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On the behavior of the solution of a nonlinear polytropic filtration problem with a source and multiple nonlinearities

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In this paper, we study the global solvability and unsolvability conditions of a nonlinear filtration problem with nonlinear boundary flux. We establish the critical global existence exponent and critical Fujita exponent of nonlinear filtration problem in inhomogeneous medium. An asymptotic representation of the solution with a compact support is obtained, which made it possible to carry out a numerical experiment.

Keywords: filtration, global solutions, blow-up, critical curve, asymptotic behavior, numerical analysis.

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1. Introduction

Consider the following parabolic equation

$$\rho\left(x\right)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)+\rho\left(x\right)u^{\beta},\qquad\left(x,t\right)\in\mathbb{R}_{+}\times\left(0,\ +\infty\right),\tag{1}$$

with nonlinear boundary flux

$$-\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\ \left(0,t\right)=u^{q}\left(0,t\right),\quad t>0,\tag{2}$$

and initial value condition

$$u\left(x,0\right)=u_{0}\left(x\right)\geq0,\quad x\in\mathbb{R}_{+},\tag{3}$$

where $p>2$, β, $q>0$, $\rho(x)=x^{-n}$, $n\in\mathbb{R}$, $u_0(x)$ – is a bounded, continuous, nonnegative and nontrivial initial data.

Equations (1) occur in various fields of natural science [1, 3–5]. For example, equation (1) arises in the mathematical modeling of thermal conductivity of nanofluids, in the study of problems of fluid flow through porous media, in problems of the dynamics of biological populations, polytropic filtration, and the formation of structures in synergetics and in nanotechnologies, and in a number of other areas [1, 4].

Equation (1) is called a parabolic equation with inhomogeneous density [1] and the case $p>2$ corresponds to the slow diffusion equation [2]. The problem (1) – (3) has been intensively studied by many authors (see [2, 6–17] and references therein) in different values of numerical parameters.

The authors in [2] considered the Cauchy problem for the equation (1) at $n=0$, proving that if $0<\beta<1$, then all solutions of (1) are global in time, while for $\beta>1$ there are solutions with finite time blow-up. Also in this paper the following statements were proved:

– If $1<\beta<2p-1$, then every solutions of Cauchy problem (1), (3) blow up in finite time;

– If $\beta>2p-1$, then the Cauchy problem (1), (3) admits nontrivial global solutions with small initial data. The value $\beta=2p-1$ is Fujita type critical exponent.

Some properties of the solutions of (1)–(3) at $\rho(x)=1$ were studied in [9] by Zhongping Li, Chunlai Mu and Li Xie. They obtained the critical global existence exponent and the critical Fujita exponent by constructing sub and supersolutions.

In [6, 7], the authors studied the unboundedness of the solution for the following reaction-diffusion model with nonlocal nonlinearities

$$u_t=\Delta u^m+u^{\beta},\quad\left(x,t\right)\in\Omega\times\left(0,T\right),$$
$$\frac{\partial u}{\partial\eta}=u^q,\quad\left(x,t\right)\in\partial\Omega\times\left(0,T\right),$$
$$u\left(x,0\right)=u_0\left(x\right),\quad x\in\Omega,$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain. Authors obtained that all positive solutions exist globally if and only if β, $q \le 1$ when $m > 1$ and if and only if $\beta \le 1$, $q \le 2m/(m+1)$ when $m \le 1$.

The authors of the work [15] have studied problem (1)–(3) at $n = 0$. They obtained the principal terms for the asymptotic of self-similar solutions.

As it is well known, degenerate equations need not possess classical solutions. Therefore, its solution is understood in the generalized sense.

Definition 1. *The function $u(x,t)$ is said to be the weak solution to problem (1)–(3) in $\Omega = \{\mathbb{R}_+ \times (0,T)\}$, if $0 \le u(x,t) \in C(\Omega)$, $\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x} \in C(\Omega)$ and if it satisfies (1)–(3) in the sense of distribution in Ω, where $T > 0$ is the maximal existence time.*

2. Main results

In this section, we show the critical global existence curve and critical Fujita curve, and characterize when all solutions to the problem (1)–(3) are global in time or they blow up.

Theorem 1. *If $0 < \beta \le 1$ and $0 < q \le \frac{(2-n)(p-1)}{p-n}$ then each solution of problem (1)–(3) is global in time.*

Proof. We look for a globally defined in time supersolution of the following self-similar form

$$\bar{u}(x,t) = e^{Bt}\left(D + e^{-L\xi}\right), \quad \xi = xe^{-Jt},$$

where $D > \|u_0\|_\infty$, $L^{(p-1)/q} - e^{-L} = D$, $B = \frac{L^p(p-1) + \left(D + e^{-L}\right)^\beta}{D}$, $J = \frac{B(p-2)}{p-n}$.

After computation, we have:

$$\rho(x)\frac{\partial \bar{u}}{\partial t} = \left(Be^{Bt}(D + e^{-L\xi}) + LJe^{(B-J)t}(1+x)\right)e^{-nJt}\xi^{-n} \ge BDe^{(B-nJ)t},$$

$$\bar{u}^\beta = e^{\beta Bt}\left(D + e^{-L\xi}\right)^\beta e^{-nJt}\xi^{-n} \le e^{(\beta B - nJ)t}\left(D + e^{-L}\right)^\beta,$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}}{\partial x}\right|^{p-2}\frac{\partial \bar{u}}{\partial x}\right) = L^p(p-1)\,e^{(B(p-1)-Jp)t}e^{-L(p-1)\xi},$$

$$-\left|\frac{\partial \bar{u}}{\partial x}\right|^{p-2}\frac{\partial \bar{u}}{\partial x}\;(0,t) = L^{p-1}e^{(B-J)(p-1)t}e^{-B(p-1)}.$$

Now, we shall show that the function $\bar{u}(x,t)$ is a supersolution of problem (1)–(3). According to the comparison principle it must satisfy the following inequality:

$$\rho(x)\frac{\partial \bar{u}}{\partial t} \ge \frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}}{\partial x}\right|^{p-2}\frac{\partial \bar{u}}{\partial x}\right) + \rho(x)\,\bar{u}^\beta, \qquad (x,t) \in \mathbb{R}_+ \times (0,\ +\infty), \tag{4}$$

$$-\left|\frac{\partial \bar{u}}{\partial x}\right|^{p-2}\frac{\partial \bar{u}}{\partial x}\;(0,t) \ge \bar{u}^q(0,t), \quad t > 0. \tag{5}$$

It is not difficult to verify that if $0 < \beta \le 1$ and $0 < q \le \frac{(2-n)(p-1)}{p-n}$, by definition B, L, D, J the inequalities (4) and (5) are valid. Hence, $\bar{u}(x,0) \ge u_0(x)$ and $\bar{u}(0,0) \ge u_0(0)$. Thus, by comparison principle, Theorem 1 is proved.

Remark 1. *Theorem 1 shows that the critical global existence exponent of the problem (1)–(3) is $\left\{\beta = 1,\ 0 < q \le \frac{(2-n)(p-1)}{p-n}\right\} \cup \left\{q = \frac{(2-n)(p-1)}{p-n},\ \beta \le 1\right\}$.*

Theorem 2. *If $\beta < 1$ and $q > \frac{(2-n)(p-1)}{p-n}$ then every solution of problem (1)–(3) blows up in time.*

Proof. We shall seek a blow up subsolution of the self-similar form:

$$u_-(x,t) = t^\sigma \varphi(\eta), \quad \eta = xt^{-\gamma} \tag{6}$$

where $\sigma = \frac{1}{1-\beta}$, $\gamma = \frac{p-1-\beta}{(1-\beta)(p-n)}$ and $\varphi(\eta)$ is the solution of the following problem

$$\frac{d}{d\eta}\left(\left|\frac{d\varphi}{d\eta}\right|^{p-2}\frac{d\varphi}{d\eta}\right) + \gamma\eta^{1-n}\frac{d\varphi}{d\eta} - \sigma\eta^{-n}\varphi + \eta^{-n}\varphi^\beta = 0, \tag{7}$$

$$\left|\frac{d\varphi}{d\eta}\right|^{p-2}\frac{d\varphi}{d\eta}(0)=\varphi^{q}(0). \tag{8}$$

In order $u_{-}(x,t)$ to be a subsolution of problem (1)-(3) function $\varphi(\eta)$ should satisfy following:

$$\frac{d}{d\eta}\left(\left|\frac{d\varphi}{d\eta}\right|^{p-2}\frac{d\varphi}{d\eta}\right)+\gamma\eta^{1-n}\frac{d\varphi}{d\eta}-\sigma\eta^{-n}\varphi+\eta^{-n}\varphi^{\beta}\geq 0, \tag{9}$$

$$\left|\frac{d\varphi}{d\eta}\right|^{p-2}\frac{d\varphi}{d\eta}(0)\leq\varphi^{q}(0). \tag{10}$$

We set:

$$\varphi(\eta)=E\left(a-\eta^{\frac{p-n}{p-1}}\right)_{+}^{\frac{p-1}{p-2}}, \tag{11}$$

where E, a are constants to be determined. It is easy to see that

$$\varphi'(\eta)=-\frac{p-n}{p-2}E\eta^{\frac{1-n}{p-1}}\left(a-\eta^{\frac{p-n}{p-1}}\right)^{\frac{p-1}{p-2}},$$

$$\frac{d}{d\eta}\left(\left|\frac{d\varphi}{d\eta}\right|^{p-2}\frac{d\varphi}{d\eta}\right)=-E^{p-1}\left(\frac{p-n}{p-2}\right)^{p-1}\eta^{-n}\left(a-\eta^{\frac{p-n}{p-1}}\right)^{\frac{p-1}{p-2}},$$

$$\left|\frac{d\varphi}{d\eta}\right|^{p-2}\frac{d\varphi}{d\eta}(0)=0.$$

By taking

$$E^{p-2}\left(\frac{p-n}{p-2}\right)^{p-1}\geq\gamma,\quad a^{\frac{(p-1)(\beta-1)}{p-2}}\geq E^{p-2-\beta}\left(\frac{p-n}{p-2}\right)^{p-1}+E^{1-\beta}\sigma$$

it is easy to check that if $\beta<1$, then (9) and (10) are valid. Thus $u_{-}(x,t)$ is a subsolution of the problem (1)–(3) with every nontrivial initial data.

Theorem 3. *If* $\beta>2p-1$ *and* $q<\dfrac{(2-n)(p-1)}{p-n}$ *then every solution of problem (1)–(3) blows up in time.*

Proof. In this case we prove that the flux condition makes the solution large enough to be in the set of initial data for which the reaction term alone is enough to cause blow up. We consider the self-similar subsolution of the problem (1)–(3) without a source:

$$u_{b}(x,t)=t^{\alpha}g(\xi),\quad \xi=xt^{-\gamma} \tag{12}$$

where $\alpha=\dfrac{p-1}{(p-1)(2-n)-q(p-n)}$, $\gamma=\dfrac{p-1-q}{(p-1)(2-n)-q(p-n)}$. Let us show that function $u_{b}(x,t)$ defined by (12) is a subsolution. Then, according to the comparison principle, function $g(\xi)$ should satisfy the following inequalities:

$$\frac{d}{d\xi}\left(\left|\frac{dg}{d\xi}\right|^{p-2}\frac{dg}{d\xi}\right)+\gamma\xi^{1-n}\frac{dg}{d\xi}-\alpha g(\xi)\geq 0, \tag{13}$$

$$-\left|\frac{dg}{d\xi}\right|^{p-2}\frac{dg}{d\xi}(0)\leq g^{q}(0). \tag{14}$$

Let us set

$$\bar{g}(\xi)=B(b-\xi)_{+}^{\frac{p-1}{p-2}},$$

where b and B are positive constants to be determined. After some computations, we have

$$\bar{g}'(\xi)=-B\frac{p-1}{p-2}(b-\xi)_{+}^{\frac{1}{p-2}},$$

$$\left|\frac{d\bar{g}}{d\xi}\right|^{p-2}\frac{d\bar{g}}{d\xi}=-B^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}(b-\xi)_{+}^{\frac{p-1}{p-2}},$$

$$\frac{d}{d\xi}\left(\left|\frac{d\bar{g}}{d\xi}\right|^{p-2}\frac{d\bar{g}}{d\xi}\right)=B^{p-1}\left(\frac{p-1}{p-2}\right)^{p}(b-\xi)_{+}^{\frac{1}{p-2}}.$$

Noting that

$$\gamma\xi^{1-n}\frac{d\bar{g}}{d\xi}-\alpha\bar{g}\left(\xi\right)\geq-\gamma B\frac{p-1}{p-2}\left(b-\xi\right)_{+}^{\frac{1}{p-2}}b-\alpha B\left(b-\xi\right)_{+}^{\frac{1}{p-2}}b=$$
$$=-Bb\left(b-\xi\right)_{+}^{\frac{1}{p-2}}\left(\gamma\frac{p-1}{p-2}+\alpha\right).$$

By taking:

$$B^{p-2}\left(\frac{p-1}{p-2}\right)^{p}\geq b\left(\gamma\frac{p-1}{p-2}+\alpha\right),\;\; B^{p-1-q}\left(\frac{p-1}{p-2}\right)^{p-1}\leq b^{\frac{(q-1)(p-1)}{p-2}}$$

and $q<\dfrac{(2-n)\left(p-1\right)}{p-n}$, it is easy to check that (13) and (14) are valid. It follows from the comparison principle that the problem (1)–(3) exists a solution blowing up in a finite time.

Theorem 4. *If $\beta>1$ and $q<(2-n)\left(p-1\right)$, then every solution of problem (1)-(3) blows up in time.*

Remark 2. *Theorem 3 and Theorem 4 shows that the critical Fujita curve of the problem (1)–(3) is $\{\beta=2p-1,\;\; q\geq2\left(p-1\right)\}\cup\{\beta\geq2p-1,\;\; q=2\left(p-1\right)\}$.*

Theorem 5. *If $\beta>\dfrac{(p-n)\,q-p+1}{p-1}$ and $q>(2-n)\left(p-1\right)$ then each solution of problem (1)–(3) is global in time.*

Theorems 4 and 5 can be proved in the same manner as it was done in [13, 16].

Let us show the asymptotics of self-similar solutions.

The case $\boldsymbol{1/(p-1)<\beta\leq1,\; q>\dfrac{(2-n)\,(p-1)}{p-n}}$. Consider the following self-similar solution of problem (1)–(3):

$$u_1\left(x,t\right)=t^{\alpha}\varphi\left(\xi\right),\;\;\xi=xt^{-\gamma},\tag{15}$$

where $\alpha=\dfrac{1}{1-\beta}$, $\gamma=\dfrac{p-1-\beta}{(p-n)\left(1-\beta\right)}$ and $\varphi\left(\xi\right)$ is solution of the following problem

$$\frac{d}{d\xi}\left(\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\right)+\gamma\xi^{1-n}\frac{d\varphi}{d\xi}-\alpha\xi^{-n}\varphi+\xi^{-n}\varphi^{\beta}=0\tag{16}$$

$$-\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\Bigg|_{\xi=0}=\varphi^{q}\left(0\right)\tag{17}$$

Consider the function

$$\bar{\varphi}\left(\xi\right)=\left(a-\frac{p-2}{p-n}\gamma^{\frac{1}{p-1}}\xi^{\frac{p-n}{p-1}}\right)_{+}^{\frac{p-1}{p-2}},\tag{18}$$

where $a>0$, $(y)_{+}=\max(0,y)$. We show that the function (18) is the asymptotics of the solutions of problem (16), (17).

Theorem 6. *The compactly supported solution of problem (16), (17) has the asymptotic*

$$\varphi\left(\xi\right)=\bar{\varphi}\left(\xi\right)\left(1+o(1)\right),$$

when $\xi\rightarrow\left(\dfrac{a\left(p-n\right)}{\left(p-2\right)\gamma^{1/(p-1)}}\right)^{(p-1)/(p-n)}$.

Proof. We seek a solution of equation (16) in the following form

$$\varphi=\bar{\varphi}\left(\xi\right)w(\tau),\tag{19}$$

with $\tau=-\ln\left(a-b\xi^{\frac{p-n}{p-1}}\right)$, $b=\dfrac{p-2}{(p-n)}\gamma^{1/(p-1)}$, where $\tau\rightarrow+\infty$ at $\xi\rightarrow(a/b)^{(p-1)/(p-n)}$.

Substituting (19) into equation (16) with regard to (18), it takes the form:

$$\frac{d}{d\tau}(L_1w)^{p-1}+\left(k_1\phi_1\left(\tau\right)-\frac{p-1}{p-2}\right)(L_1w)^{p-1}+k_2L_1w-k_3w\phi_2\left(\tau\right)-k_4w^{\beta}\phi_3\left(\tau\right)=0,\tag{20}$$

where $L_1w=\dfrac{w}{p-2}-\dfrac{w'}{p-1}$, $\phi_1\left(\tau\right)=e^{-\tau}$, $\phi_2\left(\tau\right)=e^{-\tau}\big/\left(a-e^{-\tau}\right)$, $\phi_3\left(\tau\right)=e^{-\frac{\beta(p-1)-1}{p-2}\tau}\Big/\left(a-e^{-\tau}\right)$, $k_1=\dfrac{p-1}{b\left(p-n\right)}$, $k_2=\dfrac{\gamma\left(p-1\right)}{\left(b\left(p-n\right)\right)^{p-1}}$, $k_3=\dfrac{\alpha(p-1)}{b^{p-1}(p-n)^{p}}$, $k_4=\dfrac{p-1}{\left(b\left(p-n\right)\right)^{p}}$.

We note that the study of the solutions of the last equation is equivalent to the study of those solutions of equation (1), each of which satisfies the inequalities in some interval $[\tau_0, +\infty)$:

$$w(\tau) > 0, \quad \frac{w(\tau)}{p-2} - \frac{w'(\tau)}{p-1} \neq 0.$$

We verify that the solution $w(\tau)$ of the equation (20) has a finite limit w_0 or not at $\tau \to +\infty$. Let $\nu(\tau) = (L_1 w)^{p-1}$. Then, for the derivative of the function $\nu(\tau)$, we have:

$$v' = -\left(k_1\phi_1(\tau) - \frac{p-1}{p-2}\right)v - k_2 L_1 w + k_3 w\phi_2(\tau) + k_4 w^\beta \phi_3(\tau).$$

We introduce the auxiliary function to analyze the solutions of the last equation:

$$\theta(\tau,\mu) = -\left(k_1\phi_1(\tau) - \frac{p-1}{p-2}\right)\mu - k_2 L_1 w + k_3 w\phi_2(\tau) + k_4 w^\beta \phi_3(\tau) \tag{21}$$

where $\mu \in \mathbb{R}$. Hence, it is easy to see that for each value of μ the function $\theta(\tau,\mu)$ keep the sign on some interval $[\tau_1, +\infty) \subset [\tau_0, +\infty)$ and for all $\tau \in [\tau_1, +\infty)$ holds one of the inequalities $\nu'(\tau) > 0$, $\nu'(\tau) < 0$. Then, analyzing (21) with the help of Bohl's theorem [13], we conclude that there for the function $\nu(\tau)$ exists a limit at $\tau \in [\tau_1, +\infty)$. It is easy to see that

$$\lim_{\tau\to+\infty} \phi_1(\tau) \to 0, \quad \lim_{\tau\to+\infty} \phi_2(\eta) \to 0, \quad \lim_{\tau\to+\infty} \phi_3(\eta) \to 0$$

at $\xi \to (a/b)_-^{(p-1)/(p-n)}$. Then taking into account the last limit and $w' = 0$ from (20), we obtain following algebraic equation for w

$$(p-1)\left(\frac{w}{p-2}\right)^{p-1} - k_2 w = 0.$$

From this equation, we get that $w = 1$, thus we have $\varphi(\xi) = \bar{\varphi}(\xi)(1 + o(1))$.

The case $\beta > 2p-1$, $q < \dfrac{(2-n)(p-1)}{(p-n)}$.

Theorem 7. *The compactly supported solution of problem (1)-(3) has the asymptotic:*

$$u(x,t) = Ct^\alpha \bar{g}(\xi)(1 + o(1)),$$

where $C = \left(\frac{p-2}{p-1}b\gamma\right)^{1/(p-2)} \frac{p-2}{B(p-1)}$, $\bar{g}$ *is above defined function.*

The critical case $(p-n)q = (2-n)(p-1)$. In this case, we consider following exponential form solution of problem (1)–(3):

$$u_4(x,t) = e^{\alpha(t-\tau)}\phi(\eta), \ \eta = xe^{-\gamma(t-\tau)},$$

where $\alpha = \dfrac{p-n}{2p-1}$, $\gamma = \dfrac{p-2}{2p-1}$ and $\phi(\eta)$ is the solution of the problem

$$\frac{d}{d\eta}\left(\left|\frac{d\phi}{d\eta}\right|^{p-2}\frac{d\phi}{d\eta}\right) + \gamma\eta^{1-n}\frac{d\phi}{d\eta} - \alpha\eta^{-n}\phi = 0, \tag{22}$$

$$-\left|\frac{d\phi}{d\eta}\right|^{p-2}\frac{d\phi}{d\eta}\Bigg|_{\eta=0} = \phi^q(0). \tag{23}$$

We take:

$$\bar{\phi}(\eta) = D\left(D^{p-2}\left(\frac{p-1}{p-2}\right)^p - \eta\right)_+^{\frac{p-1}{p-2}},$$

where $D > 0$.

Theorem 8. *The compactly supported solution of problem (22)–(23) has the asymptotic*

$$\phi(\eta) = C\bar{\phi}(\eta)(1 + o(1)),$$

at $\eta \to D^{p-2}\left(\frac{p-1}{p-2}\right)^p$, *where* $C = \left(\frac{p-1}{p-2}\gamma\right)^{1/(p-2)}$.

Theorem 7 and Theorem 8 are proved similarly to the proofs of Theorems 6.

Numerical experiments. It is known that choosing a suitable initial approximation preserving nonlinear properties is very important in numerical analysis. For this purpose, a computer experiment was conducted on the basis of the above qualitative properties of solutions for the case of global solvability. Since equation (1) approximated the second order of accuracy respect to x and first order of accuracy respect to t. The iterative process for numerical modeling was constructed, in the inner steps of iteration the node values are calculated by the Thomas algorithm. It is well known that, the iteration methods required a good initial approximation, which quickly converge to the exact solution and retain the qualitative properties of the studied nonlinear processes. This is the difficulty of the numerical analysis of the nonlinear problem. It depends on the values of the numerical parameters of the equation and can be overcome by successful choice initial approximations. We present some results of numerical experiments.

Figures 1–3 show that filtration process dependents on the density of the medium. Numerical experiments show rapid convergence of iterations to the exact solution. It is due to the choice of a suitable initial approximation. The number of iterations is not more than 5 for various values of numerical parameters.

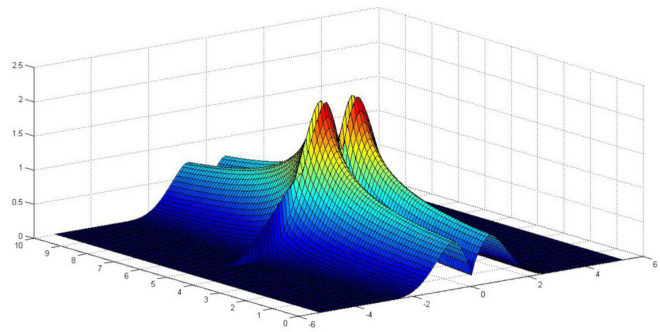

FIG. 1. Numerical solution of (1)–(3): $\beta = 2.1$, $p = 2.65$, $n = 1.2$, $q = 2.5$

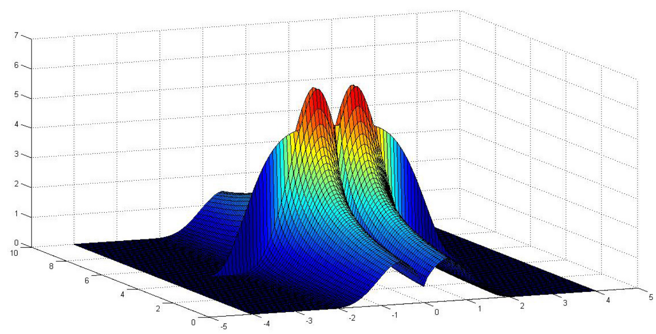

FIG. 2. Numerical solution of (1)–(3): $\beta = 2.1$, $p = 2.35$, $n = 3.5$, $q = 3$

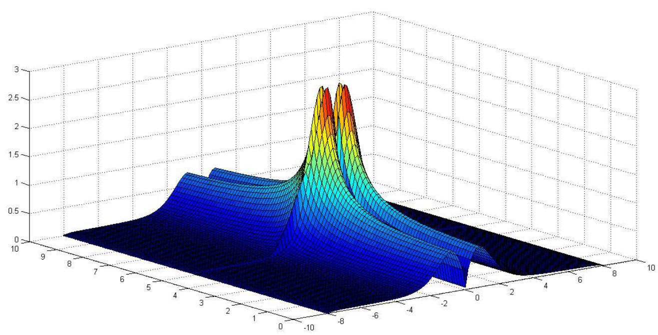

FIG. 3. Numerical solution of (1)–(3): $\beta = 1.6$, $p = 2.25$, $n = 0.8$, $q = 2.5$

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