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ is introduced by:

$$W^T \begin{pmatrix} f \\ g \end{pmatrix} := V^f(x,T) + U^g(x,T).$$

Proposition 2. *The extended control operator is an isomorphism between* $\mathcal{F}^T$ *and* $\mathcal{H}^T$.

The set $\mathcal{U}^T := W^T\mathcal{F}^T$ is called extended reachable set. The Proposition 2 says that $\mathcal{U}^T = \mathcal{H}^T$.

We consider the operator of the Dirac-type system on a half-line: let $\mathbf{D} := D^{-1}J\dfrac{d}{dx} + D^{-1}\psi$ on $L_2(\mathbb{R}_+,\mathbb{C}^2) \ni \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ with a Dirichlet condition $\Phi_1(0) = 0$. Denote by $\theta(x,z) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ a solution to the following Cauchy problem for $z \in \mathbb{C}$:

$$\begin{cases} J\theta_x + V\theta = zD\theta, & x > 0, \\ \theta_1(0,z) = 0, \quad \theta_2(0,z) = 1. \end{cases} \tag{30}$$

Let $d\rho$ be a spectral measure of $\mathbf{D}$, and $F : L_2(\mathbb{R}_+;\mathbb{C}^2) \mapsto L_{2,\rho}(\mathbb{R}_+)$ be the corresponding Fourier transform:

$$\left(F\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right)(\lambda) = F(\lambda) = \int_0^\infty \left(f_1(x)\theta_1(x,\lambda) + f_2(x)\theta_2(x,\lambda)\right)\, dx,$$

$$f_1(x) = \int_{-\infty}^{\infty} F(\lambda)\theta_1(x,\lambda)\, d\rho(\lambda), \quad f_2(x) = \int_{-\infty}^{\infty} F(\lambda)\theta_2(x,\lambda)\, d\rho(\lambda),$$

$$\int_0^\infty \left(f_1^2(x) + f_2^2(x)\right)\, dx = \int_{-\infty}^{\infty} F^2(\lambda)\, d\rho(\lambda).$$

We introduce the *extending connecting operator* $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ by the quadratic form:

$$\left(C^T\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right)_{\mathcal{F}^T} = \left(W^T\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, W^T\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right)_{\mathcal{H}^T}, \quad C^T = \left(W^T\right)^* W^T. \tag{31}$$

The important fact in the BC method is that:

Proposition 3. *The extending connecting operator is a positive isomorphism in $\mathcal{F}^T$, it admits the representation in terms of dynamic inverse data R^{2T}, and spectral inverse data $d\rho(\lambda)$.*

We introduce the linear manifold of Fourier images of extended states (Fourier image of extended reachable set) at time $t = T$:

$$B_D^T := \left\{K(\lambda)\,\middle|\, K(\lambda) = \left(FW^T\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\right)(\lambda),\ \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathcal{F}^T\right\} = F\mathcal{U}^T.$$

Equipped with the scalar product, generated by C^T:

$$[F,G]_{B_D^T} := \left(C^T\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right)_{\mathcal{F}^T}, \quad F,G \in B_D^T,$$

this linear space becomes a Hilbert space of analytic functions. It is also possible to define a reproducing kernel in this space (it is given in terms of a solution to a Krein equation), which makes B_D^T a de Branges space. Solution of dynamic and spectral IPs for (28) and construction of corresponding de Branges space will be the subject of forthcoming publications.

4.3. Dynamic approach to de Branges spaces

Based on the arguments from the previous subsection, we can formulate the hypothesis about de Branges space for canonical system (25) with general Hamiltonian. First, we introduce the auxiliary system:

$$\begin{cases} iH\dfrac{d}{dt}Z + J\dfrac{d}{dx}Z = 0, & x \geq 0,\ t \geq 0, \\ Z(x,0) = 0, & x \geq 0, \\ z^1(0,t) = g(t), & t \geq 0. \end{cases} \tag{32}$$

The *extending control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ acting in extended control space $\mathcal{F}^T := L_2(0,T;\mathbb{C}^2)$ is defined by $W^T\begin{pmatrix} f \\ g \end{pmatrix} := Y^f(x,T) + Z^g(x,T)$. The *extending connecting operator* C^T is given by analog to (31). Then, the de Branges space corresponding to (25) is a Fourier image of extended reachable set, equipped with a scalar product, generated by C^T.

We note that the construction of de Branges space by dynamic methods for general Hamiltonian in fact is equivalent to solving the dynamic IP for system (25) with general H. We note that the in studying the IP in this case, one inevitably face with two obstacles: the smoothness of H, and changing the rank of H, which reflects in the lack of the boundary controllability of the dynamical system. The authors suggest that studying the inverse dynamic problem for a Krein string [10, 19] will be instructive and can help to overcome difficulties connected with general Hamiltonian.

Acknowledgements

The research of Victor Mikhaylov was supported in part by RFBR 17-01-00529-A. Alexandr Mikhaylov was supported by RFBR 17-01-00099-A; A. S. Mikhaylov and V. S. Mikhaylov were partly supported by RFBR 18-01-00269-A and VW Foundation program "Modeling, Analysis, and Approximation Theory toward application in tomography and inverse problems". The authors are deeply indebted to Prof. R. V. Romanov and Prof. M. I. Belishev for valuable discussions.

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