

Wave dynamics on time-depending graph with Aharonov–Bohm ring

D. A. Eremin¹, E. N. Grishanov¹, D. S. Nikiforov², I. Y. Popov²

¹Department of Mathematics and IT, Ogarev Mordovia State University
Bolshevistskaya Str. 68, Saransk, Russia

²Department of Higher Mathematics, ITMO University,
Kroverkskiy pr. 49, St. Petersburg, 197101, Russia

ereminda@mail.ru, evgenyg@mail.ru, dmitrii.nikiforov@gmail.com, popov1955@gmail.com

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Aharonov–Bohm ring (AB ring) is an element frequently used in nanosystems. The paper deals with wave dynamics on quantum graph consisting of AB ring coupled to a segment. It is assumed that the lengths of the edges vary in time. Variable replacement is made to come to the problem for stationary geometric graph. The obtained equation is solved using the expansion with respect to a complete system of eigenfunctions of the unperturbed self-adjoint operator for the stationary graph. The coefficients of the expansion are found as solutions of a system of differential equations numerically. The influence of the magnetic field is studied. The comparison with the case of stable geometric graph is made.

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1. Introduction

Nanostructures in a magnetic field attract great attention of physicists due to its interesting behavior. Many of these properties, e.g., giant magnetoresistance, have found wide range of applications in nanotechnology, computer hardware, etc. One of the basic elements for nanostructures in a magnetic field is Aharonov–Bohm ring [1], i.e. a nano-sized conducting ring in a magnetic field orthogonal to the ring plane. The Aharonov–Bohm effect was observed and studied in many physical situations (see, e.g., [2–6]). For such structures, quantum graph model is actively used [7–10]. It is a rather effective mathematical model allowing one to describe the spectral and transport properties of many complex physical systems [11–15]. We consider a quantum graph with edge lengths varying in time. Although this model is very interesting from a physical point of view (see, e.g., [16]), there are only a few works devoted to this problem. One can mention papers concerning the time-dependent boundary conditions [17] or time dependent point-like interactions [18], but consideration of time-dependent graphs began recently [19–23]. Wave dynamics for time-dependent quantum graphs in a magnetic field was not studied previously. There are only works concerning Aharonov–Bohm rings in fluctuating magnetic field (see, e.g., [24]). In the present paper, we construct and study a model of time-dependent quantum graph with loop in magnetic field. We investigate the dependence of the dynamics on a magnetic field.

2. Model

2.1. Graph description

We consider quantum graph Γ with a loop (ring) shown in Fig. 1. It is assumed that the magnetic field acts at the ring. Lengths of graph edges, ring ($L_r(t)$) and segment ($L_\ell(t)$), vary in time in accordance with the following relations:

$$\begin{cases} L_r(t) = 2\pi rL(t), \\ L_\ell(t) = lL(t), \end{cases} \quad (1)$$

where r, l are constants, $L(t)$ is some twice continuously differentiable function. Obviously, $L_r(t)$ and $L_\ell(t)$ should be positive.

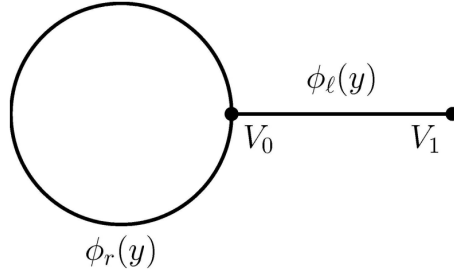


FIG. 1. Geometrical structure of the graph

2.2. Schrödinger operator on the graph

We start with the problem corresponding to the absence of a magnetic field. It means free Schrödinger operator acts on edges of the graph:

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \Psi_r(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi_r(x, t), & 0 \leq x \leq L_r(t), \\ i\hbar \frac{\partial}{\partial t} \Psi_\ell(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi_\ell(x, t), & 0 \leq x \leq L_\ell(t). \end{cases} \quad (2)$$

We look for continuous solution satisfying the Kirchhoff conditions at the central vertex V_0 and the Dirichlet conditions at the boundary one V_1 :

$$\begin{cases} \Psi_\ell(0, t) = \Psi_r(0, t) = \Psi_r(L_r(t), t), \\ \Psi_\ell(L_\ell(t), t) = 0, \\ \frac{\partial}{\partial x} \Psi_\ell(0, t) + \frac{\partial}{\partial x} \Psi_r(0, t) - \frac{\partial}{\partial x} \Psi_r(L_r(t), t) = 0. \end{cases} \quad (3)$$

We replace variables in (2), (3) to obtain a problem with non-varying edges:

$$\begin{cases} y = \frac{x}{L(t)}, \\ t_1 = t, \end{cases} \quad (4)$$

Further, t_1 is mentioned as t for simplicity. After the replacement, we come to the following equations:

$$\begin{cases} i \frac{\partial}{\partial t} \Psi_r = -\frac{1}{L^2} \frac{\partial^2}{\partial y^2} \Psi_r + i \frac{\dot{L}}{L} y \frac{\partial}{\partial y} \Psi_r, & 0 \leq x \leq 2\pi r, \\ i \frac{\partial}{\partial t} \Psi_\ell = -\frac{1}{L^2} \frac{\partial^2}{\partial y^2} \Psi_\ell + i \frac{\dot{L}}{L} y \frac{\partial}{\partial y} \Psi_\ell, & 0 \leq x \leq l. \end{cases} \quad (5)$$

Here $\dot{L} = dL/dt$. The boundary conditions remain the same as (3). The appearing factor $L(t)$ in the last condition can be omitted due to its positivity.

Thus, we obtained the problem for stable geometric graph. To solve it, we will use eigenfunctions of the stationary Schrödinger operator. The corresponding problem for such quantum graph (with the length of segment equals l and the radius of the ring equals r) has the form:

$$\begin{aligned} -\frac{d^2}{dy^2} \phi_\ell(y) &= k^2 \phi_\ell(y), 0 \leq y \leq l, \\ -\frac{d^2}{dy^2} \phi_r(y) &= k^2 \phi_r(y), 0 \leq y \leq 2\pi r, \\ \begin{cases} \phi_\ell(0) = \phi_r(0) = \phi_r(2\pi r), \\ \phi_\ell(l) = 0, \\ \frac{d}{y} \phi_l|_{y=0} + \frac{d}{dy} \phi_r|_{y=0} - \frac{d}{dy} \phi_r|_{y=2\pi r} = 0. \end{cases} \end{aligned}$$

The eigenfunctions are as follows

$$\begin{aligned}\phi_\ell^{(n)}(y) &= \frac{\sin(k_n(l-y))}{B_n \sin(k_n)}, \\ \phi_r^{(n)}(y) &= \frac{\cos(k_n(y-\pi r))}{B_n \cos(k_n \pi r)},\end{aligned}$$

the normalizing coefficient is

$$B_n^2 = \frac{1}{2 \sin^2(k_n l)} + \frac{\pi r}{\cos^2(\pi k_n r)},$$

and k_n is n -th root of characteristic equation

$$2 \tan(\pi k r) = \cot k l.$$

Dealing with stationary geometric graph, we can expand solution into a series of the eigenfunctions which form a complete set as eigenfunctions of self-adjoint operator:

$$\begin{pmatrix} \psi_\ell(y, t) \\ \psi_r(y, t) \end{pmatrix} = \sum_n c_n(t) \begin{pmatrix} \phi_\ell^{(n)}(y) \\ \phi_r^{(n)}(y) \end{pmatrix}. \quad (6)$$

After the substitution of expansion (6) into equations (5), multiplication by $\phi_r^{(m)}$ and $\phi_\ell^{(m)}$, correspondingly, summation of the both equations and integration of the obtained expression over graph Γ , we come to a system of ordinary differential equations for coefficients $c_n(t)$ of the expansion:

$$\dot{c}_m(t) + i \frac{k_m^2}{L^2} c_m - \sum_n c_n \frac{\dot{L}}{L} \int_\Gamma y \frac{\partial \phi^{(n)}}{\partial y} \phi^{(m)} dy = 0. \quad (7)$$

The system is truncated and solved numerically. We will discuss it in the next section, where the analogous procedure will be applied to the graph in a magnetic field.

2.3. Quantum graph in magnetic field

In the model, it is assumed that we have different operators acting at the loop and at the segment. While at the segment, we deal with the free Schrödinger operator, at the loop we consider the Landau operator, i.e. the Schrödinger operator with a magnetic field:

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \Psi_r(x, t) = \frac{\hbar^2}{2m} \left(-i \frac{\partial}{\partial x} + \frac{\Phi(t)}{L(t)\Phi_0} \right)^2 \Psi_r(x, t), & 0 \leq x \leq L_r(t), \\ i\hbar \frac{\partial}{\partial t} \Psi_\ell(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi_\ell(x, t), & 0 \leq x \leq L_\ell(t), \end{cases} \quad (8)$$

where the magnetic flux is $\Phi(t) = \pi r^2 L^2(t) B$, Φ_0 is the magnetic flux quantum and B is constant magnetic field.

After variables replacement (4), one obtains:

$$\begin{cases} i\hbar \left(-\frac{\dot{L}}{L} y \frac{\partial}{\partial y} \Psi_r(y, t) + \frac{\partial}{\partial t} \Psi_r(y, t) \right) = \\ -\frac{\hbar^2}{2m} \frac{1}{L^2} \frac{\partial^2}{\partial y^2} \Psi_r(y, t) - i \frac{\hbar^2}{m} \frac{\Phi}{L\Phi_0} \frac{1}{L} \frac{\partial}{\partial y} \Psi_r(y, t) + i \frac{\hbar^2}{m} \frac{\Phi^2}{L^2 \Phi_0^2} \Psi_r(y, t), & 0 \leq y \leq l_0, \\ i\hbar \left(-\frac{\dot{L}}{L} y \frac{\partial}{\partial y} \Psi_\ell(y, t) + \frac{\partial}{\partial t} \Psi_\ell(y, t) \right) = -\frac{\hbar^2}{2m} \frac{1}{L^2} \frac{\partial^2}{\partial y^2} \Psi_\ell(y, t). \end{cases} \quad (9)$$

To solve the equations, we use the expansion with respect to the complete system of eigenfunctions of the free stationary Schrödinger operator on the stationary graph:

$$\begin{pmatrix} \Psi_\ell(y, t) \\ \Psi_r(y, t) \end{pmatrix} = \sum_n C_n(t) \begin{pmatrix} \phi_\ell^{(n)}(y) \\ \phi_r^{(n)}(y) \end{pmatrix}. \quad (10)$$

We put expansion (10) into equations (9) and multiply the both sides by $\phi_r^{(m)}$ and $\phi_\ell^{(m)}$, correspondingly. Then, we summarize both equations and integrate the expression over graph Γ . Finally, we come to a system of ordinary

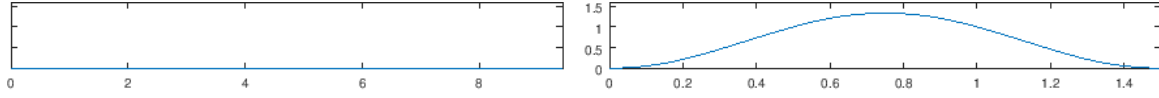


FIG. 2. Wave function modulo. Left – at the ring, right – at the segment; $t = 0.0$ (Initial function) (arbitrary units)

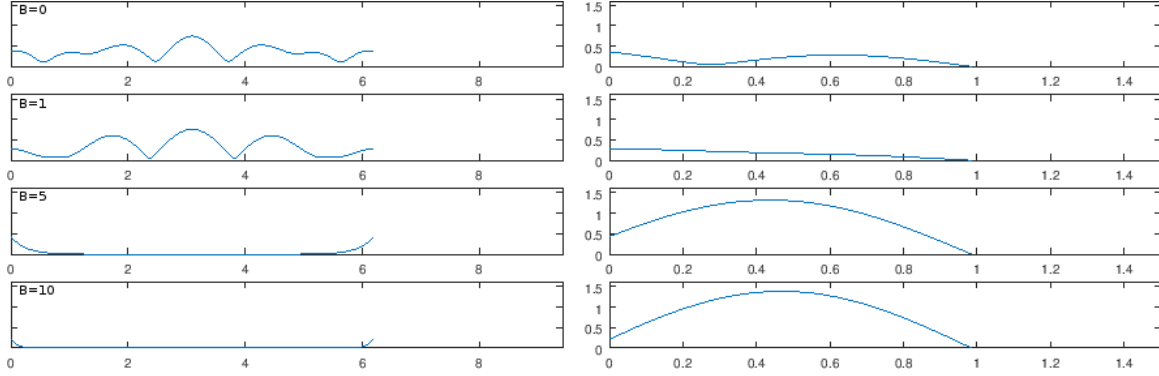


FIG. 3. Wave function modulo for different values of B . Left – at the ring, right – at the segment; $t = 1.6$ (arbitrary units)

differential equations for coefficients $C_n(t)$ of expansion (10):

$$\begin{aligned} \dot{C}_m + i \frac{\hbar}{2m} \frac{k_m^2}{L^2} C_m - \sum_n C_n \frac{\dot{L}}{L} \left(\int_0^{2\pi r} y \frac{\partial \phi_r^{(n)}}{\partial y} \phi_r^{(m)} dy + \int_0^l y \frac{\partial \phi_\ell^{(n)}}{\partial y} \phi_\ell^{(m)} dy \right) \\ + \frac{\hbar}{m} \frac{\pi r B}{\Phi_0} \sum_n C_n \int_0^{2\pi r} \frac{\partial \phi_r^{(n)}}{\partial y} \phi_r^{(m)} dy + i \frac{\hbar}{2m} \frac{\pi^2 r^2 L^2 B^2}{\Phi_0^2} \sum_n C_n \int_0^{2\pi r} \phi_\ell^{(n)} \phi_\ell^{(m)} dy = 0. \end{aligned} \quad (11)$$

3. Results and discussion

The initial conditions for system (11) are obtained from the initial condition for the wave function:

$$C_n(0) = \int_0^{2\pi r} \Psi_r(y, 0) \phi_r^{(n)} dy + \int_0^l \Psi_\ell(y, 0) \phi_\ell^{(n)} dy. \quad (12)$$

We choose the initial value of the wave function in the following form:

$$\begin{aligned} \Psi_r(y, 0) &= 0, \\ \Psi_\ell(y, 0) &= (1 - \cos 2\pi y) \sqrt{\frac{2}{3}}. \end{aligned}$$

Other parameters are chosen in the following way:

$$\begin{aligned} L(t) &= a + b \cos \omega t, \quad a = 1, \quad b = 0.5, \quad \omega = 1, \\ \hbar &= 2m = 1, \quad l = r = 1. \end{aligned}$$

System (11) is infinite. To solve it numerically, we make a truncation. We increase the number of equations up to the moment when the result becomes stable.

Results for $B = 0, 1, 5, 10$ are shown in Fig. 2–6. One can see that an increase of the magnetic field lead to greater localization of the solution at the segment and to stabilization of the wave function.

The stabilization of the wave function is observed. Comparing results for the graph having constant lengths of edges ($L = 1$) and for the time-dependent graph, one can see that in the stationary case, the magnetic field can stabilize the solution more quickly. Results for $B = 5$ are shown in Fig. 7–9.

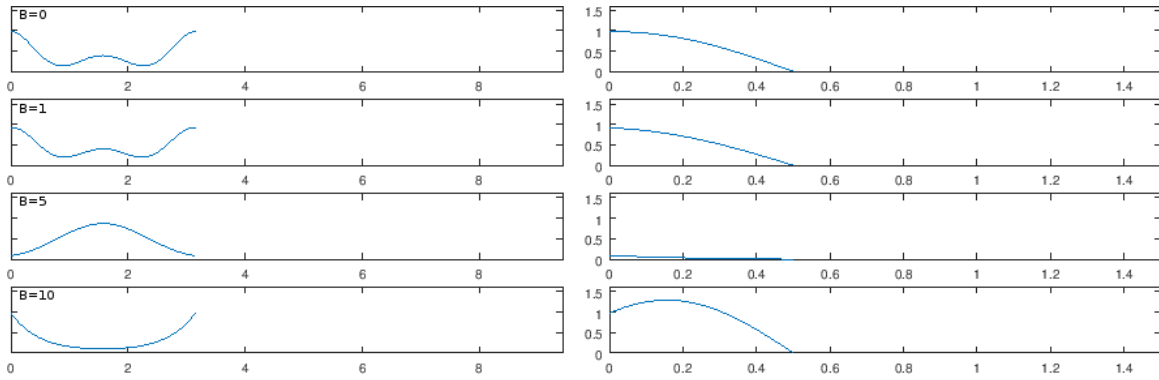


FIG. 4. Wave function modulo for different values of B . Left – at the ring, right – at the segment; $t = 3.24$ (arbitrary units)

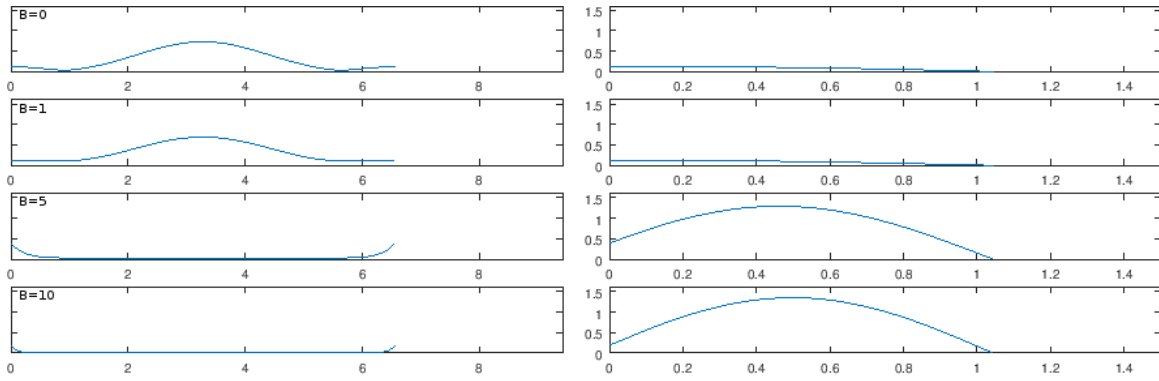


FIG. 5. Wave function modulo for different values of B . Left – at the ring, right – at the segment; $t = 4.8$ (arbitrary units)

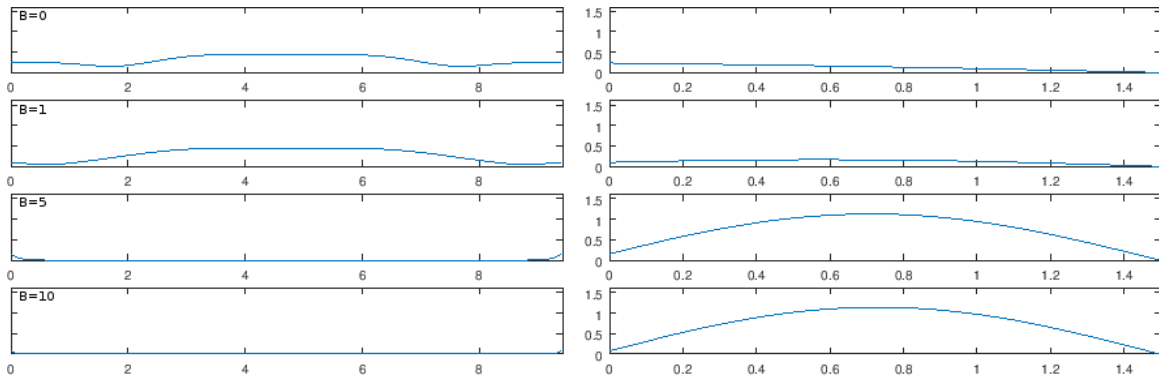


FIG. 6. Wave function modulo for different values of B . Left – at the ring, right – at the segment; $t = 6.4$ (arbitrary units)

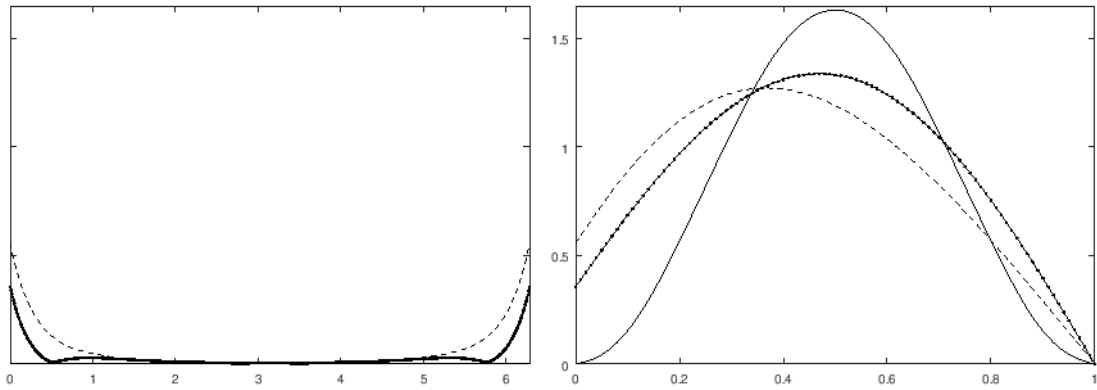


FIG. 7. Wave function modulo for $B = 5$. Left – at the ring, right – at the segment; thin curve – $t = 0.0$, dotted curve – $t = 0.2$, solid curve – $t = 0.4$ (arbitrary units)

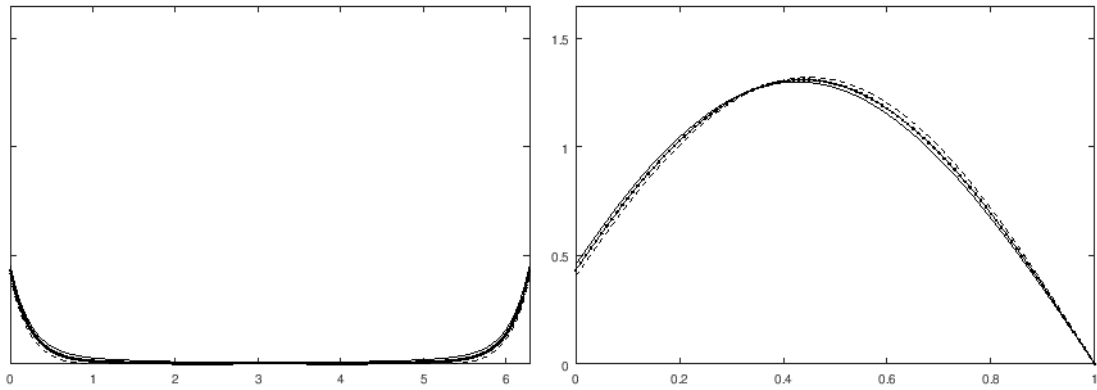


FIG. 8. Wave function modulo for $B = 5$. Left – at the ring, right – at the segment; thin curve – $t = 0.6$, dotted curve – $t = 0.8$, solid curve – $t = 1.0$ (arbitrary units)

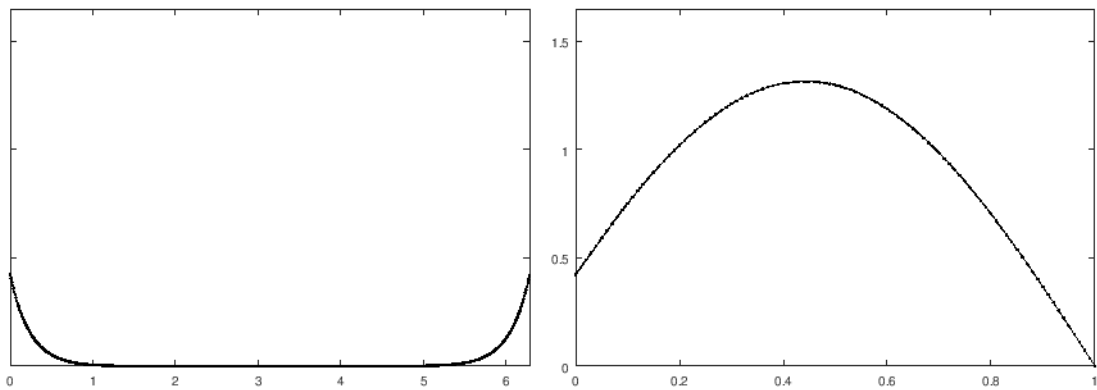


FIG. 9. Wave function modulo for $B = 5$. Left – at the ring, right – at the segment; thin curve – $t = 1.2$, dotted curve – $t = 1.4$, solid curve – $t = 1.6 \dots$ (arbitrary units)

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