NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2019, 10 (6), P. 616–622

Threshold analysis for a family of 2×2 operator matrices

T. H. Rasulov, E. B. Dilmurodov

Department of Mathematics, Faculty of Physics and Mathematics,
Bukhara State University, M. Ikbol str. 11, 200100 Bukhara, Uzbekistan

rth@mail.ru, elyor.dilmurodov@mail.ru

DOI 10.17586/2220-8054-2019-10-6-616-622

We consider a family of 2×2 operator matrices $\mathcal{A}_\mu(k)$, $k \in \mathbb{T}^3 := (-\pi, \pi]^3$, $\mu > 0$, acting in the direct sum of zero- and one-particle subspaces of a Fock space. It is associated with the Hamiltonian of a system consisting of at most two particles on a three-dimensional lattice $\mathbb{Z}^3$, interacting via annihilation and creation operators. We find a set $\Lambda := \{k^{(1)}, ..., k^{(8)}\} \subset \mathbb{T}^3$ and a critical value of the coupling constant μ to establish necessary and sufficient conditions for either $z = 0 = \min\limits_{k \in \mathbb{T}^3} \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k))$ (or $z = 27/2 = \max\limits_{k \in \mathbb{T}^3} \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k))$ is a threshold eigenvalue or a virtual level of $\mathcal{A}_\mu(k^{(i)})$ for some $k^{(i)} \in \Lambda$.

Keywords: operator matrices, Hamiltonian, generalized Friedrichs model, zero- and one-particle subspaces of a Fock space, threshold eigenvalues, virtual levels, annihilation and creation operators.

Received: 19 October 2019

Revised: 13 November 2019

1. Introduction

Operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces, see [1]. One special class of operator matrices are Hamiltonians associated with the systems of non-conserved number of quasi-particles on a lattice. In such systems the number of particles can be unbounded as in the case of spin-boson models [2,3] or bounded as in the case of "truncated" spin-boson models [4–7]. They arise, for example, in the theory of solid-state physics [8], quantum field theory [9] and statistical physics [4, 10].

The study of systems describing n particles in interaction, without conservation of the number of particles is reduced to the investigation of the spectral properties of self-adjoint operators acting in the *cut subspace* $\mathcal{H}^{(n)}$ of the Fock space, consisting of $r \leq n$ particles [4, 9, 10]. The perturbation of an operator (the generalized Friedrichs model which has a 2×2 operator matrix form acting in $\mathcal{H}^{(2)}$), with discrete and essential spectrum has played a considerable role in the study of spectral problems connected with the quantum theory of fields [9].

One of the most actively studied objects in operator theory, in many problems of mathematical physics and other related fields is the investigation of the threshold eigenvalues and virtual levels of block operator matrices, in particular, Hamiltonians on a Fock space associated with systems of non-conserved number of quasi-particles on a lattice. In the present paper, we consider a family of 2×2 operator matrices $\mathcal{A}_\mu(k)$, $k \in \mathbb{T}^3 := (-\pi, \pi]^3$, $\mu > 0$ (so - called generalized Friedrichs models) associated with the Hamiltonian of a system consisting of at most two particles on a three-dimensional lattice $\mathbb{Z}^3$, interacting via creation and annihilation operators. They are acting in the direct sum of zero-particle and one-particle subspaces of a Fock space. The main goal of the paper is to give a thorough mathematical treatment of the spectral properties of this family in three dimensions. More exactly, we find a set $\Lambda := \{k^{(1)}, ..., k^{(8)}\} \subset \mathbb{T}^3$ and prove that for a $i \in \{1, 2, \ldots, 8\}$ there is a value μ_i of the parameter μ such that only for $\mu = \mu_i$ the operator $\mathcal{A}_\mu(\bar{0})$ has a zero-energy resonance, here $0 = \min \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(\bar{0}))$ and the operator $\mathcal{A}_\mu(k^{(i)})$ has a virtual level at the point $z = 27/2 = \max \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k^{(i)}))$, where $\bar{0} := (0, 0, 0) \in \mathbb{T}^3$ and $k^{(i)} \in \Lambda$. We point out that a part of the results is typical for lattice models; in fact, they do not have analogs in the continuous case (because its essential spectrum is half-line $[E; +\infty)$, see for example [4]).

We notice that threshold eigenvalue and virtual level (threshold energy resonance) of a generalized Friedrichs model have been studied in [11–14]. The paper [15] is devoted to the threshold analysis for a family of Friedrichs models under rank one perturbations. In [16] a wide class of two-body energy operators $h(k)$ on the d-dimensional lattice $\mathbb{Z}^{\mathrm{d}}$, $\mathrm{d} \geq 3$, is considered, where k is the two-particle quasi-momentum. If the two-particle Hamiltonian $h(0)$ has either an eigenvalue or a virtual level at the bottom of its essential spectrum and the one-particle free Hamiltonians in the coordinate representation generate positivity preserving semi-groups, then it is shown that for all nontrivial values k, $k \neq 0$, the discrete spectrum of $h(k)$ below its threshold is non-empty. These results have been applied to the proof of the existence of Efimov's effect and to obtain discrete spectrum asymptotics of the corresponding Hamiltonians. We note that above mentioned results are discussed only for the bottom of the essential spectrum. The threshold eigenvalues and virtual levels of a slightly simpler version of $\mathcal{A}_\mu(k)$ were investigated in [17], and the structure of the numerical range are studied using similar results. In [18], the essential spectrum of the family of 3×3 operator matrices $H(K)$ is described by the spectrum of the family of 2×2 operator matrices. The results of the present paper are play important role in the investigations of the operator $H(K)$, see [12].

The plan of this paper is as follows: Section 1 is an introduction to the whole work. In Section 2, a family of 2×2 operator matrices are described as bounded self-adjoint operators in the direct sum of two Hilbert spaces and its spectrum is described. In Section 3, we discuss some results concerning threshold analysis of a family of 2×2 operator matrices.

We adopt the following conventions throughout the present paper. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of all positive integers, integers, real and complex numbers, respectively. We denote by $\mathbb{T}^3$ the three-dimensional torus (the first Brillouin zone, i.e., dual group of $\mathbb{Z}^3$), the cube $(-\pi, \pi]^3$ with appropriately identified sides equipped with its Haar measure. The torus $\mathbb{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi\mathbb{Z})^3$.

Denote by $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$, respectively, the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator.

2. Family of 2×2 operator matrices and its spectrum

Let $L_2(\mathbb{T}^3)$ be the Hilbert space of square-integrable (complex-valued) functions defined on the three-dimensional torus $\mathbb{T}^3$. Denote $\mathcal{H}$ by the direct sum of spaces $\mathcal{H}_0 := \mathbb{C}$ and $\mathcal{H}_1 := L_2(\mathbb{T}^3)$, that is, $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$. We write the elements f of the space $\mathcal{H}$ in the form $f = (f_0, f_1)$ with $f_0 \in \mathcal{H}_0$ and $f_1 \in \mathcal{H}_1$. Then for any two elements $f = (f_0, f_1)$ and $g = (g_0, g_1)$, their scalar product is defined by

$$(f, g) := f_0\overline{g_0} + \int_{\mathbb{T}^3} f_1(t)\overline{g_1(t)}dt.$$

The Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are zero- and one-particle subspaces of a Fock space $\mathcal{F}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively, where

$$\mathcal{F}(L_2(\mathbb{T}^3)) := \mathbb{C} \oplus L_2(\mathbb{T}^3) \oplus L_2((\mathbb{T}^3)^2) \oplus \cdots \oplus L_2((\mathbb{T}^3)^n) \oplus \cdots .$$

In the Hilbert space $\mathcal{H}$ we consider the following family of 2×2 operator matrices

$$\mathcal{A}_\mu(k) := \begin{pmatrix} A_{00}(k) & \mu A_{01} \\ \mu A_{01}^* & A_{11}(k) \end{pmatrix},$$

where $A_{ii}(k) : \mathcal{H}_i \to \mathcal{H}_i$, $i = 0, 1$, $k \in \mathbb{T}^3$ and $A_{01} : \mathcal{H}_1 \to \mathcal{H}_0$ are defined by the rules

$$A_{00}(k)f_0 = w_0(k)f_0, \quad A_{01}f_1 = \int_{\mathbb{T}^3} v(t)f_1(t)dt, \quad (A_{11}(k)f_1)(p) = w_1(k,p)f_1(p).$$

Here $f_i \in \mathcal{H}_i$, $i = 0, 1$; $\mu > 0$ is a coupling constant, the function $v(\cdot)$ is a real-valued analytic function on $\mathbb{T}^3$, the functions $w_0(\cdot)$ and $w_1(\cdot, \cdot)$ have the form

$$w_0(k) := \varepsilon(k) + \gamma, \quad w_1(k,p) := \varepsilon(k) + \varepsilon(k+p) + \varepsilon(p)$$

with $\gamma \in \mathbb{R}$ and the dispersion function $\varepsilon(\cdot)$ is defined by

$$\varepsilon(k) := \sum_{i=1}^{3}(1 - \cos k_i),\ k = (k_1, k_2, k_3) \in \mathbb{T}^3. \tag{2.1}$$

Under these assumptions the operator matrix $\mathcal{A}_\mu(k)$ is a bounded and self-adjoint in $\mathcal{H}$.

We remark that the operators A_{01} and A_{01}^* are called annihilation and creation operators [9], respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

Let $\mathcal{A}_0(k) := \mathcal{A}_\mu(k)|_{\mu=0}$. The perturbation $\mathcal{A}_\mu(k) - \mathcal{A}_0(k)$ of the operator $\mathcal{A}_0(k)$ is a self-adjoint operator of rank 2. Therefore, in accordance with the invariance of the essential spectrum under the finite rank perturbations [19], the essential spectrum $\sigma_{\rm ess}(\mathcal{A}_\mu(k))$ of $\mathcal{A}_\mu(k)$ fills the following interval on the real axis

$$\sigma_{\rm ess}(\mathcal{A}_\mu(k)) = [m(k), M(k)],$$

where the numbers $m(k)$ and $M(k)$ are defined by

$$m(k) := \min_{p \in \mathbb{T}^3} w_1(k,p), \quad M(k) := \max_{p \in \mathbb{T}^3} w_1(k,p). \tag{2.2}$$

Remark 2.1. *We remark that the essential spectrum of $\mathcal{A}_\mu(\bar{\pi})$, $\bar{\pi} := (\pi,\pi,\pi) \in \mathbb{T}^3$ is degenerate to the set consisting of the unique point $\{12\}$ and hence we can not state that the essential spectrum of $\mathcal{A}_\mu(k)$ is absolutely continuous for any $k \in \mathbb{T}^3$.*

For any $\mu > 0$ and $k \in \mathbb{T}^3$ we define an analytic function $\Delta_\mu(k\,;\cdot)$ in $\mathbb{C} \setminus \sigma_{\rm ess}(\mathcal{A}_\mu(k))$ by

$$\Delta_\mu(k\,;z) := w_0(k) - z - \mu^2 \int\limits_{\mathbb{T}^3} \frac{v^2(t)dt}{w_1(k,t)-z}, \ z \in \mathbb{C} \setminus \sigma_{\rm ess}(\mathcal{A}_\mu(k)).$$

Usually the function $\Delta_\mu(k\,;\cdot)$ is called the Fredholm determinant associated to the operator matrix $\mathcal{A}_\mu(k)$.

The following statement establishes connection between the eigenvalues of the operator $\mathcal{A}_\mu(k)$ and zeros of the function $\Delta_\mu(k\,;\cdot)$, see [11, 14].

Lemma 2.2. *For any $\mu > 0$ and $k \in \mathbb{T}^3$ the operator $\mathcal{A}_\mu(k)$ has an eigenvalue $z_\mu(k) \in \mathbb{C} \setminus \sigma_{\rm ess}(\mathcal{A}_\mu(k))$ if and only if $\Delta_\mu(k\,;z_\mu(k)) = 0$.*

From Lemma 2.2 it follows that

$$\sigma_{\rm disc}(\mathcal{A}_\mu(k)) = \{z \in \mathbb{C} \setminus \sigma_{\rm ess}(\mathcal{A}_\mu(k)) : \ \Delta_\mu(k\,;z) = 0\}.$$

Since the function $\Delta_\mu(k\,;\cdot)$ is a monotonically decreasing function on $(-\infty; m(k))$ and $(M(k); +\infty)$, for $\mu > 0$ and $k \in \mathbb{T}^3$ the operator $\mathcal{A}_\mu(k)$ has no more than 1 simple eigenvalue in $(-\infty; m(k))$ and $(M(k); +\infty)$.

Let $\Lambda := \{k = (k_1,k_2,k_3) : k_i \in \{-2\pi/3, 2\pi/3\}, i = 1,2,3\}$. Since the set $\Lambda \subset \mathbb{T}^3$ consists 8 points for a convenience we rewrite the set Λ as $\Lambda = \{k^{(1)}, k^{(2)}, \ldots, k^{(8)}\}$.

It is easy to verify that the function $w_1(\cdot,\cdot)$ has a non-degenerate minimum at the point $(\bar{0},\bar{0}) \in (\mathbb{T}^3)^2$, $\bar{0} := (0,0,0)$ and has non-degenerate maximum at the points of the form $(k^{(i)},k^{(i)}) \in (\mathbb{T}^3)^2$, $i = 1,\ldots,8$, such that

$$\min_{k,p \in \mathbb{T}^3} w_1(k,p) = w_1(\bar{0},\bar{0}) = 0, \quad \max_{k,p \in \mathbb{T}^3} w_1(k,p) = w_2(k^{(i)},k^{(i)}) = 27/2, \quad i = 1,\ldots,8.$$

Simple calculations show that

$$\sigma_{\rm ess}(\mathcal{A}_\mu(\bar{0})) = [0;12];$$
$$\sigma_{\rm ess}(\mathcal{A}_\mu(k^{(i)})) = [\frac{15}{2}; \frac{27}{2}], \quad i = 1,\ldots,8.$$

Therefore,

$$\min_{k \in \mathbb{T}^3} \sigma_{\rm ess}(\mathcal{A}_\mu(k)) = 0, \quad \max_{k \in \mathbb{T}^3} \sigma_{\rm ess}(\mathcal{A}_\mu(k)) = \frac{27}{2}.$$

3. Threshold eigenvalues and virtual levels

In this Section, we prove that for any $i \in \{1,\ldots,8\}$ there is a value μ_i of the parameter (coupling constant) μ such that only for $\mu = \mu_i$ the operator $\mathcal{A}_\mu(\bar{0})$ has a virtual level at the point $z = 0$ (zero-energy resonance) and the operator $\mathcal{A}_\mu(k^{(i)})$ has a virtual level at the point $z = 27/2$ under the assumption that $v(\bar{0}) \neq 0$ and $v(k^{(i)}) \neq 0$. For the case $v(\bar{0}) = 0$ and $v(k^{(i)}) = 0$ we show that the number $z = 0$ $(z = 27/2)$ is a threshold eigenvalue of $\mathcal{A}_\mu(\bar{0})$ $(\mathcal{A}_\mu(k^{(i)}))$.

Denote by $C(\mathbb{T}^3)$ and $L_1(\mathbb{T}^3)$ the Banach spaces of continuous and integrable functions on $\mathbb{T}^3$, respectively.

Definition 3.1. *Let $\gamma \neq 0$. The operator $\mathcal{A}_\mu(\bar{0})$ is said to have a virtual level at $z = 0$ (or zero-energy resonance), if the number 1 is an eigenvalue of the integral operator*

$$(G_\mu \psi)(p) = \frac{\mu^2 v(p)}{2\gamma} \int\limits_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{\varepsilon(t)}, \quad \psi \in C(\mathbb{T}^3)$$

and the associated eigenfunction $\psi(\cdot)$ (up to constant factor) satisfies the condition $\psi(\bar{0}) \neq 0$.

Definition 3.2. *Let* $\gamma \neq 9$ *and* $i \in \{1, \ldots, 8\}$. *The operator* $\mathcal{A}_\mu(k^{(i)})$ *is said to have a virtual level at* $z = 27/2$, *if the number* 1 *is an eigenvalue of the integral operator*

$$(G_\mu^{(i)}\varphi)(p) = \frac{\mu^2 v(p)}{\gamma - 9} \int_{\mathbb{T}^3} \frac{v(t)\varphi(t)dt}{\varepsilon(k^{(i)} + t) + \varepsilon(t) - 9}, \quad \varphi \in C(\mathbb{T}^3)$$

and the associated eigenfunction $\varphi(\cdot)$ (*up to constant factor*) *satisfies the condition* $\varphi(k^{(i)}) \neq 0$.

Using the extremal properties of the function $\varepsilon(\cdot)$, and the Lebesgue dominated convergence theorem, we obtain that there exist the positive finite limits

$$\lim_{z \to -0} \int_{\mathbb{T}^3} \frac{v^2(t)dt}{\varepsilon(t) - z} = \int_{\mathbb{T}^3} \frac{v^2(t)dt}{\varepsilon(t)};$$

$$\lim_{z \to 9+0} \int_{\mathbb{T}^3} \frac{v^2(t)dt}{z - \varepsilon(k^{(i)} + t) - \varepsilon(t)} = \int_{\mathbb{T}^3} \frac{v^2(t)dt}{9 - \varepsilon(k^{(i)} + t) - \varepsilon(t)}.$$

For the next investigations, we define the following quantities

$$\mu_l(\gamma) := \sqrt{2\gamma} \left(\int_{\mathbb{T}^3} \frac{v^2(t)dt}{\varepsilon(t)} \right)^{-1/2} \quad \text{for } \gamma > 0;$$

$$\mu_r^{(i)}(\gamma) := \sqrt{9 - \gamma} \left(\int_{\mathbb{T}^3} \frac{v^2(t)dt}{9 - \varepsilon(k^{(i)} + t) - \varepsilon(t)} \right)^{-1/2} \quad \text{for } \gamma < 9, \ i = 1, \ldots, 8.$$

Let $\gamma_i \in (0; 9)$ be an unique solution of $\mu_l(\gamma) = \mu_r^{(i)}(\gamma)$. It follows immediately that

$$\gamma_i := 9 \left(2 \int_{\mathbb{T}^3} \frac{v^2(t)dt}{9 - \varepsilon(k^{(i)} + t) - \varepsilon(t)} + \int_{\mathbb{T}^3} \frac{v^2(t)dt}{\varepsilon(t)} \right)^{-1} \int_{\mathbb{T}^3} \frac{v^2(t)dt}{\varepsilon(t)}.$$

In the following, we compare the values of $\mu_l(\gamma)$ and $\mu_r(\gamma)$ depending on $\gamma \in (0; 9)$.

Remark 3.3. *Let* $i \in \{1, \ldots, 8\}$. *By the definition of the quantities* $\mu_l(\gamma)$ *and* $\mu_r^{(i)}(\gamma)$ *one can conclude that*
if $\gamma \in (0; \gamma_i)$, *then* $\mu_l(\gamma) < \mu_r^{(i)}(\gamma)$;
if $\gamma = \gamma_i$, *then* $\mu_l(\gamma) = \mu_r^{(i)}(\gamma)$;
if $\gamma \in (\gamma_i; 9)$, *then* $\mu_l(\gamma) > \mu_r^{(i)}(\gamma)$.

From the Definition 3.1 (resp. 3.2) we obtain that the number 1 is an eigenvalue of G_μ (resp. $G_\mu^{(i)}$) if and only if $\mu = \mu_l(\gamma)$ (resp. $\mu = \mu_r^{(i)}(\gamma)$).

We notice that in the Definition 3.2, the requirement of the presence of an eigenvalue 1 of $G_\mu^{(i)}$ corresponds to the existence of a solution of the equation $\mathcal{A}_\mu(k^{(i)})f = (27/2)f$ and the condition $\psi(k^{(i)}) \neq 0$ implies that the solution $f = (f_0, f_1)$ of this equation does not belong to $\mathcal{H}$. More exactly, if the operator $\mathcal{A}_\mu(k^{(i)})$ has a virtual level at $z = 27/2$, then the vector-function $f = (f_0, f_1)$, where

$$f_0 = \text{const} \neq 0, \quad f_1(q) = -\frac{\mu v(q) f_0}{\varepsilon(k^{(i)} + q) + \varepsilon(q) - 9}, \tag{3.1}$$

satisfies the equation $\mathcal{A}_\mu(k^{(i)})f = (27/2)f$ and $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$ (see assertion (i) of Theorem 3.4).

If the number $z = 27/2$ is an eigenvalue of the operator $\mathcal{A}_\mu(k^{(i)})$ then the vector-function $f = (f_0, f_1)$, where f_0 and f_1 are defined in (3.1), satisfies the equation $\mathcal{A}_\mu(k^{(i)})f = (27/2)f$ and $f_1 \in L_2(\mathbb{T}^3)$ (see assertion (ii) of Theorem 3.4).

The same assertions are true for the operator $\mathcal{A}_\mu(\bar{0})$ at the point $z = 0$.

Henceforth, we shall denote by C_1, C_2, C_3 different positive numbers and for each $\delta > 0$, the notation $U_\delta(p_0)$ is used for the $\delta-$neighborhood of the point $p_0 \in \mathbb{T}^3$:

$$U_\delta(p_0) := \{p \in \mathbb{T}^3 : |p - p_0| < \delta\}.$$

Now we formulate the first main result of the paper.

Theorem 3.4. *Let $\gamma < 9$ and $i \in \{1, \dots, 8\}$.*
(i) *The number $z = 27/2$ is an eigenvalue of the operator $\mathcal{A}_\mu(k^{(i)})$ if and only if $\mu = \mu_r^{(i)}(\gamma)$ and $v(k^{(i)}) = 0$;*
(ii) *The operator $\mathcal{A}_\mu(k^{(i)})$ has a virtual level at the point $z = 27/2$ if and only if $\mu = \mu_r^{(i)}(\gamma)$ and $v(k^{(i)}) \neq 0$.*

Proof. Suppose $\gamma < 9$ and $i \in \{1, \dots, 8\}$.

(i) "Only If Part". Let the number $z = 27/2$ be an eigenvalue of $\mathcal{A}_\mu(k^{(i)})$ and $f = (f_0, f_1) \in \mathcal{H}$ be an associated eigenvector. Then f_0 and f_1 are satisfy the system of equations

$$\begin{aligned} &(\gamma - 9) f_0 + \mu \int\limits_{\mathbb{T}^3} v(t) f_1(t) dt = 0; \\ &\mu v(p) f_0 + (\varepsilon(k^{(i)} + p) + \varepsilon(p) - 9) f_1(p) = 0. \end{aligned} \tag{3.2}$$

This implies that f_0 and f_1 are of the form (3.1) and the first equation of system (3.2) yields $\Delta_\mu(k^{(i)}; 27/2) = 0$, therefore, $\mu = \mu_r^{(i)}(\gamma)$.

Now let us show that $f_1 \in L_2(\mathbb{T}^3)$ if and only if $v(k^{(i)}) = 0$. Indeed, if $v(k^{(i)}) = 0$ (resp. $v(k^{(i)}) \neq 0$), from analyticity of the function $v(\cdot)$ it follows that there exist $C_1, C_2, C_3 > 0$, $\theta_i \in \mathbb{N}$ and $\delta > 0$ such that

$$C_1 |p - k^{(i)}|^{\theta_i} \leq |v(p)| \leq C_2 |p - k^{(i)}|^{\theta_i}, \quad p \in U_\delta(k^{(i)}), \tag{3.3}$$

respectively

$$|v(p)| \geq C_3, \quad p \in \mathbb{T}^3 \setminus U_\delta(k^{(i)}). \tag{3.4}$$

Since the function $\varepsilon(k^{(i)} + p) + \varepsilon(p)$ has an unique non-degenerate maximum at the point $k^{(i)} \in \mathbb{T}^3$ there exist $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that

$$C_1 |p - k^{(i)}|^2 \leq |\varepsilon(k^{(i)} + p) + \varepsilon(p) - 9| \leq C_2 |p - k^{(i)}|^2, \quad p \in U_\delta(k^{(i)}), \tag{3.5}$$

$$|\varepsilon(k^{(i)} + p) + \varepsilon(p) - 9| \geq C_3, \quad p \in \mathbb{T}^3 \setminus U_\delta(k^{(i)}). \tag{3.6}$$

We have

$$\begin{aligned} \int\limits_{\mathbb{T}^3} |f_1(t)|^2 dt &= \mu^2 |f_0|^2 \int\limits_{U_\delta(k^{(i)})} \frac{v^2(t) dt}{(\varepsilon(k^{(i)} + t) + \varepsilon(t) - 9)^2} \\ &+ \mu^2 |f_0|^2 \int\limits_{\mathbb{T}^3 \setminus U_\delta(k^{(i)})} \frac{v^2(t) dt}{(\varepsilon(k^{(i)} + t) + \varepsilon(t) - 9)^2}. \end{aligned} \tag{3.7}$$

Let $v(k^{(i)}) = 0$. Then by (3.3) and (3.5) for the first summand on the right-hand side of (3.7) we have

$$\int\limits_{U_\delta(k^{(i)})} \frac{v^2(t) dt}{(\varepsilon(k^{(i)} + t) + \varepsilon(t) - 9)^2} \leq C_1 \int\limits_{U_\delta(k^{(i)})} \frac{|t - k^{(i)}|^{2\theta_i} dt}{|t - k^{(i)}|^4} < +\infty.$$

It follows from the continuity of $v(\cdot)$ on a compact set $\mathbb{T}^3$ and (3.6) that

$$\int\limits_{\mathbb{T}^3 \setminus U_\delta(k^{(i)})} \frac{v^2(t) dt}{(\varepsilon(k^{(i)} + t) + \varepsilon(t) - 9)^2} \leq C_1 \int\limits_{\mathbb{T}^3 \setminus U_\delta(k^{(i)})} dt < +\infty.$$

So, in this case $f_1 \in L_2(\mathbb{T}^3)$.

For the case $v(k^{(i)}) \neq 0$ there exsist the numbers $\delta > 0$ and $C_1 > 0$ such that $|v(p)| \geq C_1$ for any $p \in U_\delta(k^{(i)})$. Then from (3.5) we obtain

$$\int\limits_{\mathbb{T}^3} |f_1(t)|^2 dt \geq C_1 \int\limits_{U_\delta(k^{(i)})} \frac{dt}{|t - k^{(i)}|^4} = +\infty.$$

Therefore, $f_1 \in L_2(\mathbb{T}^3)$ if and only if $v(k^{(i)}) = 0$.

"If Part". Suppose that $\mu = \mu_r^{(i)}(\gamma)$ and $v(k^{(i)}) = 0$. It is easy to verify that the vector-function $f = (f_0, f_1)$ with f_0 and f_1 defined in (3.1) satisfies the equation $\mathcal{A}_\mu(k^{(i)}) f = (27/2) f$. We proved above that if $v(k^{(i)}) = 0$, then $f_1 \in L_2(\mathbb{T}^3)$.

(ii) "Only If Part". Suppose that the operator $\mathcal{A}_\mu(k^{(i)})$ has a virtual level at $z = 27/2$. Then by Definition 3.2 the equation

$$\varphi(p) = \frac{\mu^2 v(p)}{\gamma - 9} \int\limits_{\mathbb{T}^3} \frac{v(t) \varphi(t) dt}{\varepsilon(k^{(i)} + t) + \varepsilon(t) - 9}, \quad \varphi \in C(\mathbb{T}^3) \tag{3.8}$$

has a nontrivial solution $\varphi \in C(\mathbb{T}^3)$, which satisfies the condition $\varphi(k^{(i)}) \neq 0$.

This solution is equal to the function $v(p)$ (up to a constant factor) and hence

$$\Delta_\mu(k^{(i)}, 27/2) = \gamma - 9 - \mu^2 \int\limits_{\mathbb{T}^3} \frac{v^2(t)dt}{\varepsilon(k^{(i)}+t) + \varepsilon(t) - 9} = 0,$$

that is, $\mu = \mu_r^{(i)}(\gamma)$.

"If Part". Let now $\mu = \mu_r^{(i)}(\gamma)$ and $v(k^{(i)}) \neq 0$. Then the function $v \in C(\mathbb{T}^3)$ is a solution of (3.8), and consequently, by Definition 3.2 the operator $\mathcal{A}_\mu(k^{(i)})$ has a virtual level at $z = 27/2$. □

The following result may be proved in much the same way as Theorem 3.4.

Theorem 3.5. *Let* $\gamma > 0$.
(i) *The operator* $\mathcal{A}_\mu(\bar{0})$ *has an zero eigenvalue if and only if* $\mu = \mu_l(\gamma)$ *and* $v(\bar{0}) = 0$;
(ii) *The operator* $\mathcal{A}_\mu(\bar{0})$ *has a zero energy resonance if and only if* $\mu = \mu_l(\gamma)$ *and* $v(\bar{0}) \neq 0$.

Since $\mu_l(\gamma_i) = \mu_r^{(i)}(\gamma_i)$, setting $\mu_i := \mu_l(\gamma_i)$, from Theorems 3.4 and 3.5 we obtain the following

Corollary 3.6. *Let* $\gamma \in (0; 9)$ *and* $i \in \{1, \ldots, 8\}$.
(i) *The operator* $\mathcal{A}_\mu(\bar{0})$ *has a zero eigenvalue and the number* $z = 27/2$ *is an eigenvalue of* $\mathcal{A}_\mu(k^{(i)})$ *iff* $\mu = \mu_i$ *and* $v(\bar{0}) = v(k^{(i)}) = 0$;
(ii) *The operator* $\mathcal{A}_\mu(\bar{0})$ *has zero-energy resonance and the operator* $\mathcal{A}_\mu(k^{(i)})$ *has a virtual level at the point* $z = 27/2$ *iff* $\mu = \mu_i$, $v(\bar{0}) \neq 0$ *and* $v(k^{(i)}) \neq 0$;
(iii) *The operator* $\mathcal{A}_\mu(\bar{0})$ *has a zero eigenvalue and the operator* $\mathcal{A}_\mu(k^{(i)})$ *has a virtual level at the point* $z = 27/2$ *iff* $\mu = \mu_i$, $v(\bar{0}) = 0$ *and* $v(k^{(i)}) \neq 0$;
(iv) *The operator* $\mathcal{A}_\mu(\bar{0})$ *has a zero-energy resonance and the number* $z = 27/2$ *is an eigenvalue of* $\mathcal{A}_\mu(k^{(i)})$ *iff* $\mu = \mu_i$, $v(\bar{0}) \neq 0$ *and* $v(k^{(i)}) = 0$.

Next we will consider some applications of the results. Denote by $\mathcal{H}_2 := L_2^{\mathrm{s}}((\mathbb{T}^3)^2)$ the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. In the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ we consider a 2×2 operator matrix

$$\mathcal{A}_\mu := \begin{pmatrix} A_{11} & \sqrt{2}\mu A_{12} \\ \sqrt{2}\mu A_{12}^* & A_{22} \end{pmatrix},$$

where $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i = 1, 2$ are defined by the rules

$$(A_{11}f_1)(k) = w_1(k)f_1(k), \quad (A_{12}f_2)(k) = \int\limits_{\mathbb{T}^3} v(t)f_2(k,t)dt,$$

$$(A_{22}f_2)(k,p) = w_1(k,p)f_2(k,p) \quad f_i \in \mathcal{H}_i, \quad i = 1, 2.$$

Here $A_{12}^* : \mathcal{H}_1 \to \mathcal{H}_2$ denotes the adjoint operator to A_{12} and

$$(A_{12}^*f_1)(k,p) = \frac{1}{2}(v(k)f_1(p) + v(p)f_1(k)), \quad f_1 \in \mathcal{H}_1.$$

Under these assumptions the operator $\mathcal{A}_\mu$ is bounded and self-adjoint.

The main results of the present paper plays crucial role in the study of the spectral properties of the operator matrix $\mathcal{A}_\mu$. In particular, the essential spectrum of $\mathcal{A}_\mu$ can be described via the spectrum of $\mathcal{A}_\mu(k)$ the following equality holds

$$\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) = [0; 27/2] \cup \bigcup_{k \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathcal{A}_\mu(k)).$$

Since the operator $\mathcal{A}_\mu(k)$ has at most 2 simple eigenvalues, the set $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$ consists at least one and at most three bounded closed intervals, for similar results see [7].

Using Theorems 3.4 and 3.5 one can investigate [14] the number of eigenvalues of $\mathcal{A}_\mu$ and find its discrete spectrum asymptotics.

We note that the case

$$v(p) = \sqrt{\mu} = const, \quad w_1(k,p) = \varepsilon(k) + \varepsilon(\frac{1}{2}(k+p)) + \varepsilon(p)$$

is studied in [13], and it is shown that the bounds $\min\limits_{k \in \mathbb{T}^3} \sigma_{ess}(\mathcal{A}_\mu(\bar{0}))$ and $\max\limits_{k \in \mathbb{T}^3} \sigma_{ess}(\mathcal{A}_\mu(\bar{\pi}))$ are only virtual levels. This paper generalizes the results of the paper [13] and it is proved that these bounds are threshold eigenvalues or virtual levels depending on the values of the function $v(\cdot)$.

Acknowledgements

The authors thank the anonymous referee for reading the manuscript carefully and for making valuable suggestions.

References

[1] Tretter C. *Spectral Theory of Block Operator Matrices and Applications*. Imperial College Press, 2008.
[2] Huebner M.,Spohn H. Spectral properties of spin-boson Hamiltonian. *Annl. Inst. Poincare*, 1995, **62**(3), P. 289–323.
[3] Spohn H. Ground states of the spin-boson Hamiltonian. *Comm. Math. Phys.*, 1989, **123**, P. 277–304.
[4] Minlos R.A., Spohn H. The three-body problem in radioactive decay: the case of one atom and at most two photons. Topics in Statistical and Theoretical Physics. *Amer. Math. Soc. Transl.*, Ser. 2, **177**, AMS, Providence, RI, 1996, P. 159–193.
[5] Muminov M., Neidhardt H., Rasulov T. On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case. *Journal of Mathematical Physics*, 2015, **56**, P. 053507.
[6] Ibrogimov O.I. Spectral Analysis of the Spin-Boson Hamiltonian with Two Photons for Arbitrary Coupling. *Ann. Henri Poincaré*, 2018, **19**(11), P. 3561–3579.
[7] Rasulov T.Kh. Branches of the essential spectrum of the lattice spin-boson model with at most two photons. *Theoretical and Mathematical Physics*, 2016, **186**(2), P. 251–267.
[8] Mogilner A.I. Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: problems and results. *Advances in Sov. Math.*, 1991, **5**, P. 139–194.
[9] Friedrichs K.O. *Perturbation of spectra in Hilbert space*. Amer. Math. Soc., Providence, Rhole Island, 1965.
[10] Malishev V.A., Minlos R.A. *Linear infinite-particle operators*. Translations of Mathematical Monographs. 143, AMS, Providence, RI, 1995.
[11] Albeverio S., Lakaev S.N., Rasulov T.H. On the spectrum of an Hamiltonian in Fock space. Discrete spectrum asymptotics. *J. Stat. Phys.*, 2007, **127**(2), P. 191–220.
[12] Muminov M.I., Rasulov T.H. On the number of eigenvalues of the family of operator matrices. *Nanosystems: Physics, Chemistry, Mathematics*, 2014, **5**(5), P. 619–625.
[13] Rasulov T.H., Dilmurodov E.B. Eigenvalues and virtual levels of a family of 2×2 operator matrices. *Methods of Functional Analysis and Topology*, 2019, **25**(3), P. 273–281.
[14] Rasulov T.Kh. On the number of eigenvalues of a matrix operator. *Siberian Math. J.*, 2011, **52**(2), P. 316–328.
[15] Albeverio S., Lakaev S.N., Muminov Z.I. The threshold effects for a family of Friedrichs models under rank one perturbations. *J. Math. Anal. Appl.*, 2007, **330**, P. 1152–1168.
[16] Albeverio S., Lakaev S.N., Makarov K.A., Muminov Z.I. The threshold effects for the two-particle Hamiltonians on lattices. *Commun. Math. Phys.*, 2006, **262**, P. 91–115.
[17] Rasulov T.Kh., Dilmurodov E.B. Investigations of the numerical range of a operator matrix. *J. Samara State Tech. Univ., Ser. Phys. and Math. Sci.*, 2014, **35**(2), P. 50–63.
[18] Rasulov T.H., Tosheva N.A. Analytic description of the essential spectrum of a family of 3×3 operator matrices. *Nanosystems: Physics, Chemistry, Mathematics*, 2019, **10**(5), P. 511–519.
[19] Reed M., Simon B. *Methods of modern mathematical physics*. IV: Analysis of Operators. Academic Press, New York, 1979.