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Inverse dynamic problem for the wave equation with periodic boundary conditions

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We consider the inverse dynamic problem for the wave equation with a potential on an interval $(0, 2\pi)$ with periodic boundary conditions. We use a boundary triplet to set up the initial-boundary value problem. As inverse data we use a response operator (dynamic Dirichlet-to-Neumann map). Using the auxiliary problem on the whole line, we derive equations of the inverse problem. We also establish the relationships between dynamic and spectral inverse data.

Keywords: inverse problem, Boundary Control method, Schrödinger operator.

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1. Introduction

Inverse problems for one-dimensional continuous and discrete systems plays an important role for the creation of new nano-devices, to mention just [1, 2] and references therein. In the present paper, we set up and study the inverse dynamic problem for a wave equation with a potential on an interval with periodic boundary conditions. The control problems for dynamical systems for wave equation with periodic boundary conditions (the density allows certain dependence on time) were considered in [3, 4]. The spectral problem for a Schrödinger operator on an interval with periodic and anti-periodic boundary conditions are used for treating the spectral problem for a Schrödinger operator with periodic potential on $\mathbb{R}$, see [5]. The inverse spectral problem with periodic boundary conditions for Schrödinger operator plays an important role for studying inverse problems on graphs with cycles [6].

In the previous papers by the authors, the "dynamic" approach to inverse spectral problems based on ideas of the Boundary Control method [7, 8] was developed in the cases of Schrödinger operator on a half-line [9–12] and finite and semi-infinite Jacobi matrices [13, 14]. We believe that our "dynamic" methods will help us to establish new relationships and develop new tools for studying the inverse problems with periodic potential, and will also stimulate studying inverse problems on graphs with cycles [6, 15].

For a potential $q \in C^2(0, 2\pi)$ we consider an operator H in $L_2(0, 2\pi)$ given by:

$$(Hf)(x) = -f''(x) + q(x)f(x), \quad x \in (0, 2\pi),$$
$$\operatorname{dom} H = \left\{ f \in H^2(0, 2\pi) \,|\, f(0) = f'(0) = f(2\pi) = f'(2\pi) = 0 \right\}.$$

Then

$$(H^*f)(x) = -f''(x) + q(x)f(x), \quad x \in (0, 2\pi),$$
$$\operatorname{dom} H^* = \left\{ f \in H^2(0, 2\pi) \right\}.$$

For a continuous function g we introduce the notations:

$$g_0 := \lim_{\varepsilon \to 0} g(0 + \varepsilon), \quad g_{2\pi} := \lim_{\varepsilon \to 0} g(2\pi - \varepsilon).$$

Let $B := \mathbb{R}^2$. The *boundary operators* $\Gamma_{0,1} : \operatorname{dom} H^* \mapsto B$ are introduced by the rules:

$$\Gamma_0 w := \begin{pmatrix} w_0 - w_{2\pi} \\ w_0' - w_{2\pi}' \end{pmatrix}, \quad \Gamma_1 w := \frac{1}{2} \begin{pmatrix} w_0' + w_{2\pi}' \\ -w_0 - w_{2\pi} \end{pmatrix}.$$

Integration by parts for $u, v \in \operatorname{dom} H^*$ shows that the abstract second Green identity holds:

$$(H^*u, v)_{L_2(0,2\pi)} - (u, H^*v)_{L_2(0,2\pi)} = (\Gamma_1 u, \Gamma_0 v)_B - (\Gamma_0 u, \Gamma_1 v)_B.$$

The mapping

$$\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \operatorname{dom} H^* \mapsto B \times B$$

is surjective. Then a triplet $\{B, \Gamma_0, \Gamma_1\}$ is a *boundary triplet* for H^* (see [16]).

Let $T > 0$ be fixed. We use the triplet $\{B, \Gamma_0, \Gamma_1\}$ to set up the following initial-boundary value problem:

$$\begin{cases} u_{tt} + H^* u = 0, \quad t > 0, \\ (\Gamma_0 u)(t) = \begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, \quad t > 0, \\ u(\cdot, 0) = u_t(\cdot, 0) = 0. \end{cases} \tag{1}$$

Here the vector function $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $f_1\, f_2 \in L_2(0, T)$, is interpreted as a *boundary control*. The solution to (1) is denoted by u^F. The *response operator* is introduced by the rule

$$\left(R^T F\right)(t) := \left(\Gamma_1 u^F\right)(t), \quad t > 0.$$

The speed of the wave propagation in the system (1) equal to one, which is why the natural set up of the dynamic inverse problem (IP) is to find a potential $q(x)$, $x \in (0, 2\pi)$ from the knowledge of a response operator $R^{2\pi}$ (see also [7, 8, 17, 18]).

In the second section, we derive the representation formula for the solution u^F, introduce the auxiliary dynamical system on the real line (see also [19]), and use the finiteness of the speed of wave propagation to establish relationships between the problem with periodic boundary conditions and problem on $\mathbb{R}$. In the third section, on the basis of this relationship, we obtain the suitable version of Krein and Gelfand-Levitan equations of the dynamic inverse problem. In the last section we derive the spectral representation of the response operator and dynamic representation of a Weyl function associated with $\{B, \Gamma_0, \Gamma_1\}$.

2. Forward problem, auxiliary dynamical system

We introduce the *outer space* of the system (1), the space of controls as $\mathcal{F}^T := L_2(0, T; \mathbb{R}^2)$, $F \in \mathcal{F}^T$, $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. By q we also denote the same potential, periodically continued to the whole real line: $q(x+2\pi) = q(x)$, $x \in \mathbb{R}$.

Theorem 1. *The solution to (1) with a control $F \in \mathcal{F}^T \cap C_0^\infty(\mathbb{R}_+)$, admits the following representation:*

1) *For* $0 < t < 2\pi$

$$\begin{aligned} u^F(x,t) &= u_1^{F_+}(x,t) + u_1^{F_-}(x,t) \\ &= \frac{1}{2} f_1(t-x) - \frac{1}{2} f_2(t-x) + \int_x^t w_1^0(x,s) f_1(t-s) + w_2^0(x,s) f_2(t-s)\, ds \\ &- \frac{1}{2} f_1(t+x-2\pi) - \frac{1}{2} f_2(t+x-2\pi) + \int_{2\pi - x}^t w_1^{2\pi}(x,s) f_1(t-s) + w_2^{2\pi}(x,s) f_2(t-s)\, ds. \end{aligned} \tag{2}$$

where kernels $w_{1,2}^{0,2\pi}(x,t)$ *satisfy the following Goursat problems:*

$$\begin{cases} w^0_{1\,tt}(x,t) - w^0_{1\,xx}(x,t) + q(x)w^0_1(x,t) = 0, \quad 0 < x < t, \\ \dfrac{d}{dx}w^0_1(x,x) = -\dfrac{q(x)}{4}, \quad x > 0, \\ w_1^{2\pi}{}_{tt}(x,t) - w_1^{2\pi}{}_{xx}(x,t) + q(x)w_1^{2\pi}(x,t) = 0, \quad 0 < 2\pi - x < t, \\ \dfrac{d}{dx}w_1^{2\pi}(x, 2\pi - x) = -\dfrac{q(x)}{4}, \quad x > 0, \\ w^0_1(0,s) = w_1^{2\pi}(2\pi, s), \\ w^0_{1\,x}(0,s) = w_1^{2\pi}{}_x(2\pi, s). \end{cases} \tag{3}$$

$$\begin{cases} w^0_{2\,tt}(x,t) - w^0_{2\,xx}(x,t) + q(x)w^0_2(x,t) = 0, \quad 0 < x < t, \\ \dfrac{d}{dx}w^0_2(x,x) = \dfrac{q(x)}{4}, \quad x > 0, \\ w_2^{2\pi}{}_{tt}(x,t) - w_2^{2\pi}{}_{xx}(x,t) + q(x)w_2^{2\pi}(x,t) = 0, \quad 0 < 2\pi - x < t, \\ \dfrac{d}{dx}w_2^{2\pi}(x, 2\pi - x) = -\dfrac{q(x)}{4}, \quad x > 0, \\ w^0_2(0,s) = w_2^{2\pi}(2\pi, s), \\ w^0_{2x}(0,s) = w_2^{2\pi}{}_x(2\pi, s). \end{cases} \tag{4}$$

2) *On* $0 < t < 4\pi$

$$\begin{aligned} u^F(x,t) &= u_1^{F_+}(x,t) + u_1^{F_-}(x,t) + u_2^{F_+}(x,t) + u_2^{F_-}(x,t) \\ &= \frac{1}{2}f_1(t-x) - \frac{1}{2}f_2(t-x) + \int_x^t w^0_1(x,s)f_1(t-s) + w^0_2(x,s)f_2(t-s)\,ds \\ &- \frac{1}{2}f_1(t+x-2\pi) - \frac{1}{2}f_2(t+x-2\pi) + \int_{2\pi-x}^t w_1^{2\pi}(x,s)f_1(t-s) + w_2^{2\pi}(x,s)f_2(t-s)\,ds \\ &+ \frac{1}{2}f_1(t-2\pi-x) - \frac{1}{2}f_2(t-2\pi-x) \\ &+ \int_x^{t-2\pi} \widetilde{w}^0_1(x,s)f_1(t-2\pi-s) + \widetilde{w}^0_2(x,s)f_2(t-2\pi-s)\,ds \\ &- \frac{1}{2}f_1(t+x-4\pi) - \frac{1}{2}f_2(t+x-4\pi) \\ &+ \int_{2\pi-x}^{t-2\pi} \widetilde{w}_1^{2\pi}(x,s)f_1(t-2\pi-s) + \widetilde{w}_2^{2\pi}(x,s)f_2(t-2\pi-s)\,ds. \end{aligned}$$

where the integral kernels $w_{1,2}^{0,2\pi}$, $\widetilde{w}_{1,2}^{0,2\pi}$ *satisfy certain Goursat problems and the following compatibility conditions:*

$$\begin{aligned} w^0_{1,2}(0,s) = w^{2\pi}_{1,2}(2\pi,s),\ w^0_{1,2_x}(0,s) = w^{2\pi}_{1,2_x}(2\pi,s), &\quad 0 < s < 4\pi, \\ w^0_{1,2}(2\pi,s) = \widetilde{w}^0_{1,2}(0,s-2\pi),\ w^0_{1,2_x}(2\pi,s) = \widetilde{w}^0_{1,2_x}(0,s-2\pi), &\quad 0 < s < 4\pi, \\ w^{2\pi}_{1,2}(0,s) = \widetilde{w}^{2\pi}_{1,2}(2\pi,s-2\pi),\ w^{2\pi}_{1,2_x}(0,s) = \widetilde{w}^{2\pi}_{1,2_x}(2\pi,s-2\pi), &\quad 0 < s < 4\pi. \end{aligned}$$

3) *On* $0 < t < 2n\pi$, $n > 1$:

$$u^F(x,t) = u_1^{F_+}(x,t) + u_1^{F_-}(x,t) + \ldots + u_n^{F_+}(x,t) + u_n^{F_-}(x,t), \tag{5}$$

where

$$\begin{aligned} u_k^{F_+}(x,t) &= \frac{1}{2}f_1(t-x-2(k-1)\pi) - \frac{1}{2}f_2(t-x-2(k-1)\pi) \\ &+ \int_{x+2(k-1)\pi}^t w_1(x+2(k-1)\pi,s)f_1(t-s) + w_2(x+2(k-1)\pi,s)f_2(t-s)\,ds \\ u_k^{F_-}(x,t) &= -\frac{1}{2}f_1(t+x-2k\pi) - \frac{1}{2}f_2(t+x-2k\pi) \\ &+ \int_{2k\pi-x}^t w_1(x-2k\pi,s)f_1(t-s) + w_2(x-2k\pi,s)f_2(t-s)\,ds \end{aligned}$$

and kernels $w_{1,2}$ satisfy the following Goursat problem:

$$\begin{cases} w_{1tt}(x,t) - w_{1xx}(x,t) + q(x)w_1(x,t), \quad 0 < |x| < t < 2n\pi, \\ \dfrac{d}{dx}w_1(x,x) = -\dfrac{q(x)}{4}, \quad x > 0, \\ \dfrac{d}{dx}w_1(x,-x) = -\dfrac{q(x)}{4}, \quad x < 0, \end{cases} \tag{6}$$

$$\begin{cases} w_{2tt}(x,t) - w_{2xx}(x,t) + q(x)w_2(x,t), \quad 0 < |x| < t < 2n\pi, \\ \dfrac{d}{dx}w_2(x,x) = \dfrac{q(x)}{4}, \quad x > 0, \\ \dfrac{d}{dx}w_2(x,-x) = -\dfrac{q(x)}{4}, \quad x < 0. \end{cases} \tag{7}$$

Several remarks have to be made.

Remark 1. *The proof of the representation (2) is straightforward and similar to one in [19]. If $F \in \mathcal{F}^T$, the function u^F defined by (2) is a generalized solution to (1) for $t \in (0, 2\pi)$.*

Remark 2. *The compatibility conditions in (3), (4) is used in the next subsection to relate the solution of the problem with periodic boundary conditions with one of the problem on the whole line.*

Since we consider the periodic boundary conditions, sometimes it would be convenient for us to interpret the interval as a ring:

Remark 3. *The compatibility conditions in 2) allows one to construct the "general" Goursat problems in 3). The physical meaning of the representation (5) is clear: the members of the sum indexed with "+" corresponds to waves that move clockwise, ones indexed with "−" correspond to waves moving counterclockwise.*

The *response operator* $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ with the domain $D_R = \left\{\mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)\right\}$ is defined by the rule

$$(R^T F)(t) := \left(\Gamma_1 u^F\right)(t), \quad 0 < t < T.$$

Representation (2) implies the following

Corollary 1. *The response operator has a form:*

1) *on an interval* $(0, 2\pi)$*:*

$$\left(R^T F\right)(t) = -\frac{1}{2}\begin{pmatrix} f_1'(t) \\ -f_2(t) \end{pmatrix} + R * \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{8}$$

where

$$R(t) := \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix} = \begin{pmatrix} w^0_{1x}(0,t) & w^0_{2x}(0,t) \\ -w^0_1(0,t) & -w^0_2(0,t) \end{pmatrix} = \begin{pmatrix} w^{2\pi}_1{}_x(0,t) & w^{2\pi}_2{}_x(0,t) \\ -w^{2\pi}_1(0,t) & -w^{2\pi}_2(0,t) \end{pmatrix}$$

is a response matrix*,*

2) *on an interval* $(0, 2n\pi)$:

$$\left(R^T F\right)(t) = \left(-\frac{1}{2}\sum_{k=1}^{n}\begin{pmatrix} \delta'(t-2k\pi) & 0 \\ 0 & -\delta(t-2k\pi) \end{pmatrix} + \widetilde{R}(t)\right) * \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{9}$$

where the integral kernel $\widetilde{R}$ is expressed in terms of solutions to Goursat problems (6), (7).

Remark 4. *Due to the finite speed of wave propagation in system (1), the natural set up of IP is to recover the potential on $(0, 2\pi)$ from $R^{2\pi}$, that is why, for solving IP we can consider the system for times less or equal 2π.*

2.1. Auxiliary problem on $\mathbb{R}$

We introduce the the potential $\widetilde{q}$ by the rule

$$\widetilde{q}(x) = \begin{cases} q(x), \quad 0 < x < 2\pi, \\ 0, \quad x > 2\pi, \\ q(x+2\pi), \quad -2\pi < x < 0, \\ 0, \quad x < -2\pi, \end{cases} \tag{10}$$

For this potential, we consider an operator $\widetilde{H}$ in $L_2(\mathbb{R})$ given by:

$$(\widetilde{H}f)(x) = -f''(x) + \widetilde{q}(x)f(x), \quad x \in \mathbb{R},$$
$$\operatorname{dom}\widetilde{H} = \left\{ f \in H^2(\mathbb{R}) \,|\, f(0) = f'(0) = 0 \right\}.$$

Then:

$$(\widetilde{H}^* f)(x) = -f''(x) + \widetilde{q}(x)f(x), \quad x \in \mathbb{R},$$
$$\operatorname{dom}\widetilde{H}^* = \left\{ f \in L_2(\mathbb{R}) \,|\, f \in H^2(-\infty,0),\, f \in H^2(-\infty,0) \right\}.$$

For a continuous function g we denote:

$$g_\pm := \lim_{\varepsilon\to 0} g(0 \pm \varepsilon).$$

The *boundary operators* $\widetilde{\Gamma}_{0,1} : \operatorname{dom} H^* \mapsto B$ are introduced by the rules

$$\widetilde{\Gamma}_0 w := \begin{pmatrix} w_+ - w_- \\ w'_+ - w'_- \end{pmatrix}, \quad \widetilde{\Gamma}_1 w := \frac{1}{2}\begin{pmatrix} w'_+ + w'_- \\ -w_+ - w'_- \end{pmatrix}.$$

We consider the initial boundary value problem for an auxiliary dynamical system on $\mathbb{R}$:

$$\begin{cases} v_{tt} + v_{xx} + \widetilde{q}v = 0, \quad x \in \mathbb{R}, \quad 0 < t < 2\pi, \\ (\Gamma_0 v)(t) = \begin{pmatrix} f_1(t) \\ f'_2(t) \end{pmatrix}, \quad 0 < t < 2\pi, \\ v(\cdot,0) = v_t(\cdot,0) = 0. \end{cases} \tag{11}$$

In [19] the dynamic IP for (11) was studied, where as a inverse data the authors used the *response operator*, introduced by the rule:

$$\left(\widetilde{R}^T F\right)(t) := \left(\widetilde{\Gamma}_1 v^F\right)(t), \quad t > 0.$$

On comparing the representation (2) with one obtained in [19] in Theorem 1, one deduce that for $0 < t < 2\pi$ the following equality holds:

$$v^F(x,t) = \begin{cases} u_1^{F_+}(x,t), \quad 0 < x < 2\pi, \\ u_1^{F_-}(x+2\pi,t), \quad -2\pi < x < 0. \end{cases} \tag{12}$$

Moreover, one has that:

$$R^{2\pi}F = \Gamma_1 u^F = \widetilde{\Gamma}_1 v^F = \widetilde{R}^{2\pi}F, \quad 0 < t < 2\pi. \tag{13}$$

Thus we reduced our initial IP to the IP for dynamical system (11) of recovering the potential $\widetilde{q}(x)$, on the interval $-\pi < x < \pi$ from $\widetilde{R}^{2\pi}$.

3. Equations of IP

In this section, we briefly outline the results of [19] in applying to our situation. Fix a parameter $0 < T \leqslant \pi$ and introduce the *inner space*, the space of states of the system (11) as $\mathcal{H}^T := L_2(-T,T)$. The representation (12) and Theorem 1 imply that $v^F(\cdot,T) \in \mathcal{H}^T$.

A *control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ is defined by the formula $W^T F := v^F(\cdot,T)$. The *reachable set* is defined by the rule:

$$U^T := W^T\mathcal{F}^T = \left\{ v^F(\cdot,T) \,\middle|\, F \in \mathcal{F}^T \right\}.$$

It will be convenient for us to associate the outer space $\mathcal{H}^T = L_2(-T,T)$ with a vector space $L_2(0,T;\mathbb{R}^2)$ by setting for $a \in L_2(-T,T)$ (we keep the same notation for a function)

$$a = \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} \in L_2(0,T;\mathbb{R}^2), \quad a_1(x) := a(x),\, a_2(x) := a(-x),\, x \in (0,T).$$

Theorem 2. *The control operator is a boundedly invertible isomorphism between $\mathcal{F}^T$ and $\mathcal{H}^T$, and $U^T = \mathcal{H}^T$.*

The *connecting operator* $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is introduced via the quadratic form:

$$\left(C^T F_1, F_2\right)_{\mathcal{F}^T} = \left(v^{F_1}(\cdot,T), v^{F_2}(\cdot,T)\right)_{\mathcal{H}^T}.$$

The crucial fact in the Boundary Control method is that the connecting operator is expressed in terms of inverse dynamic data:

Theorem 3. *The connecting operator C^T admits the following representation:*

$$\left(C^T F\right)(t) = \frac{1}{2}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + \int_0^T C(t,s)\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds,$$

where

$$C_{1,1}(t,s) = p_1(2T-t-s) - p_1(|t-s|), \quad p_1(s) = \int_0^s r_{11}(\alpha)\,d\alpha,$$

$$C_{1,2}(t,s) = \widetilde{p}_1(2T-t-s) - \widetilde{p}_1(t-s), \quad \widetilde{p}_1(s) = \begin{cases} \displaystyle\int_0^s r_{12}(\alpha)\,d\alpha, \ s > 0, \\ \displaystyle -\int_0^{-s} r_{12}(\alpha)\,d\alpha, \ s < 0, \end{cases}$$

$$C_{2,1}(t,s) = -\widetilde{r}_{21}(t-s) - \widetilde{r}_{21}(2T-t-s), \quad \widetilde{r}_{21}(s) = \begin{cases} r_{21}(s), \ s > 0, \\ -r_{21}(-s), \ s < 0, \end{cases}$$

$$C_{2,2}(t,s) = -r_{22}(|t-s|) - r_{22}(2T-t-s).$$

3.1. Krein equations

Let $y(x)$ be a solution to the following Cauchy problem:

$$\begin{cases} -y'' + \widetilde{q}y = 0, \quad x \in (-T,T), \\ y(0) = 0, \ y'(0) = 1. \end{cases} \tag{14}$$

We set up the *special control problem*: to find $F \in \mathcal{F}^T$ such that $W^T F = y$ in $\mathcal{H}^T$. By the Theorem 2, such a control F exists, but we can say even more:

Theorem 4. *The solution to a special control problem is a unique solution to the following Krein equation:*

$$\left(C^T F\right)(t) = (T-t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \in (0,T). \tag{15}$$

Representation formulas (2) and (12) imply that that the solution F to a special control problem satisfies relations:

$$y(T) = v^F(T,T) = \frac{1}{2}f_1(0) - \frac{1}{2}f_2(0),$$
$$y(-T) = v^F(-T,T) = -\frac{1}{2}f_1(0) - \frac{1}{2}f_2(0).$$

Thus solving (15) for all $T \in (0,\pi)$, we recover the solution $y(x)$ to (14) on the interval $(-\pi,\pi)$. Then the potential $\widetilde{q}(x)$, $x \in (-\pi,\pi)$ can be recovered as $\widetilde{q}(x) = \dfrac{y''(x)}{y(x)}$, $x \in (-\pi,\pi)$, and consequently

$$q(x) = \begin{cases} \widetilde{q}(x), \quad 0 < x < \pi, \\ \widetilde{q}(x-2\pi), \quad \pi < x < 2\pi. \end{cases}$$

3.2. Gelfand-Levitan equations

We introduce the notations:

$$C^T = \frac{1}{2}(I + C), \quad (Cf)(t) = 2\int_0^T C(t,s)\begin{pmatrix} f(s) \\ g(s) \end{pmatrix} ds,$$
$$J^T : \mathcal{F}^T \mapsto \mathcal{F}^T, \quad \left(J^T F\right)(t) = F(T-t),$$
$$\widetilde{C} = J^T C J^T, \quad \left(\widetilde{C}F\right)(t) = \int_0^T \widetilde{C}(t,s)F(s)\,ds. \tag{16}$$

Let $m(x,t) \in C\left((0,\pi)^2, R^{2\times 2}\right)$ denotes a matrix-valued function such that $m(x,t) = 0$ when $x > t$. In [13] it was proved the following

Theorem 5. *The unique solution to the Gelfand-Levitan equation*

$$m(x,s)+\widetilde{C}(x,s)+\int_0^\pi \widetilde{C}(x,\alpha)m(\alpha,s)\,d\alpha=0,\quad 0<x<s<\pi.$$

where the kernel $\widetilde{C}$ is defined by (16), determines the potential by the formula:

$$q(x)=\begin{cases}2\dfrac{d}{dx}\left(m_{11}(x,x)-m_{12}(x,x)\right), & x\in(0,\pi),\\ -2\dfrac{d}{dx}\left(m_{11}(2\pi-x,2\pi-x)+m_{12}(2\pi-x,2\pi-x)\right), & x\in(\pi,2\pi).\end{cases}$$

4. Relationship between dynamic and spectral inverse data

The problem of finding relationships between different types of inverse data is very important in inverse problems theory. We can mention [9, 10, 13, 20–22] on some recent results in this direction. Below we show the relationships between the dynamic response function, matrix spectral measure and Weyl matrix.

4.1. Response function and spectral measure

Consider two solutions to the equation:

$$-\phi''+q(x)\phi=\lambda\phi,\quad 0<x<2\pi, \tag{17}$$

satisfying the Cauchy data:

$$\varphi(0,\lambda)=0,\ \varphi'(0,\lambda)=1,\ \theta(0,\lambda)=1,\ \theta'(0,\lambda)=0.$$

The eigenvalues and normalized eigenfunctions of (17) with periodic boundary conditions:

$$\phi(0)=\phi(2\pi),\quad \phi'(0)=\phi'(2\pi). \tag{18}$$

are denoted by $\{\lambda_n,y_n\}_{n=1}^\infty$. Let $\beta_n,\gamma_n\in\mathbb{R}$ be such that:

$$y_n(x)=\beta_n\varphi(x,\lambda_n)-\gamma_n\theta(x,\lambda_n),$$

we point out that there can be eigenvalues of multiplicity two.

We evaluate:

$$\begin{aligned} y_n(0)=-\gamma_n,\quad y_n(2\pi)=\beta_n\varphi(2\pi,\lambda_n)+\gamma_n\theta(2\pi,\lambda_n),\\ y_n'(0)=\beta_n,\quad y_n'(2\pi)=\beta_n\varphi'(2\pi,\lambda_n)+\gamma_n\theta'(2\pi,\lambda_n).\end{aligned}$$

Then:

$$\Gamma_1 y_n=\frac12\begin{pmatrix} y_n'(0)+y_n'(2\pi)\\ -y_n(0)-y_n(2\pi)=\end{pmatrix}=\begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}. \tag{19}$$

Let $F\in\mathcal{F}^T\cap C_0^\infty(0,T;\mathbb{R}^2)$, and u^F be a solution to (1). On multiplying (1) by y_n and integrating by parts, we get the following relation:

$$\begin{aligned}0=\int_0^{2\pi}u_{tt}^F y_n\,dx-\int_0^{2\pi}u_{xx}^F y_n\,dx+\int_0^{2\pi}q(x)u^F y_n\,dx=\int_0^{2\pi}u_{tt}^F y_n\,dx\\ +\left(u^F,Hy_n\right)+\left(\Gamma_1 u^F,\Gamma_0 y_n\right)_B-\left(\Gamma_0 u^F,\Gamma_1 y_n\right)_B\\ =\int_0^{2\pi}u_{tt}^F y_n\,dx+\lambda_n\left(u^F,y_n\right)-\left(\begin{pmatrix}f_1(t)\\ f_2'(t)\end{pmatrix},\begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}\right)_B.\end{aligned}$$

Looking for the solution to (1) in the form:

$$u^F=\sum_{k=1}^\infty c_k(t)y_k(x), \tag{20}$$

we plug (20) into (1) and multiply by y_n and integrate over $(0,2\pi)$ to get:

$$\int_0^{2\pi}\sum_{k=1}^\infty c_k''(t)y_k(x)y_n(x)\,dx+\int_0^{2\pi}\sum_{k=1}^\infty c_k(t)y_k(x)\lambda_n y_n(x)\,dx=\left(\begin{pmatrix}f_1(t)\\ f_2'(t)\end{pmatrix},\begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}\right)_B.$$

Thus we obtain that $c_n(t)$, $n \geqslant 1$, satisfies the following Cauchy problem:

$$\begin{cases} c_n''(t) + \lambda_n c_n(t) = \left(\begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \right)_B, \\ c_n(0) = 0,\, c_n'(0) = 0. \end{cases}$$

the solution of which is given by the formula:

$$c_n(t) = \int_0^t \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(f_1(s)\beta_n + f_2'(s)\gamma_n\right) ds.$$

Then for u^F (20) we have the expansion:

$$\begin{aligned} u^F(x,t) &= \sum_{k=1}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(f_1(s)\beta_n + f_2'(s)\gamma_n\right) ds \left(\beta_n \varphi(x,\lambda_n) - \gamma_n \theta(x,\lambda_n)\right) \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(\begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \otimes \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2'(s) \end{pmatrix}, \begin{pmatrix} \varphi(x,\lambda_n) \\ -\theta(x,\lambda_n) \end{pmatrix} \right) \\ &= \int_{-\infty}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} \left(d\Sigma(\lambda) \begin{pmatrix} f_1(s) \\ f_2'(s) \end{pmatrix}, \begin{pmatrix} \varphi(x,\lambda) \\ -\theta(x,\lambda) \end{pmatrix} \right). \end{aligned} \tag{21}$$

Where $d\Sigma(\lambda)$ is a matrix measure (see [5]) introduced by the rule:

$$\Sigma(\lambda) = \sum_{\{k \,|\, \lambda_k < \lambda\}} \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} \otimes \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix}. \tag{22}$$

Thus, the response operator R^T is given by:

$$\begin{aligned} (RF)(t) = \Gamma_1 v^F &= \sum_{k=1}^{\infty} c_k(t) \Gamma_1 y_k = \sum c_k(t) \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda_k}(t-s)}{\sqrt{\lambda_k}} \left(f_1(s)\beta_k + f_2'(s)\gamma_k\right) ds \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} \\ &= \int_{-\infty}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} d\Sigma(\lambda) \begin{pmatrix} f_1(s) \\ f_2'(s) \end{pmatrix} ds, \quad 0 < t. \end{aligned} \tag{23}$$

4.2. Weyl function and response function

Let $N_\lambda := \ker(H^* - \lambda I)$, we observe that any $\psi(x,\lambda) \in N_\lambda$ is given by:

$$\psi(x,\lambda) = c_1 \varphi(x,\lambda) + c_2 \theta(x,\lambda). \tag{24}$$

We evaluate:

$$\begin{aligned} \psi_0 = c_2, \quad \psi_{2\pi} &= c_1 \varphi(2\pi) + c_2 \theta(2\pi), \\ \psi_0' = c_1, \quad \psi_{2\pi}' &= c_1 \varphi'(2\pi) + c_2 \theta'(2\pi). \end{aligned}$$

Thus the following relations hold:

$$\begin{aligned} \Gamma_0 \psi &= \begin{pmatrix} -\varphi(2\pi) & 1 - \theta(2\pi) \\ 1 - \varphi'(2\pi) & -\theta'(2\pi) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \\ \Gamma_1 \psi &= \frac{1}{2} \begin{pmatrix} 1 + \varphi'(2\pi) & \theta'(2\pi) \\ -\varphi(2\pi) & -(1 + \theta(2\pi)) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

The Weyl matrix is given by (see [16]):

$$M(\lambda) = \Gamma_1 \left(\Gamma_0|_{N_\lambda}\right)^{-1},$$

so we have:

$$M(\lambda) = \frac{1}{2} \begin{pmatrix} 1 + \varphi'(2\pi) & \theta'(2\pi) \\ -\varphi(2\pi) & -(1 + \theta(2\pi)) \end{pmatrix} \frac{1}{\det \Gamma_0} \begin{pmatrix} -\theta'(2\pi) & -(1 - \varphi'(2\pi)) \\ -(1 - \theta(2\pi)) & -\varphi(2\pi) \end{pmatrix}^T.$$

Evaluating the last expression we get the following formula for the Weyl matrix:

$$M(\lambda)=\frac{1}{2\left(F(2\pi,\lambda)-2\right)}\begin{pmatrix}-2\theta'(2\pi,\lambda)\left(1-\varphi'(2\pi,\lambda)\right) & -\varphi'(2\pi,\lambda)+\theta(2\pi,\lambda)\\ -\varphi'(2\pi,\lambda)+\theta(2\pi,\lambda) & 2\varphi(2\pi,\lambda)\end{pmatrix},$$

where

$$F(x,\lambda)=\varphi'(x,\lambda)+\theta(x,\lambda)$$

is a Lyapunov function.

In [9] the authors established the relationship between the Weyl function and the kernel of dynamic response operator (see also [10,13,22]). Note that one needs to know the response for all $t>0$. Then, cf. (9):

$$M(k^2)=\int_0^\infty\left(-\frac{1}{2}\sum_{k=1}^{\infty}\begin{pmatrix}\delta'(t-2k\pi) & 0\\ 0 & -\delta(t-2k\pi)\end{pmatrix}+\widetilde{R}(t)\right)e^{ikt}\,dt,$$

where this equality is understood in a weak sense.

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