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On a problem for an elliptic type equation of the second kind with a conormal and integral condition

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In this paper, we prove the uniqueness of a solution of the boundary value problem for an elliptic type equation of the second kind with the conormal and integral condition.

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1. Introduction

The Poincaré problem with the conormal derivative are studied by Tricomi, Lavrentyev–Bitsadze and Gellerstedt [1–5].

Boundary value problems with the conormal derivative for the elliptic type equation with one and two lines of degeneration of the first kind is considered when on the lines of degeneration a function or its derivative is given by M. A. Usanatashvili [6], M. S. Salokhitdinov and B. Islomov [7, 8], H. Islomov [9].

In this paper, we prove the unique solvability of a boundary value problem with the conormal and integral condition for the elliptic type equation of the second kind.

Stationary processes of a various physical nature (oscillations, heat conductivity, diffusion, electrostatics, etc.) are described by equations of the elliptic type [10]. In particular, in some nanophysical models such as hydrodynamics and gas dynamics elliptic equations are considered.

2. Boundary value problem with a conormal and integral condition

We consider the boundary value problem with a conormal and integral conditions for the following elliptic type equation of the second kind:

$$y^m u_{xx} + u_{yy} = 0, \tag{1}$$

where $-1 < m < 0$ is a real number. Let D be a simply connected domain in the plane (x, y) bounded by a curve σ at the first quadrant $(x > 0, y > 0)$ with its end points $A(0, 0)$, $B(1, 0)$ and with the line segment AB of the real axis Ox.

Let us introduce the following notations:

$$J = \{(x, y) : \ \ 0 < x < 1, \ y = 0\}, \ \ \partial D = \vec{\sigma} \cup \overline{AB}, \quad 2\beta = \frac{m}{m+2},$$

note that as it is defined, we have:

$$-\frac{1}{2} < \beta < 0. \tag{2}$$

We denote by $C\,(D)$ the space of continuous functions defined on a set D on the (x, y) plane or on the real line and $C^k\,(D)$ denotes the space of k continuously differentiable functions on D.

In the domain D, we consider the following Problem Conormal (Problem CN) for the equation (1):

Problem CN. Find a function $u(x, y)$ with the following properties:

(1) $u(x, y) \in C\left(\bar{D}\right) \cup C^1\left(D \cup \sigma \cup J\right)$ and u_x, u_y can tend to infinity of the order less than -2β at points $A\,(0, 0)$ and $B\,(1, 0)$;

(2) $u(x, y) \in C^2(D)$ is a solution of the equation (1) in D;

(3) $u(x,y)$ – satisfies the following boundary conditions:
for all $0 < s < l$

$$\left\{\delta(s)A_s[u] + \rho(s)u\right\}\Big|_{\sigma} = \varphi(s) \tag{3}$$

for all $0 < x < 1$

$$a_0(x)u_y(x,0) + \sum_{j=1}^{n} a_j(x)D_{0x}^{\alpha_j}u(x,0) + a_{n+1}(x)u(x,0) = b(x), \tag{4}$$

where $\delta(s)$, $\rho(s)$, $\varphi(s)$, $a_j(x)$, and $b(x)$ are given functions with the following conditions:

$$b(0) = 0, \quad a_0(1) \neq 0, \tag{5}$$

$$a_0(x) \neq 0, \quad \forall x \in \bar{J}, \tag{6}$$

$$\delta^2(s) + \rho^2(s) \neq 0, \quad \forall s \in [0,l], \tag{7}$$

$$\sum_{j=0}^{n+1} a_j^2\,(x) \neq 0, \quad \forall x \in \overline{J}, \tag{8}$$

$$\delta(s),\, \rho(s),\, \varphi(s) \in C[0,l], \tag{9}$$

$$a_j(x),\, b(x) \in C\left(\overline{J}\right) \cap C^2\left(J\right), \quad (j = \overline{0, n+1}), \tag{10}$$

and

$$A_s[u] = y^m \frac{dy}{ds}\frac{\partial u}{\partial x} - \frac{dx}{ds}\frac{\partial u}{\partial y},$$

$\dfrac{dx}{ds} = -\cos(n,y)$, $\dfrac{dy}{ds} = \cos(n,x)$, where n is the external normal to the curve σ, l is the length of the curve σ, s is the length of an arc of the curve σ, starting from the point $B\,(1,0)$, and $D_{0x}^{\alpha_j}[*]$ is the Riemann–Liouville integral operator of a fractional order α [11]:

$$D_{0x}^{\alpha_j} f(x) = \frac{1}{\Gamma(-\alpha_j)}\int_0^x \frac{f(t)dt}{(x-t)^{\alpha_j+1}}, \quad -1 < \alpha_j < 0. \tag{11}$$

Let $\alpha = \max\limits_{1\le j\le n}\{|\alpha_j|\}$, and

$$\alpha \le -2\beta. \tag{12}$$

We assume that the curve σ satisfies the following conditions:

(1) functions $x(s)$, $y(s)$, which are the parametric representations of the curve σ, have continuous derivatives $x'(s)$, $y'(s)$, and don't tend to zero at the same time, moreover, they have the second derivatives that satisfy Hölder condition of an exponent $\kappa\,(0 < \kappa < 1)$ in the interval $0 \le s \le l$;
(2) near the endpoints of the curve σ, they satisfy inequalities:

$$\left|\frac{dx}{ds}\right| \le const \cdot y^{m+1}(s), \tag{13}$$

and $x(l) = y(0) = 0$, $x(0) = 1$, $y(l) = 0$.

3. Uniqueness of the solution of Problem CN

Assume we have:

$$-1 < \alpha_j < 0, \quad \delta(s) \neq 0, \quad \forall s \in [0,l], \tag{14}$$

then the uniqueness of a solution of Problem CN can be proved by the method of integral energy. We have the following.

Theorem 1. *If conditions (2), (5), (6), (7), (8) are satisfied and:*

$$\delta(s)\rho(s) \ge 0, \quad 0 \le s \le l, \tag{15}$$

$$\frac{a_{n+1}(x)}{a_0(x)} \le 0, \tag{16}$$

$$\left(\frac{a_j(x)}{a_0(x)}\right)'_x \ge 0, \quad \frac{a_j(1)}{a_0(1)} \le 0, \quad \left(j = \overline{1,n}\right), \tag{17}$$

then Problem CN in the domain D can't have more than one solution.

Proof of Theorem 1. Let (x, y) be a point inside the domain D. Consider a domain $D_\delta^\varepsilon \in D$, bounded by a curve σ_ε, parallel to σ, and a segment of the straight line $y = \delta$ $(\delta > \varepsilon > 0)$. We choose δ, ε small enough such that the point (x, y) belongs to the domain D_δ^ε and $u\,(x, y) \in C^2\left(\bar{D}_\delta^\varepsilon\right)$.

We use the following identity:

$$u[y^m u_{xx} + u_{yy}] = \frac{\partial}{\partial x}[y^m u\, u_x] + \frac{\partial}{\partial y}[u\, u_y] - y^m u_x^2 - u_y^2. \tag{18}$$

Integrating (18) on domain D_δ^ε results in:

$$0 = \iint\limits_{D_\delta^\varepsilon} u[y^m u_{xx} + u_{yy}]dxdy = \iint\limits_{D_\delta^\varepsilon} \left\{\frac{\partial}{\partial x}[y^m u u_x] + \frac{\partial}{\partial y}[u u_y]\right\}dxdy - \iint\limits_{D_\delta^\varepsilon} \left[y^m u_x^2 + u_y^2\right]dxdy,$$

and applying Gauss–Ostrogradsky formula (the Green's theorem) (see [11]):

$$\iint\limits_{D_\delta^\varepsilon} \left\{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right\} dxdy = \int\limits_{\partial D_\delta^\varepsilon} Qdx + Pdy$$

we obtain:

$$0 = \iint\limits_{D_\delta^\varepsilon} u\,[y^m u_{xx} + u_{yy}]dxdy = -\iint\limits_{D_\delta^\varepsilon} \left[y^m u_x^2 + u_y^2\right]dxdy + \int\limits_{\partial D_\delta^\varepsilon} u\,[y^m u_x dy - u_y dx]\,.$$

From here on, by considering AB: $y = 0 \Rightarrow dy = 0$, and $dy = \cos(n, x)ds$, $dx = -\cos(n, y)ds$ we have:

$$0 = -\iint\limits_{D_\delta^\varepsilon} \left[y^m u_x^2 + u_y^2\right]dxdy - \int\limits_{x_1}^{x_2} u(x,\delta)u_y(x,\delta)dx + \int\limits_{\sigma_\varepsilon} u\, A_s[u]ds, \tag{19}$$

where x_1, x_2 – are the abscissas of the points of the intersection of the straight line $y = \delta$ with the curve σ_ε. Taking into account the conditions (1) of Problem CN and $\varphi(s) \equiv b(x) \equiv 0$ from (19) with (3) at $\delta(s) \neq 0$ and $\varepsilon \to 0$, $\delta \to 0$ we get the following:

$$\iint\limits_{D} \left[y^m u_x^2 + u_y^2\right] dxdy + \int\limits_0^1 \tau(x)\nu(x)dx + \int\limits_{\sigma} \frac{\delta(s)\rho(s)}{\delta^2(s)} u^2 ds = 0, \tag{20}$$

where:

$$u(x,0) = \tau(x), \quad (x,0) \in \overline{J}, \qquad u_y(x,0) = \nu(x), \quad (x,0) \in J. \tag{21}$$

Due to the condition (15), the third integral of equality (20) implies that:

$$R_3 = \int\limits_{\sigma} \frac{\delta(s)\rho(s)}{\delta^2(s)} u^2 ds \geq 0. \tag{22}$$

Now, we show that the second term of the left-hand side of (20) is nonnegative.

By (8) and (21), taking into account (4), we get:

$$\nu(x) = -\left[\sum_{j=1}^{n} \frac{a_j(x)}{a_0(x)} D_{0x}^{\alpha_j}\tau(x) + \frac{a_{n+1}(x)}{a_0(x)}\tau(x)\right].$$

Using (12), we rewrite the second term of (20) in the form:

$$R_2 = \int\limits_0^1 \tau(x)\nu(x)dx = -\int\limits_0^1 \sum_{j=1}^{n} \frac{a_j(x)\tau(x)}{\Gamma\left(-\alpha_j\right) a_0(x)}dx \int\limits_0^x \frac{\tau(t)dt}{(x-t)^{1+\alpha_j}} - \int\limits_0^1 \frac{a_{n+1}(x)}{a_0(x)}\tau^2(x)dx = R_{21} + R_{22}. \tag{23}$$

Using the following formula (see [1]):

$$|x-t|^{-\gamma} = \frac{1}{\Gamma\left(\gamma\right)\cos\frac{\pi\gamma}{2}} \int\limits_0^\infty z^{\gamma-1}\cos[z(x-t)]dz, \quad 0 < \gamma < 1$$

and from the equality (23), we obtain:

$$R_{21}=-\int\limits_0^1\sum_{j=1}^n\frac{a_j(x)\tau(x)dx}{a_0(x)\Gamma\left(-\alpha_j\right)\Gamma(1+\alpha_j)\cos\pi(1+\alpha_j)/2}\int\limits_0^x\tau(t)dt\int\limits_0^\infty z^{\alpha_j}\cos z(x-t)dz$$

$$=-2\sum_{j=1}^n\cos\frac{\pi\alpha_j}{2}\int\limits_0^\infty z^{\alpha_j}dz\int\limits_0^1\frac{a_j(x)\tau(x)}{a_0(x)}dx\int\limits_0^x[\cos zx\cos zt+\sin zx\sin zt]\tau(t)dt$$

$$=-\sum_{j=1}^n\cos\frac{\pi\,\alpha_j}{2}\int\limits_0^\infty z^{\alpha_j}dz\int\limits_0^1\frac{a_j(x)}{a_0(x)}\cdot\frac{\partial}{\partial x}\left[\left(\int\limits_0^x\tau(t)\cos ztdt\right)^2+\left(\int\limits_0^x\tau(t)\sin ztdt\right)^2\right].$$

Integrating the last integrals by parts on x, we have:

$$R_{21}=-\sum_{j=1}^n\cos\frac{\pi\alpha_j}{2}\int\limits_0^\infty z^{\alpha_j}dz\left\{\frac{a_j(x)}{a_0(x)}\left[\left(\int\limits_0^x\tau(t)\cos ztdt\right)^2+\left(\int\limits_0^x\tau(t)\sin ztdt\right)^2\right]\Bigg|_{x=0}^{x=1}\right.$$

$$\left.-\int\limits_0^1\left(\frac{a_j(x)}{a_0(x)}\right)'_x\left[\left(\int\limits_0^x\tau(t)\cos ztdt\right)^2+\left(\int\limits_0^x\tau(t)\sin ztdt\right)^2\right]dx\right\}$$

$$=-\sum_{j=1}^n\cos\frac{\pi\,\alpha_j}{2}\int\limits_0^\infty z^{\alpha_j}\frac{a_j(1)}{a_0(1)}\left[\left(\int\limits_0^x\tau(t)\cos ztdt\right)^2+\left(\int\limits_0^x\tau(t)\sin ztdt\right)^2\right]dz$$

$$+\sum_{j=1}^n\cos\frac{\pi\,\alpha_j}{2}\int\limits_0^\infty z^{\alpha_j}dz\int\limits_0^1\left(\frac{a_j(x)}{a_0(x)}\right)'_x\left[\left(\int\limits_0^x\tau(t)\cos ztdt\right)^2+\left(\int\limits_0^x\tau(t)\sin ztdt\right)^2\right]dx.$$

Hence, by (17) we have

$$R_{21}\geq 0. \tag{24}$$

Furthermore, by (16) and (23), we have:

$$R_{22}\geq 0. \tag{25}$$

Finally, using (24) and (25), by (23), we obtain:

$$R_2\geq 0. \tag{26}$$

Using the relations (22) and (26), by (20) it follows that $u_x=u_y=0$ in D, that is, $u=const$ for all $(x,y)\in D$. The fact that each term in (20) tends to zero concludes $u=0$ on $\bar{\sigma}$. Thus, $u\equiv 0$ in $\bar{D}$ for $\delta(s)\neq 0$.

Remark. The uniqueness of a solution of Problem CN for $\rho(s)\neq 0$, $\forall s\in[0,\,l]$ is proved using the maximum principle [7].

Theorem 1 is proved.

4. Existence of a solution of Problem CN for $\delta(s)\neq 0$

We consider the following auxiliary problem.

Problem DK. Find a solution $u(x,y)\in C(\bar{D})\cap C^1(D\cup\sigma\cup J)\cap C^2(D)$ of the equation (1) in the domain D that satisfies conditions (3) and the following:

$$u|_{y=0}=\tau(x),\quad 0\leq x\leq 1, \tag{27}$$

where $\tau(x)$ is a continuous function that satisfies Hölder condition with the exponent $\gamma_0\geq 1-2\beta$ in the interval $(0,1)$ and it have the following representation:

$$\tau(x)=\int\limits_x^1(t-x)^{-2\beta}\,T(t)dt, \tag{28}$$

where a function $T(t)$ is continuous in $(0,1)$ and it is integrated in $[0,1]$.

The uniqueness of a solution of Problem DK follows from identity (20).

The solution of Problem DK that satisfies conditions (3) and (27) for the equation (1) in the domain D exists and unique, moreover it has the following representation (see [11, eq. (10.78)]:

$$u(x,y)=\int_0^1\tau(\xi)\frac{\partial}{\partial\eta}G_2(\xi,0;x,y)d\xi+\int_0^l\frac{\varphi(s)}{\delta(s)}G_2(\xi,\eta;x,y)ds, \tag{29}$$

where $G_2(\xi,\eta;x,y)$ is the Green's function of Problem DK for equation (1), and it has the following form (see, [11]):

$$G_2(\xi,\eta;x,y)=G_{02}(\xi,\eta;x,y)+H_2(\xi,\eta;x,y), \tag{30}$$

where $G_{02}(\xi,\eta;x,y)$ is the Green's function of Problem DK for equation (1) on the normal domain D_0 bounded with the segment $\overline{AB}$ and the normal curve $\sigma_0:\left(x-\frac{1}{2}\right)^2+\frac{4}{(m+2)^2}y^{m+2}=\frac{1}{4}$

$$\begin{aligned}H_2\left(\xi,\eta;x,y\right)&=G_2\left(\xi,\eta;x,y\right)-G_{02}\left(\xi,\eta;x,y\right)\\&=\int_0^l\lambda_2(s;\xi,\eta)\left\{A_s[G_{02}\left(\xi\left(s\right),\eta\left(s\right);x,y\right)]+\frac{\rho(s)}{\delta(s)}G_{02}\left(\xi\left(s\right),\eta\left(s\right);x,y\right)\right\}ds,\end{aligned} \tag{31}$$

where $\lambda_2\left(s;\xi,\eta\right)$ is a solution of the integral equation:

$$\begin{aligned}\lambda_2\left(s;\xi,\eta\right)+2\int_0^l\lambda_2\left(t;\xi,\eta\right)\left\{A_s[q_2\left(\xi\left(t\right),\eta\left(t\right);x(s),y(s)\right)]+\frac{\rho(s)}{\delta(s)}q_2\left(\xi\left(t\right),\eta\left(t\right);x(s),y(s)\right)\right\}dt\\=-2q_2(\xi(s),\eta(s);\xi,\eta),\end{aligned} \tag{32}$$

where $q_2\left(x,y,x_0,y_0\right)$ is the fundamental solution of the equation (1) and it has the following form:

$$q_2\left(x,\ y,x_0,y_0\right)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\left(r_1^2\right)^{-\beta}\left(1-\sigma\right)^{1-2\beta}F\left(1-\beta,\ 1-\beta,\ 2-2\beta;\ 1-\sigma\right), \tag{33}$$

where:

$$\left.\begin{matrix}r^2\\r_1^2\end{matrix}\right\}=\left(x-x_0\right)^2+\frac{4}{\left(m+2\right)^2}\left(y^{\frac{m+2}{2}}\mp y_0^{\frac{m+2}{2}}\right)^2,$$

$$\sigma=\frac{r^2}{r_1^2},\quad\beta=\frac{m}{2(m+2)}<0,\quad k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2\left(1-\beta\right)}{\Gamma\left(2-2\beta\right)},$$

where $F\left(a,b,c;z\right)$ is the Gauss hypergeometric function [7].

Differentiating the equation (29) by y, then by tending y to zero and by (30) and (33), we obtain a functional relation between $\tau(x)$ and $\nu(x)$, transferred from the domain D to J:

$$\nu(x)=k_2\int_0^1|t-x|^{2\beta-2}\tau(t)dt-k_2\int_0^1\frac{\tau(t)dt}{\left(t+x-2tx\right)^{2-2\beta}}+\int_0^1\tau(t)\frac{\partial^2H_2(t,0;x,0)}{\partial\eta\partial y}dt+\int_0^l\chi(s)\frac{\partial q_2(t,\eta;x,0)}{\partial y}ds, \tag{34}$$

where $\chi(s)$ is a solution of the integral equation:

$$\chi(s)+2\int_0^l\chi(t)\left\{A_s\left[q_2\left(\xi(t),\eta(t);x(s),y(s)\right)\right]+\frac{\rho(s)}{\delta(s)}q_2\left(\xi(t),\eta(t);x(s),y(s)\right)\right\}dt=\frac{2\varphi(s)}{\delta(s)}. \tag{35}$$

Lemma 1. *Let a function $\tau(x)$ belongs to the class $C^{(1,\gamma_1)}(0,1)$, $\gamma_1\geq-2\beta$, then the following identities hold on $(0,1)$:*

$$\int_0^x(x-t)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta\left(2\beta-1\right)}\frac{d^2}{dx^2}\int_0^x\left(x-t\right)^{2\beta}\tau(t)dt, \tag{36}$$

$$\int_x^1(t-x)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta\left(2\beta-1\right)}\frac{d^2}{dx^2}\int_x^1\left(t-x\right)^{2\beta}\tau(t)dt, \tag{37}$$

where $0<-2\beta<1$.

Proof. We rewrite identities (36) and (37) as follows:

$$T_1(x) = \lim_{\varepsilon\to 0} T_{1\varepsilon}(x) = \lim_{\varepsilon\to 0}\left\{\frac{d^2}{dx^2}\int\limits_0^{x-\varepsilon}(x-t)^{2\beta}\,\tau(t)dt\right\}$$

and

$$T_2(x) = \lim_{\varepsilon\to 0} T_{2\varepsilon}(x) = \lim_{\varepsilon\to 0}\left\{\frac{d^2}{dx^2}\int\limits_{x+\varepsilon}^{1}(t-x)^{2\beta}\,\tau(t)dt\right\}.$$

It follows that:

$$\begin{aligned} T_{1\varepsilon}(x) &= \frac{d}{dx}\left[\varepsilon^{2\beta}\tau(x-\varepsilon) + 2\beta\int\limits_0^{x-\varepsilon}(x-t)^{2\beta-1}\,\tau(t)dt\right] \\ &= \varepsilon^{2\beta}\tau'(x-\varepsilon) + 2\beta\varepsilon^{2\beta-1}\tau(x-\varepsilon) + 2\beta\,(2\beta-1)\int\limits_0^{x-\varepsilon}(x-t)^{2\beta-2}\,\tau(t)dt, \end{aligned} \tag{38}$$

$$\begin{aligned} T_{2\varepsilon} &= \frac{d}{dx}\left[-\varepsilon^{2\beta}\tau(x+\varepsilon) - 2\beta\int\limits_{x+\varepsilon}^{1}(t-x)^{2\beta-1}\tau(t)dt\right] \\ &= -\varepsilon^{2\beta}\tau'(x+\varepsilon) + 2\beta\varepsilon^{2\beta-1}\tau(x+\varepsilon) + 2\beta\,(2\beta-1)\int\limits_{x+\varepsilon}^{1}(t-x)^{2\beta-2}\,\tau(t)dt. \end{aligned} \tag{39}$$

The conditions of the Lemma 1 and relations (38) and (39) at $\varepsilon\to 0$ imply the identities (36) and (37). This completes the proof.

Lemma 2. *Let* $\tau(x)\in C^{(1,\gamma_1)}(0,1)$, $\gamma_1\geq -2\beta$ *and it is representable in the form of (28), then the following identities hold on* $(0,1)$*:*

$$\begin{aligned} \int\limits_0^x (x-t)^{2\beta-2}\,\tau(t)dt &= \frac{1}{2\beta\,(2\beta-1)}\frac{d^2}{dx^2}\int\limits_0^x (x-t)^{2\beta}\,\tau(t)dt \\ &= \frac{\Gamma\,(1+2\beta)\,\Gamma\,(1-2\beta)}{2\beta\,(2\beta-1)} D_{0x}^{1-2\beta} D_{x1}^{2\beta-1} T(x) \\ &= \frac{\pi\cot\pi 2\beta}{1-2\beta}T(x) - \frac{1}{1-2\beta}\int\limits_0^1\left(\frac{t}{x}\right)^{1-2\beta}\frac{T(t)}{t-x}dt, \end{aligned} \tag{40}$$

$$\begin{aligned} \int\limits_x^1 (t-x)^{2\beta-2}\,\tau(t)dt &= \frac{1}{2\beta\,(2\beta-1)}\frac{d^2}{dx^2}\int\limits_x^1 (t-x)^{2\beta}\,\tau(t)dt \\ &= \frac{\Gamma\,(1+2\beta)\,\Gamma\,(1-2\beta)}{2\beta\,(2\beta-1)} D_{x1}^{1-2\beta} D_{x1}^{2\beta-1} T(x) \\ &= \frac{\pi}{(2\beta-1)\sin 2\beta\pi}T(x), \end{aligned} \tag{41}$$

$$\begin{aligned} \int\limits_0^1 (t+x-2tx)^{2\beta-2}\,\tau(t)dt &= \int\limits_0^1 (t+x-2tx)^{2\beta-2}\,dt\int\limits_t^1 (z-t)^{-2\beta}\,T(z)dz \\ &= \frac{1}{1-2\beta}\int\limits_0^1\left(\frac{t}{x}\right)^{1-2\beta}\frac{T(t)dt}{x+t-2xt}, \end{aligned} \tag{42}$$

where $(0<-2\beta<1)$.

Proof. Using definitions of the integro-differential operator with a fractional order (see [11, §4, (4.1), (4.6)]):

$$D_{px}^{\sigma}f(x)=\begin{cases}\dfrac{1}{\Gamma(-\sigma)}\displaystyle\int_p^x (x-t)^{-\sigma-1}f(t)dt, & x\in(p,q),\ \sigma<0,\\ f(x), & \sigma=0,\\ \dfrac{d^n}{dx^n}\left[D_{px}^{\sigma-n}f(x)\right], & n-1<\sigma\le n,\ n\in\mathbb{N},\end{cases}$$

and representation (28) and the identity $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ from the right-hand side of (40) implies that:

$$\begin{aligned}A_1&=\int_0^x (x-t)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_0^x (x-t)^{2\beta}\tau(t)dt\\&=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_0^x (x-t)^{2\beta}dt\int_t^1 (z-t)^{-2\beta}T(z)dz\\&=\frac{\Gamma(1+2\beta)\Gamma(1-2\beta)}{2\beta(2\beta-1)}\frac{d^2}{dx^2}D_{0x}^{-(1+2\beta)}D_{x1}^{2\beta-1}T(x)=\frac{\pi}{(2\beta-1)\sin 2\beta\pi}D_{0x}^{1-2\beta}D_{x1}^{2\beta-1}T(x).\end{aligned}\tag{43}$$

Using the formula (see [11, p. 24, lemma 4.5]):

$$D_{ax}^{\alpha}D_{xb}^{-\alpha}\Phi(x)=\cos\pi\alpha\,\Phi(x)+\frac{\sin\pi\alpha}{\pi}\int_a^b\left(\frac{t-a}{x-a}\right)^{\alpha}\frac{\Phi(t)}{t-x}dt,\quad 0<\alpha<1$$

and the equality (43) we have:

$$A_1=\frac{\pi\cot 2\beta\pi}{1-2\beta}T(x)-\frac{1}{1-2\beta}\int_0^1\left(\frac{t}{x}\right)^{1-2\beta}\frac{T(t)}{t-x}dt.$$

The last equality implies the identity (40).

Now we prove the identity (41). Due to the representation (28) we have:

$$\begin{aligned}A_2&=\int_x^1 (t-x)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_x^1 (t-x)^{2\beta}\tau(t)dt\\&=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_x^1 (t-x)^{2\beta}\tau(t)dt\int_t^1 (z-t)^{-2\beta}T(z)dz.\end{aligned}$$

Using definitions of the integro-differential operator with a fractional order (see, [11, §4, see (4.13), (4.14)]):

$$D_{xq}^{\sigma}f(x)=\begin{cases}\dfrac{1}{\Gamma}(-\sigma)\displaystyle\int_x^b (t-x)^{-\sigma-1}f(t)dt, & x\in(p,q),\ \sigma<0,\\ f(x), & \sigma=0,\\ (-1)^n\dfrac{d^n}{dx^n}\left[D_{xb}^{-(n-\sigma)}f(x)\right], & n-1<\sigma\le n,\ n\in\mathbb{N},\end{cases}\tag{44}$$

and the formula $D_{xb}^{\alpha}D_{xb}^{-\alpha}f(x)=f(x)$, we have:

$$A_2=\frac{\pi}{(2\beta-1)\sin 2\beta\pi}D_{x1}^{1-2\beta}D_{x1}^{-(1-2\beta)}T(x)=\frac{\pi}{(2\beta-1)\sin 2\beta\pi}T(x).$$

The last equality implies the identity (41).

We consider the expression:

$$A_3=\int_0^1 (t+x-2tx)^{2\beta-2}\tau(t)dt.\tag{45}$$

Substituting (28) into (42), and changing the order of the integrations, we have:

$$A_3 = \int_0^1 (t + x - 2tx)^{2\beta-2}\, dt \int_t^1 (z-t)^{-2\beta}\, T(z)dz = \int_0^1 T(z)dz \int_0^z (z-t)^{-2\beta}\, (t+x-2tx)^{2\beta-2}\, dt.$$

Putting $t = z(1-s)$ in internal integral and using the formula (see [11, p.8, (2.10)])

$$\int_0^1 t^{a-1}\,(1-t)^{c-a-1}\,(1-zt)^{-b}\, dt = \frac{\Gamma(a)\,\Gamma(c-a)}{\Gamma(c)} F(a,b,c;z),$$

$$0 < \operatorname{Re} a < \operatorname{Re} c, \quad |\arg(1-z)| < \pi. \tag{46}$$

We obtain:

$$A_3 = \int_0^1 T(z)\, z^{-2\beta+1}\,(z + x - 2x\,z)^{2\beta-2}\, dz \int_0^1 s^{-2\beta}\left[1 - \frac{(1-2x)\,z\,s}{(z+x-2x\,z)}\right]^{2\beta-2} ds$$

$$= \frac{\Gamma(1-2\beta)\,\Gamma(1)}{\Gamma(2-2\beta)} \int_0^1 \left(\frac{z}{x+z-2x\,z}\right)^{1-2\beta} \frac{T(z)}{x+z-2x\,z} F\left(1-2\beta, 2-2\beta, 2-2\beta; \frac{(1-2x)\,z}{x+z-2x\,z}\right) dz.$$

Using the formulas $F(a,b,b;z) = (1-z)^{-a}$ and $\Gamma(1+a) = a\Gamma(a)$, from the last expression we have:

$$A_3 = \frac{1}{1-2\beta} \int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} \frac{T(t)dt}{x+t-2xt}.$$

The last equality implies the identity (42). Lemma 2 is proved.

Thus, putting (40), (41), (42) and (28) into (34), we get a functional relation between $T(x)$ and $\nu(x)$, transferred from the domain$D$ to J:

$$\begin{aligned} \nu(x) = &- \frac{k_2 \pi \tan \beta\pi}{1-2\beta} T(x) \\ &+ \frac{k_2}{1-2\beta} \int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} T(t)\left[\frac{1}{x-t} - \frac{1}{x+t-2xt}\right] dt \int_0^1 T(t)dt \int_0^t \frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\,\partial y}\,(t-z)^{-2\beta}\, dz \\ &+ \int_0^l \frac{\partial q_2(t,\eta;x,0)}{\partial y} \chi(s)ds, \quad (x,0) \in J. \end{aligned} \tag{47}$$

By the conditions (21) and the relations (8), (11), (28), from (4) on the interval J, we get a functional relation between $\tau(x)$ and $\nu(x)$:

$$\begin{aligned} \nu(x) = &- \sum_{j=1}^{n} \frac{a_j(x)}{a_0(x)} \cdot \frac{1}{\Gamma(-\alpha_j)} \int_0^x (x-t)^{-\alpha_j - 1}\, dt \int_t^1 (z-t)^{-2\beta}\, T(z)dz \\ &- \frac{a_{n+1}(x)}{a_0(x)} \int_x^1 (t-x)^{-2\beta}\, T(t)dt + \frac{b(x)}{a_0(x)}, \quad (x,0) \in J. \end{aligned} \tag{48}$$

Theorem 2. *If the conditions (2), (5)–(10) and (12)–(14) are satisfied, then there exists the solution of Problem CN in the domain D.*

Proof. Excluding $\nu(x)$ from relation (47) and (48) we have:

$$T(x) - \frac{\cot\beta\pi}{\pi}\int\limits_0^1 \left(\frac{t}{x}\right)^{1-2\beta}\left[\frac{1}{x-t} - \frac{1}{x+t-2xt}\right]T(t)dt$$

$$- \frac{(1-2\beta)\cot\beta\pi}{k_2\pi}\int\limits_0^1 T(t)dt\int\limits_0^t (t-z)^{-2\beta}\,\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz$$

$$- \frac{(1-2\beta)\cot\beta\pi}{k_2\pi}\sum_{j=1}^n \frac{a_j(x)}{a_0(x)}\cdot\frac{1}{\Gamma(-\alpha_j)}\int\limits_0^x T(t)dt\int\limits_0^t (x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz$$

$$- \frac{(1-2\beta)\cot\beta\pi}{k_2\pi}\sum_{j=1}^n \frac{a_j(x)}{a_0(x)}\cdot\frac{1}{\Gamma(-\alpha_j)}\int\limits_x^1 T(t)dt\int\limits_0^x (x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz$$

$$- \frac{(1-2\beta)\cot\beta\pi}{k_2\pi}\cdot\frac{a_{n+1}(x)}{a_0(x)}\int\limits_x^1 (t-x)^{-2\beta}\,T(t)dt$$

$$= \frac{(2\beta-1)\cot\beta\pi}{k_2\pi}\cdot\frac{b(x)}{a_0(x)} + \frac{(1-2\beta)\cot\beta\pi}{k_2\pi}\int\limits_0^l \chi(s)\frac{\partial q_2(\xi(s),\eta(s);x,0)}{\partial y}ds,$$

or

$$\tilde{T}(x) - \gamma_3\int\limits_0^1\left[\frac{1}{x-t} - \frac{1}{x+t-2xt}\right]\tilde{T}(t)dt - \int\limits_0^1 K(x,t)\tilde{T}(t)dt = F(x), \tag{49}$$

where, $\gamma_3 = \dfrac{1}{\pi}\cot\beta\pi$, $\tilde{T}(x) = x^{1-2\beta}T(x)$,

$$K(x,t) = \begin{cases} K_1(x,t), & 0\le t\le 1,\\ K_2(x,t), & 0\le t\le x,\\ K_3(x,t)+K_4(x,t), & x\le t\le 1,\end{cases} \tag{50}$$

$$K_1(x,t) = \frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\int\limits_0^t (t-z)^{-2\beta}\cdot\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz, \tag{51}$$

$$K_2(x,t) = \frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^n\frac{a_j(x)}{a_0(x)}\frac{1}{\Gamma(-\alpha_j)}\int\limits_0^t (x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz, \tag{52}$$

$$K_3(x,t) = \frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^n\frac{a_j(x)}{a_0(x)}\frac{1}{\Gamma(-\alpha_j)}\int\limits_0^x (x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz, \tag{53}$$

$$K_4(x,t) = \frac{(1-2\beta)\,\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\frac{a_{n+1}(x)}{a_0(x)}(t-x)^{-2\beta}, \tag{54}$$

$$F(x) = \frac{(2\beta-1)\,\gamma_3 x^{1-2\beta}}{k_2}\left[\frac{b(x)}{a_0(x)} - \int\limits_0^l \chi(s)\frac{\partial q_2(\xi(s),\eta(s);x,0)}{\partial y}ds\right]. \tag{55}$$

We investigate the kernel and the right-hand side of the singular integral equation (49).

Lemma 3. *Let $0<x<1,\ 0<z<1$, then the following inequality holds:*

$$\left|\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}\right| < C_1(x+z-2xz)^{2\beta-1}. \tag{56}$$

where C_1 is a constant depending only on the domain D.

The proof of Lemma 3 is similar to that of Lemma 18.1(see [11, p. 133–136]).

Using (56) by (51) we have

$$|K_1(x,t)| \leq C_1 \frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\left|\int\limits_0^t (t-z)^{-2\beta}(x+z-2xz)^{2\beta-1}dz\right|. \tag{57}$$

Changing variables $z = t(1-\sigma)$ and using formulas (46) from (57) we have:

$$|K_1(x,t)| \leq C_1 \frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta}\int\limits_0^1 \sigma^{-2\beta}\left[1-\frac{(1-2x)t}{x+t-2xt}\sigma\right]^{2\beta-1}d\sigma. \tag{58}$$

Using (46) from (57), we get:

$$|K_1(x,t)| \leq C_1 \frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta} F\left(1-2\beta,\ 1-2\beta,\ 2-2\beta; \frac{t\,(1-2x)}{x+t-2xt}\right). \tag{59}$$

Since $c-a-b = 2-2\beta-2+4\beta = 2\beta < 0$, using a formula [7]:

$$F\left(a,b,c,z\right) = (1-z)^{c-a-b}\,F\left(c-a,c-b,c;z\right), \quad |\arg(1-z)| < \pi \tag{60}$$

and an estimate

$$F\left(a,b,c,z\right) \leq \begin{cases} const & at \quad c-a-b>0, \quad 0\leq z\leq 1, \\ const(1-z)^{c-a-b} & at \quad c-a-b<0, \quad 0<z<1, \\ const[1+l\,(1-z)] & at \quad c-a-b=0, \quad 0<z<1 \end{cases} \tag{61}$$

from (59) we have the following:

$$|K_1(x,t)| \leq C_1C_2 \frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta}\left(\frac{x}{x+t-2xt}\right)^{2\beta} \leq \frac{C_3x}{x+t-2xt}. \tag{62}$$

We consider the kernel (52). We make a replacement $z = t\mu$ in (52) and taking into account (46) we get:

$$\begin{aligned} K_2\left(x,t\right) &= \frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^n \frac{a_j\left(x\right)}{a_0\left(x\right)}\frac{x^{-\alpha_j-1}t^{1-2\beta}}{\Gamma\left(-\alpha_j\right)}\int\limits_0^1 (1-s)^{-2\beta}\left(1-\frac{t}{x}s\right)^{-\alpha_j-1}ds \\ &= \frac{(1-2\beta)\cdot\gamma_3}{k_2}\frac{\Gamma(1-2\beta)}{\Gamma(2-2\beta)}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^n \frac{a_j\left(x\right)}{a_0\left(x\right)}\frac{x^{-\alpha_j-1}t^{1-2\beta}}{\Gamma\left(-\alpha_j\right)}F\left(1,\ 1+\alpha_j, 2-2\beta;\ \frac{t}{x}\right) \\ &= \frac{\gamma_3}{k_2}\sum_{j=1}^n \frac{a_j\left(x\right)}{a_0\left(x\right)}\frac{x^{-\alpha_j-2\beta}}{\Gamma\left(-\alpha_j\right)}F\left(1,\ 1+\alpha_j, 2-2\beta; \frac{t}{x}\right). \end{aligned} \tag{63}$$

Considering (6), (10), (12), (61) and taking into account $c-a-b = 2-2\beta-1-1-\alpha_j = -2\beta-\alpha_j \geq -2\beta-\alpha \geq 0$, $0 \leq t \leq x \leq 1$ from (63) we have an estimation:

$$|K_2\left(x,t\right)| \leq \frac{\gamma_3}{k_2}\sum_{j=1}^n \frac{C_4C_6}{C_5}\frac{x^{-\alpha_j-2\beta}}{\Gamma\left(-\alpha_j\right)} \leq \frac{\gamma_3C_4C_6}{k_2C_5}x^{-2\beta}\sum_{j=1}^n \frac{1}{x^{\alpha_j}\Gamma\left(-\alpha_j\right)} \leq C_7x^{-\alpha-2\beta},$$

or

$$|K_2\left(x,t\right)| \leq\ C_7\ x^{-(\alpha+2\beta)} \leq C_8. \tag{64}$$

Similarly, we estimate $K_3\left(x,t\right)$. Due to (6), (10), (12), (61) and taking into account $c-a-b = 1-\alpha_j-1-2\beta = -2\beta-\alpha_j$, $0 \leq x \leq t \leq 1$ from (53) we have

$$|K_3\left(x,t\right)| \leq \frac{\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^n \frac{C_4C_9}{C_5}\frac{x^{-\alpha_j}t^{-2\beta}}{\Gamma\left(-\alpha_j\right)} \leq \frac{\gamma_3C_4C_9}{k_2C_5}\cdot\frac{x}{t}\cdot x^{-2\beta}\sum_{j=1}^n \frac{1}{x^{\alpha_j}\Gamma\left(-\alpha_j\right)} \leq C_{10}\ x^{-(\alpha+2\beta)} \leq C_{11},$$

or

$$|K_3\left(x,t\right)| \leq C_{11}. \tag{65}$$

Due to (6), (10) with $x \leq t \leq 1$, from (54) it follows that:

$$|K_4\left(x,t\right)| \leq \frac{(1-2\beta)\ \gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\frac{C_{12}}{C_5(t-x)^{2\beta}} \leq C_{13}(t-x)^{-2\beta},$$

or

$$|K_4\left(x,t\right)| \leq C_{13}(t-x)^{-2\beta} \leq C_{14}. \tag{66}$$

Now we estimate the right-hand side of the equality (49). Differentiating (33) with respect to y and at $y=0$, we obtain:

$$\frac{\partial q_2\left(\xi,\ \eta;\ t,\ 0\right)}{\partial y}=k_2\eta\left[\left(\xi-t\right)^2+\frac{4}{\left(m+2\right)^2}\eta^{m+2}\right]^{\beta-1}. \tag{67}$$

Substituting (67) to (55) we have:

$$F(x)=\frac{\left(2\beta-1\right)\gamma_3 x^{1-2\beta}}{k_2}\left[\frac{b(x)}{a_0(x)}-k_2\int\limits_0^l\frac{\eta(s)\chi(s)}{\left[\left(\xi(s)-t\right)^2+\frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{1-\beta}}ds\right]. \tag{68}$$

It is clear that the function $F\left(x\right)$ has derivatives of any order in the interval (0,1). Let us study the behavior of the function $F\left(x\right)$ and its derivative at $x\to 0$ and $x\to 1$.

In this regard, we consider the following expression:

$$F_1(x)=k_2\int\limits_0^l\frac{\eta(s)\chi(s)}{\left[\left(\xi(s)-t\right)^2+\frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{1-\beta}}ds. \tag{69}$$

By (9) for the sufficiently small $x>0$, we have

$$\begin{aligned}\left|F_1\left(x\right)\right|&\le k_2\int\limits_{l-\varepsilon}^l\left|\chi(s)\right|\frac{\eta}{\left[\left(\xi-x\right)^2+\frac{4}{(m+2)^2}\eta^{m+2}\right]^{1-\beta}}ds+O(1)\\&<C_{15}\int\limits_{l-\varepsilon}^l\frac{\eta}{\left[\left(\xi-x\right)^2+\frac{4}{(m+2)^2}\eta^{m+2}\right]^{1-\beta}}ds+O(1).\end{aligned}$$

Hence, using (13) for sufficiently small $\varepsilon>0$, we get

$$\left|F_1\left(x\right)\right|<C_{16}\int\limits_{l-\varepsilon}^l\frac{\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|}{\left[x^2+\frac{4}{(m+2)^2}\eta^{m+2}\right]^{\frac{1}{2}+\beta}}ds+O(1)<C_{17}\int\limits_0^\delta\frac{d\tilde{\eta}}{\left[x^2+\tilde{\eta}^2\right]^{\frac{1}{2}+\beta}}+O(1). \tag{70}$$

Substituting $\mu^2=\omega$ in (70) and by relations (46), (61) and formulas [11, (2.17), (2.14), (2.22), p. 10-13], we have:

$$\begin{aligned}&\left|F_1\left(x\right)\right|<\\&\frac{\delta^2}{x^{2\beta+1}}\left|\frac{\Gamma(1,5)\Gamma(\beta)}{\Gamma(0,5+\beta)}\right|\left(\frac{\delta^2}{x^2}\right)^{-\frac{1}{2}}+\frac{\delta}{x^{2\beta+1}}\left|\frac{\Gamma(1,5)\Gamma(-\beta)}{\Gamma(0,5)\Gamma(1-\beta)}\right|(x^2+\delta^2)^{-\beta}x^{2\beta+1}F\left(\beta,\frac{1}{2},\ 1+\ \beta;\ \frac{x^2}{x^2+\delta^2}\right)\\&<\frac{\delta}{x^{2\beta}}\left|\frac{\Gamma(1,5)\Gamma(\beta)}{\Gamma(0,5+\beta)}\right|+\left|\frac{\delta\,\Gamma(1,5)\Gamma(-\beta)}{\Gamma(0,5)\Gamma(1-\beta)}\right|(x^2+\delta^2)^{-\beta}<C_{18}\,x^{-2\beta},\end{aligned}$$

or

$$\left|F_1\left(x\right)\right|<C_{18}x^{-2\beta}. \tag{71}$$

If $1-x$ it is sufficiently small, then as before, we get:

$$\left|F_1\left(x\right)\right|=C_{19}\left(1-x\right)^{-2\beta}. \tag{72}$$

By the same arguments, we obtain:

$$\left|F_1'\left(x\right)\right|<C_{20}x^{-2\beta-1},\quad\left|F_1'\left(x\right)\right|=C_{21}\left(1-x\right)^{-2\beta-1}. \tag{73}$$

From the relations (6), (10), (71), (72), (73) from (68), we conclude that:

$$F(x)\in C(\bar{J})\cap C^1(J). \tag{74}$$

The function $F'(x)$ tends to infinity of the order less $2\beta+1$ at $x\to 1$, and when $x\to 0$ it is bounded.

Introducing new variables:

$$\zeta=\frac{t^2}{1-2t+2t^2},\quad z=\frac{x^2}{1-2x+2x^2}, \tag{75}$$

in the equation (49), we have:

$$\omega(z) + \gamma_3 \int\limits_0^1 \frac{\omega(\zeta)d\zeta}{\zeta - z} - \int\limits_0^1 \bar{K}(z,\zeta)\omega(\zeta)d\zeta = \tilde{F}(z), \tag{76}$$

where:

$$\omega(z) = (1-2x+2x^2)\tilde{T}(x), \quad \tilde{F}(z) = (1-2x+2x^2)F(x),$$
$$\bar{K}(z,\zeta) = \frac{1-2t+2t^2}{2t(1-t)(1-2x+2x^2)}K(x,t) + \gamma_3 \frac{(1-2x+2x^2)(1-2t+2t^2)}{(1-t)(t+x-2xt)},$$
$$x = \frac{\sqrt{z}}{\sqrt{z}+\sqrt{1-z}}, \quad t = \frac{\sqrt{\zeta}}{\sqrt{\zeta}+\sqrt{1-\zeta}}.$$

Since $1+\gamma_3^2 \neq 0$, the equation (49) is the normal type. Its index is equal to zero in a class of h_2 functions $\omega(z) \in H(0,1)$, bounded at the ends of the segment $\bar{J}$ (see [12]).

We apply the method of regularization of Carleman–Vekua [12] to the equation (76). This method is developed by S. G. Mikhlin [13] and M. M. Smirnov [11, p. 258]. This results in the Fredholm's integral equation of the second kind, solvability of which follows from the uniqueness of a solution of Problem CN. By the notations $\tilde{T}(x) = x^{1-2\beta}T(x)$, $\omega(z) = (1-2x+2x^2)\tilde{T}(x)$, we obtain a function $T(x)$ that is continuous on $(0,1)$ and is integrable in $[0,1]$.

Substituting the solution $T(x)$ of Fredholm's equation of the second kind in (28), we get $\tau(x)$. Furthermore, knowing the function $\tau(x)$, the solution of Problem CN for equation (1) in the domain D is defined as a solution of Problem DK for equation (1) with conditions (3) and (27).

Thus, the existence of a solution of Problem CN for $\delta(s) \neq 0$ is proved.

Theorem 2 is proved.

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