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Metric graph version of the FitzHugh–Nagumo model

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The FitzHugh–Nagumo model on a metric graph is studied. System of delayed differential equations is used to model a pair of FitzHugh- Nagumo excitable systems with time-delayed fast threshold modulation coupling. The model can be used for description of signal transmission in different nanostructures, microsystems or neural networks. The effect of time delay on the impulse transmission is studied.

Keywords: FitzHugh–Nagumo system, time-delayed coupling, travelling wave, metric graph, neural network.

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1. Introduction

Coupled oscillating systems with time delay are presented in many problems related to nano- and micro-systems: oscillation reactions, delay-sustained pattern formation in subexcitable media, coupled lasers, signal transmission through biological neurons [1–5]. The most bright example of such system is a biological neural network. The signal transmission in such a system is based on a number of physical-chemical reactions in complex molecular structures. An appropriate but rather complicated model for the process was suggested by A. L. Hodgkin and A. F. Huxley [6]. The model is widely used (see, e.g., [7, 8]) although it is complex for computations. To reduce the computational complexity, FitzHugh and Nagumo suggested a more simple model [9,10] created, initially, for electronics. It is, really, a modification of the well-known van der Pol model [11]. The approach has been intensively used last decade (see, e.g., [12–16]). The model possesses the main features of the Hodgkin-Huxley model and quite accurately describes the dynamics of a biological neuron and at the same time has a relatively small computational complexity. In this article, we suggest a metric graph type model of a simple neural network including three neurons forming a lasso graph. The model describes the actual movement of an impulse through axons from one neuron to another and vice versa. The delay time plays an important role in the dynamics of the system. The delay time, really, corresponds to the length of the axons. As shown, the types of system behavior fundamentally depend on these parameters. One can observe oscillation or relaxation regime. The corresponding critical values of the parameters was found numerically.

2. Metric graph model

We now consider the model of three neurons. The system is modelled as a metric graph with a loop shown in Fig. 1. At each edge of the graph, the FitzHugh–Nagumo partial differential equations are treated:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} &= D\dfrac{\partial^2 u_1}{\partial x^2} - au_1 + (a+1)u_1^2 - u_1^3 - v_1, \\ \dfrac{\partial v_1}{\partial t} &= bu_1 - \gamma v_1, \\ \dfrac{\partial u_2}{\partial t} &= D\dfrac{\partial^2 u_2}{\partial x^2} - au_2 + (a+1)u_2^2 - u_2^3 - v_2, \\ \dfrac{\partial v_2}{\partial t} &= bu_2 - \gamma v_2, \\ \dfrac{\partial u_3}{\partial t} &= D\dfrac{\partial^2 u_3}{\partial x^2} - au_3 + (a+1)u_3^2 - u_3^3 - v_3, \\ \dfrac{\partial v_3}{\partial t} &= bu_3 - \gamma v_3, \end{cases} \tag{1}$$

where $i = 1, 2, 3$, functions $u_i(t, x)$ and $v_i(t, x)$ describe the states of the corresponding neurons at time t at the axon point x, a, b, γ and D are constant parameters. At the graph vertices, we pose the following conditions ensuring a proper coupling (namely, the transmission from the first neuron to the second neuron only, and from the third neuron to the second neuron only, see Fig. 1):

$$\begin{cases} u_2(t,0) &=& u_1(t,L)+u_3(t,L), \\ u_3(t,0) &=& u_2(t,L), \\ \dfrac{\partial u_1}{\partial x}(t,0) &=& 0,\ \dfrac{\partial v_1}{\partial x}(t,0) \ =\ 0, \\ \dfrac{\partial u_1}{\partial x}(t,L) &=& 0,\ \dfrac{\partial v_1}{\partial x}(t,L) \ =\ 0, \\ \dfrac{\partial u_2}{\partial x}(t,L) &=& 0,\ \dfrac{\partial v_2}{\partial x}(t,L) \ =\ 0, \\ \dfrac{\partial u_3}{\partial x}(t,L) &=& 0,\ \dfrac{\partial v_3}{\partial x}(t,L) \ =\ 0, \end{cases} \tag{2}$$

where L is the axon length (we assume that all axons have the same length, this assumption is not essential). These vertex coupling conditions differ from the conventional conditions for stationary and non-stationary metric graphs (see, e.g., [21, 22]).

Thus, the system describes three neurons: the first is the start, the second and the third are tied to each other (see. Fig. 1).

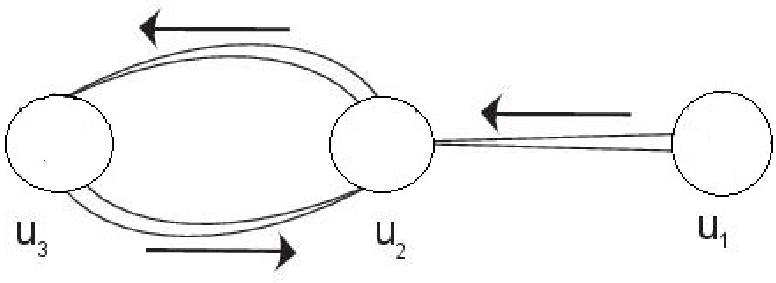

FIG. 1. Neuron connections

To numerically solve the system (1), the method described below was used. Consider the first neuron:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - D\dfrac{\partial^2 u_1}{\partial x^2} &=& f(u_1)-v_1; \\ \dfrac{\partial v_1}{\partial t} &=& bu_1-\gamma v_1; \\ \dfrac{\partial u_1}{\partial x}(t,0) &=& 0,\ \dfrac{\partial v_1}{\partial x}(t,0) \ =\ 0; \\ \dfrac{\partial u_1}{\partial x}(t,L) &=& 0,\ \dfrac{\partial v_1}{\partial x}(t,L) \ =\ 0. \end{cases}, \tag{3}$$

where, $f(x) = -ax + (a+1)x^2 - x^3$. We divide the segment $[0, L]$ into n equal segments (with lengths $h = \dfrac{1}{n}$) by points $x_0 = 0, x_1 = \dfrac{L}{n}, \ldots, x_{n-1} = \dfrac{L(n-1)}{n}, x_n = L$. We introduce a similar grid along the time axis with the step τ and the points $t_k = k\tau$. Next, we introduce the notation: $u_1^{(i,k)} = u_1(x_i, t_k)$. Now, system (3) can be rewritten in the form of the following difference equations:

$$\begin{cases} \dfrac{u_1^{(i,k)} - u_1^{(i,k-1)}}{\tau} - \dfrac{D}{2}\left(\dfrac{u_1^{(i+1,k)} - 2u_1^{(i,k)} + u_1^{(i-1,k)}}{h^2} + \dfrac{u_1^{(i+1,k-1)} - 2u_1^{(i,k-1)} + u_1^{(i-1,k-1)}}{h^2}\right) = \\ = f(u_1^{(i,k-1)}) - v_1^{(i,k-1)}, \qquad i = 1\ldots n-1; \\ \dfrac{v_1^{(i,k)} - v_1^{(i,k-1)}}{\tau} = bu_1^{(i,k-1)} - \gamma v_1^{(i,k-1)}, \qquad i = 1\ldots n-1; \\ -3u_1^{(0,k)} + 4u_1^{(1,k)} - u_1^{(2,k)} = 0; \\ u_1^{(n-2,k)} - 4u_1^{(n-1,k)} + 3u_1^{(n,k)} = 0; \\ -3v_1^{(0,k)} + 4v_1^{(1,k)} - v_1^{(2,k)} = 0; \\ v_1^{(n-2,k)} - 4v_1^{(n-1,k)} + 3v_1^{(n,k)} = 0, \end{cases}. \tag{4}$$

Initial values $u_1^{i,0}, v_1^{i,0}, i = 0 \dots n$ being given, the system can be solved by the three-diagonal matrix algorithm. Similarly, one can obtain solutions for the second and the third neurons.

This scheme was implemented in the Python programming language. Zero initial values were taken at all points, except for the vicinity of the beginning of the first axon, where the above-threshold disturbance was considered as the starting impulse of the system.

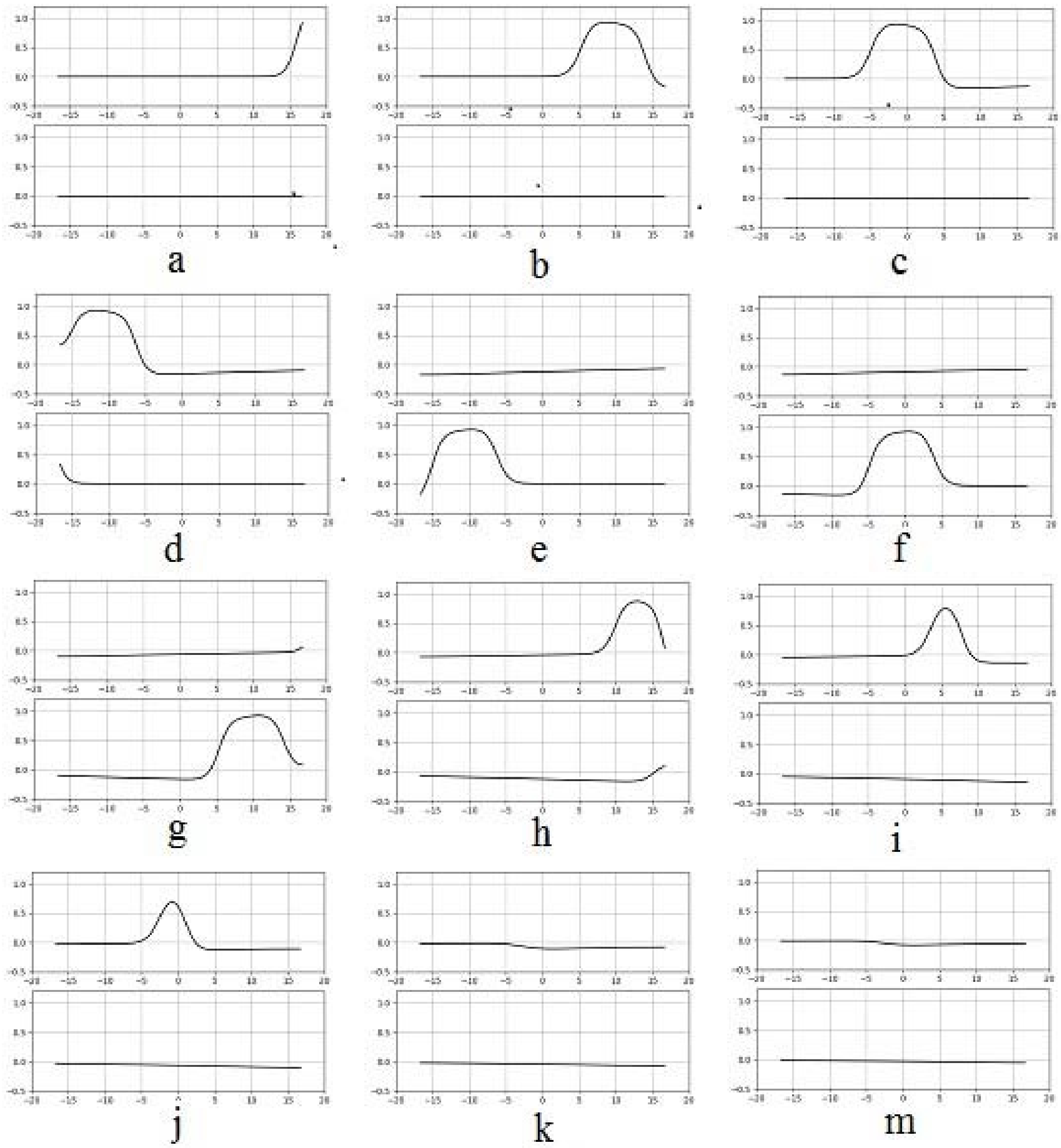

FIG. 2. Signal movement through neural system shown in Fig. 1. At each fragment, upper curve shows the signal in upper neuron from Fig. 1, lower curve corresponds to lower neuron. Different fragments correspond to different time moments (arbitrary units): a) $t = 5$, b) $t = 65$, c) $t = 125$, d) $t = 185$, e) $t = 245$, f) $t = 305$, g) $t = 365$, h) $t = 425$, i) $t = 485$, j) $t = 545$, k) $t = 605$, m) $t = 665$

Taking the following values of the parameters: $a = 0.25$, $b = 0.002$, $\gamma = 0.002$, $D = 0.3$ (as in [19]), we constructed several solutions for various axon lengths L. Fig. 2 shows several successive states of the system at different times for $L = 16.7$ (the upper part corresponds to the potential of the second axon, and the lower to the third). The figures show that, while returned to the second neuron, the signal decays. Numerical simulation shows that at $L = 16.8$, the attenuation does not occur. That is, the same situation is observed as in the discrete system, and the critical value lies between $L = 16.7$ and $L = 16.8$.

3. Conclusion

We suggest a mathematical model of graph type for the FitzHugh–Nagumo system. Particularly, it can be implemented to a neural network or to an invertible chemical or physical transformations spreading along a system of long molecules. The model showed a significant effect of the delay time on the impulse transfer and on the dynamics of the network as a whole. For the corresponding values of this parameter, one has a quick decay of the impulse or a periodic transmission. From a physical point of view, it turns out that small delays in the transmission of an impulse do not allow the impulse to pass through recursive systems. It is not essential why is the time delay small: because of short lengths of the graph edges or fast transmission speed along them. The main reason is simple. if the impulse comes to the next element of the system (e.g., a neuron) during the refractory period, it can not pass through it without an attenuation. It leads to a limitation in number of signals travelling inside the network.

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