NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2020, 11 (2), P. 138–144

Analysis of the spectrum of a 2×2 operator matrix. Discrete spectrum asymptotics

T. H. Rasulov, E. B. Dilmurodov

Faculty of Physics and Mathematics, Bukhara State University
M. Ikbol str. 11, 200100 Bukhara, Uzbekistan

rth@mail.ru, elyor.dilmurodov@mail.ru

DOI 10.17586/2220-8054-2020-11-2-138-144

We consider a 2×2 operator matrix $\mathcal{A}_\mu$, $\mu > 0$ related with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. We obtain an analog of the Faddeev equation and its symmetric version for the eigenfunctions of $\mathcal{A}_\mu$. We describe the new branches of the essential spectrum of $\mathcal{A}_\mu$ via the spectrum of a family of generalized Friedrichs models. It is established that the essential spectrum of $\mathcal{A}_\mu$ consists the union of at most three bounded closed intervals and their location is studied. For the critical value μ_0 of the coupling constant μ we establish the existence of infinitely many eigenvalues, which are located in the both sides of the essential spectrum of $\mathcal{A}_\mu$. In this case, an asymptotic formula for the discrete spectrum of $\mathcal{A}_\mu$ is found.

Keywords: operator matrix, bosonic Fock space, coupling constant, dispersion function, essential and discrete spectrum, Birman–Schwinger principle, spectral subspace, Weyl creterion.

Received: 30 December 2019

Revised: 17 February 2020

1. Introduction and statement of the problem

It is well-known that [1], if H is a bounded linear operator in a Hilbert space $\mathcal{H}$ and a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ is given, then H always admits a block operator matrix representation

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

with bounded linear operators $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 1, 2$. In addition, $H = H^*$ if and only if $H_{ii} = H_{ii}^*$, $i = 1, 2$ and $H_{21} = H_{12}^*$. Such operator matrices often arise in mathematical physics, e.g., in quantum field theory, condensed matter physics, fluid mechanics, magnetohydrodynamics and quantum mechanics. One of the special class of 2×2 block operator matrices is the Hamiltonians acting in the one- and two-particle subspaces of a Fock space. It is related with a system describing three-particles in interaction without conservation of the number of particles in Fock space. Here, off-diagonal entries of such block operator matrices are annihilation and creation operators.

Operator matrices of this form play a key role for the study of the energy operator of the spin-boson Hamiltonian with two bosons on the torus. In fact, the latter is a 6×6 operator matrix which is unitarily equivalent to a 2×2 block diagonal operator with two copies of a particular case of H on the diagonal, see e.g. [2]. Consequently, the location of the essential spectrum and finiteness of discrete eigenvalues of the spin-boson Hamiltonian are determined by the corresponding spectral information on the operator matrix H. We recall that the spin-boson model is a well-known quantum-mechanical model which describes the interaction between a two-level atom and a photon field. We refer to [3] and [4] for excellent reviews from physical and mathematical perspectives, respectively. Independently of whether the underlying domain is a torus $\mathbb{T}^{\rm d}$ or the whole space $\mathbb{R}^{\rm d}$, the full spin-boson Hamiltonian is an infinite operator matrix in Fock space for which rigorous results are very hard to obtain. One line of attack is to consider the compression to the truncated Fock space with a finite number N of bosons, and in fact most of the existing literature concentrates on the case $N \leq 2$. For the case of $\mathbb{R}^{\rm d}$ there are some exceptions, e.g. [5, 6] for arbitrary finite N and [7] for $N = 3$, where a rigorous scattering theory was developed for small coupling constants.

For the case when the underlying domain is a torus, the spectral properties of some versions of H were investigated in [8–11]. An important problem of the spectral theory of such matrix operators is the infiniteness of the number of eigenvalues located outside the essential spectrum. We mention that, the infiniteness of the discrete eigenvalues below the bottom of the essential spectrum of the Hamiltonian in Fock space, which has a block operator matrix representation, and corresponding eigenvalue asymptotics were discussed in [8]. These results were obtained using the machinery developed in [12] by Sobolev.

In the present paper we consider a 2×2 operator matrix $\mathcal{A}_\mu$, ($\mu > 0$ is a coupling constant) related with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. This operator acts in the direct sum of one- and two-particle subspaces of the bosonic Fock space and it is related with the lattice spin-boson Hamiltonian [2, 13]. We find the critical value μ_0 of the coupling constant μ, to establish the existence of infinitely many eigenvalues lying in **both** sides of essential spectrum of $\mathcal{A}_{\mu_0}$ and to obtain an asymptotics for the number of these eigenvalues.

We point out that the latter assertion seems to be quite new for the discrete models and similar result have not been obtained yet for the three-particle discrete Schrödinger operators and operator matrices in Fock space. In all papers devoted to the infiniteness of the number of eigenvalues (Efimov's effects), the situation on the neighborhood of the left edge of essential spectrum are discussed, see for example [8–10, 14–16]. Since the essential spectrum of the three-particle continuous Schrödinger operators [12, 17, 18] and standard spin-boson model with at most two photons [19, 20] coincides with half-axis $[\kappa; +\infty)$, the main results of the present paper are typical only for lattice case, and they do not have analogs in the continues case.

Now, we formulate the problem. Let $\mathbb{T}^3$ be the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides equipped with its Haar measure. Let $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^{\mathrm{s}}((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_1 := L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 := L_2^{\mathrm{s}}((\mathbb{T}^3)^2)$, that is, $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. The spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are called one- and two-particle subspaces of a bosonic Fock space $\mathcal{F}_{\mathrm{s}}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively.

Let us consider a 2×2 operator matrix $\mathcal{A}_\mu$ acting in the Hilbert space $\mathcal{H}$ as:

$$\mathcal{A}_\mu := \begin{pmatrix} A_{11} & \mu A_{12} \\ \mu A_{12}^* & A_{22} \end{pmatrix}$$

with the entries

$$(A_{11} f_1)(k) = w_1(k) f_1(k), \quad (A_{12} f_2)(k) = \int_{\mathbb{T}^3} f_2(k, s) ds,$$

$$(A_{22} f_2)(k, p) = w_2(k, p) f_2(k, p), \quad f_i \in \mathcal{H}_i, \quad i = 1, 2.$$

Here, $\mu > 0$ is a coupling constant, the functions $w_1(\cdot)$ and $w_2(\cdot, \cdot)$ have the form

$$w_1(k) := \varepsilon(k) + \gamma, \quad w_2(k, p) := \varepsilon(k) + \varepsilon(\frac{1}{2}(k + p)) + \varepsilon(p)$$

with $\gamma \in \mathbb{R}$ and the dispersion function $\varepsilon(\cdot)$ is defined by:

$$\varepsilon(k) := \sum_{i=1}^{3} (1 - \cos k_i), \; k = (k_1, k_2, k_3) \in \mathbb{T}^3, \tag{1.1}$$

A_{12}^* denotes the adjoint operator to A_{12} and

$$(A_{12}^* f_1)(k, p) = \frac{1}{2}(f_1(k) + f_1(p)), \quad f_1 \in \mathcal{H}_1.$$

Under these assumptions, the operator $\mathcal{A}_\mu$ is bounded and self-adjoint.

We remark that the operators A_{12} and A_{12}^* are called annihilation and creation operators [21], respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

2. Faddeev's equation and essential spectrum of $\mathcal{A}_\mu$

In this section, we obtain an analog of the Faddeev type integral equation for eigenvectors of $\mathcal{A}_\mu$ and investigate the location and structure of the essential spectrum of $\mathcal{A}_\mu$.

Throughout the present paper we adopt the following conventions: Denote by $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$, respectively, the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator.

Let $H_0 := \mathbb{C}$. To study the spectral properties of the operator $\mathcal{A}_\mu$, we introduce a family of bounded self-adjoint operators (generalized Friedrichs models) $\mathcal{A}_\mu(k)$, $k \in \mathbb{T}^3$ which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as 2×2 operator matrices:

$$\mathcal{A}_\mu(k) := \begin{pmatrix} A_{00}(k) & \frac{\mu}{\sqrt{2}} A_{01} \\ \frac{\mu}{\sqrt{2}} A_{01}^* & A_{11}(k) \end{pmatrix},$$

with matrix elements:

$$A_{00}(k)f_0 = w_1(k)f_0,\ (A_{01}f_1) = \int\limits_{\mathbb{T}^3} f_1(t)dt,$$
$$(A_{11}(k)f_2)(p) = w_2(k,p)f_1(p), \quad f_i \in \mathcal{H}_i, \quad i = 1, 2.$$

From the simple discussions it follows that $\sigma_{\text{ess}}(\mathcal{A}_\mu(k)) = [m(k), M(k)]$, where the numbers $m(k)$ and $M(k)$ are defined by:

$$m(k) := \min_{p \in \mathbb{T}^3} w_2(k,p), \quad M(k) := \max_{p \in \mathbb{T}^3} w_2(k,p). \tag{2.1}$$

For any $k \in \mathbb{T}^3$ we define an analytic function: $I(k\,;\cdot)$ in $\mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{A}_\mu(k))$ by

$$I(k\,;z) := \int\limits_{\mathbb{T}^3} \frac{dt}{w_2(k,t) - z}.$$

Then the Fredholm determinant associated to the operator $\mathcal{A}_\mu(k)$ is defined by:

$$\Delta_\mu(k\,;z) := w_1(k) - z - \frac{\mu^2}{2}\, I(k\,;z),\ z \in \mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{A}_\mu(k)).$$

A simple consequence of the Birman–Schwinger principle and the Fredholm theorem implies that for the discrete spectrum of $\mathcal{A}_\mu(k)$, the equality:

$$\sigma_{\text{disc}}(\mathcal{A}_\mu(k)) = \{z \in \mathbb{C} \setminus [m(k); M(k)] :\ \Delta_\mu(k\,;z) = 0\}$$

holds.

We set:

$$m := \min_{k,p \in \mathbb{T}^3} w_2(k,p), \quad M := \max_{k,p \in \mathbb{T}^3} w_2(k,p),$$
$$\Lambda_\mu := \bigcup_{k \in \mathbb{T}^3} \sigma_{\text{disc}}(\mathcal{A}_\mu(k)), \quad \Sigma_\mu := [m; M] \cup \Lambda_\mu.$$

For each $\mu > 0$ and $z \in \mathbb{C} \setminus \Sigma_\mu$ we define the integral operator $T_\mu(z)$ acting in the Hilbert spaces $L_2(\mathbb{T}^3)$ by

$$(T_\mu(z)g)(p) = \frac{\mu^2}{2\Delta_\mu(p;\,z)} \int\limits_{\mathbb{T}^3} \frac{g(t)dt}{w_2(p,t) - z}.$$

The following theorem is an analog of the well-known Faddeev's result for the operator $\mathcal{A}_\mu$ and establishes a connection between eigenvalues of $\mathcal{A}_\mu$ and $T_\mu(z)$.

Theorem 2.1. *The number $z \in \mathbb{C} \setminus \Sigma_\mu$ is an eigenvalue of the operator $\mathcal{A}_\mu$ if and only if the number $\lambda = 1$ is an eigenvalue of the operator $T_\mu(z)$. Moreover, the eigenvalues z and 1 have the same multiplicities.*

We point out that the integral equation $g = T_\mu(z)g$ is an analog of the Faddeev type system of integral equations for eigenfunctions of the operator $\mathcal{A}_\mu$ and it is played crucial role in the analysis of the spectrum of $\mathcal{A}_\mu$. For the proof of Theorem 2.1 we show the equivalence of the eigenvalue problem $\mathcal{A}_\mu f = zf$ to the equation $g = T_\mu(z)g$.

The following theorem describes the location of the essential spectrum of the operator $\mathcal{A}_\mu$ by the spectrum of the family of generalized Friedrichs models $\mathcal{A}_\mu(k)$.

Theorem 2.2. *For the essential spectrum of $\mathcal{A}_\mu$, the equality $\sigma_{\rm ess}(\mathcal{A}_\mu) = \Sigma_\mu$ holds. Moreover, the set Σ_μ consists of no more than three bounded closed intervals.*

The inclusion $\Sigma_\mu \subset \sigma_{\rm ess}(\mathcal{A}_\mu)$ in the proof of Theorem 2.2 is established with the use of a well-known Weyl creterion, see for example [11]. An application of Theorem 2.1 and analytic Fredholm theorem (see, e.g., Theorem VI.14 in [18]) proves inclusion $\sigma_{\rm ess}(\mathcal{A}_\mu) \subset \Sigma_\mu$.

In the following we introduce the new subsets of the essential spectrum of $\mathcal{A}_\mu$.

Definition 2.3. *The sets Λ_μ and $[m; M]$ are called two- and three-particle branches of the essential spectrum of $\mathcal{A}_\mu$, respectively.*

The definition of the set Λ_μ and the equality

$$\bigcup_{k \in \mathbb{T}^3} [m(k); M(k)] = [m; M]$$

together with Theorem 2.2 give the equality

$$\sigma_{\rm ess}(\mathcal{A}_\mu) = \bigcup_{k \in \mathbb{T}^3} \sigma(\mathcal{A}_\mu(k)). \tag{2.2}$$

Here the family of operators $\mathcal{A}_\mu(k)$ have a simpler structure than the operator $\mathcal{A}_\mu$. Hence, in many instances, (2.2) provides an effective tool for the description of the essential spectrum.

Using the extremal properties of the function $w_2(\cdot, \cdot)$, and the Lebesgue dominated convergence theorem one can show that the integral $I(\bar{0}; 0)$ is finite, where $\bar{0} := (0, 0, 0) \in \mathbb{T}^3$, see [22, 23].

For the next investigations we introduce the following quantities

$$\mu_l^0(\gamma) := \sqrt{2\gamma} \left(I(\bar{0}, 0)\right)^{-1/2} \text{ for } \gamma > 0;$$
$$\mu_r^0(\gamma) := \sqrt{24 - 2\gamma} \left(I(\bar{0}, 0)\right)^{-1/2} \text{ for } \gamma < 12.$$

Since $\mathbb{T}^3$ is compact, and the functions $\Delta_\mu(\cdot; 0)$ and $\Delta_\mu(\cdot; 18)$ are continuous on $\mathbb{T}^3$, there exist points $k_0, k_1 \in \mathbb{T}^3$ such that the equalities

$$\max_{k \in \mathbb{T}^3} \Delta_\mu(k; 0) = \Delta_\mu(k_0; 0), \quad \min_{k \in \mathbb{T}^3} \Delta_\mu(k; 18) = \Delta_\mu(k_1; 18)$$

hold.

Let us define the following notations:

$$\gamma_0 := \left(12 \frac{I(k_0; 0)}{I(\bar{0}; 0)} - \varepsilon(k_0)\right) \left(1 + \frac{I(k_0; 0)}{I(\bar{0}; 0)}\right)^{-1};$$
$$\gamma_1 := (18 - \varepsilon(k_1)) \left(1 - \frac{I(k_1; 18)}{I(\bar{0}; 0)}\right).$$

We denote:

$$E_\mu^{(1)} := \min\left\{\Lambda_\mu \cap (-\infty; 0]\right\}; E_\mu^{(2)} := \max\left\{\Lambda_\mu \cap (-\infty; 0]\right\};$$
$$E_\mu^{(3)} := \min\left\{\Lambda_\mu \cap [18; \infty)\right\}; E_\mu^{(4)} := \max\left\{\Lambda_\mu \cap [18; \infty)\right\}.$$

We formulate the results, which precisely describe the structure of the essential spectrum of $\mathcal{A}_\mu$. The structure of the essential spectrum depends on the location of the parameters $\mu > 0$ and $\gamma \in \mathbb{R}$.

Theorem 2.4. *Let $\mu = \mu_r^0(\gamma)$, with $\gamma < 12$. The following equality holds*

$$\sigma_{\rm ess}(\mathcal{A}_\mu) = \begin{cases} [E_1; E_2] \bigcup [0; 18], & \text{if } \gamma < \gamma_0; \\ [E_1; 18], & \text{if } \gamma_0 \leq \gamma < 6; \\ [0; 18], & \text{if } 6 \leq \gamma < 12. \end{cases}$$

Theorem 2.5. *Let $\mu = \mu_l^0(\gamma)$, with $\gamma > 0$. The following equality holds:*

$$\sigma_{\rm ess}(\mathcal{A}_\mu) = \begin{cases} [0; 18], & \text{if } 0 < \gamma \leq 6; \\ [0; E_\mu^{(4)}], & \text{if } 6 < \gamma \leq \gamma_1; \\ [0; 18] \bigcup [E_\mu^{(3)}; E_\mu^{(4)}], & \text{if } \gamma > \gamma_1. \end{cases}$$

The proof of these two theorems are based on the existence conditions of the eigenvalue $z_\mu(k)$ of the operator $\mathcal{A}_\mu(\cdot)$ and the continuity of $z_\mu(\cdot)$ on its domain.

3. Birman–Schwinger principle and discrete spectrum asymptotics of the operator $\mathcal{A}_\mu$

Let us denote by $\tau_{\min}(\mathcal{A}_\mu)$ and $\tau_{\max}(\mathcal{A}_\mu)$ the lower and upper bounds of the essential spectrum $\sigma_{\rm ess}(\mathcal{A}_\mu)$ of the operator $\mathcal{A}_\mu$, respectively, that is,

$$\tau_{\min}(\mathcal{A}_\mu) :\equiv \min \sigma_{\rm ess}(\mathcal{A}_\mu), \quad \tau_{\max}(\mathcal{A}_\mu) :\equiv \max \sigma_{\rm ess}(\mathcal{A}_\mu).$$

For an interval $\Delta \subset \mathbb{R}$, $E_\Delta(\mathcal{A}_\mu)$ stands for the spectral subspace of $\mathcal{A}_\mu$ corresponding to Δ. Let us denote by $\sharp\{\cdot\}$ the cardinality of a set and by $N_{(a,b)}(\mathcal{A}_\mu)$ the number of eigenvalues of the operator $\mathcal{A}_\mu$, including multiplicities, lying in $(a, b) \subset \mathbb{R} \setminus \sigma_{\rm ess}(\mathcal{A}_\mu)$, that is,

$$N_{(a,b)}(\mathcal{A}_\mu) := \dim E_{(a,b)}(\mathcal{A}_\mu).$$

For a $\lambda \in \mathbb{R}$, we define the number $n(\lambda, A_\mu)$ as follows

$$n(\lambda, A_\mu) := \sup\{\dim F : (A_\mu u, u) > \lambda,\ u \in F \subset \mathcal{H},\ ||u|| = 1\}.$$

The number $n(\lambda, A_\mu)$ is equal to the infinity if $\lambda < \max \sigma_{\rm ess}(A_\mu)$; if $n(\lambda, A_\mu)$ is finite, then it is equal to the number of the eigenvalues of A_μ bigger than λ.

By the definition of $N_{(a;b)}(\mathcal{A}_\mu)$, we have

$$N_{(-\infty;z)}(\mathcal{A}_\mu) = n(-z, -\mathcal{A}_\mu),\ -z > -\tau_{\min}(\mathcal{A}_\mu),$$
$$N_{(z;+\infty)}(\mathcal{A}_\mu) = n(z, \mathcal{A}_\mu),\ z > \tau_{\max}(\mathcal{A}_\mu).$$

In our analysis of the discrete spectrum of $\mathcal{A}_\mu$, the crucial role is played by the compact operator $\widehat{T}_\mu(z)$, $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu); \tau_{\max}(\mathcal{A}_\mu)]$ in the space $L_2(\mathbb{T}^3)$ as integral operator

$$(\widehat{T}_\mu(z)g)(p) = \frac{\mu^2}{2\sqrt{\Delta_\mu(p;\, z)}} \int_{\mathbb{T}^3} \frac{g(t)dt}{\sqrt{\Delta_\mu(t;\, z)}(w_2(p,t) - z)}, \quad \text{for} \quad z < \tau_{\min}(\mathcal{A}_\mu),$$
$$(\widehat{T}_\mu(z)g)(p) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(p;\, z)}} \int_{\mathbb{T}^3} \frac{g(t)dt}{\sqrt{-\Delta_\mu(t;\, z)}(w_2(p,t) - z)}, \quad \text{for} \quad z > \tau_{\max}(\mathcal{A}_\mu).$$

The following lemma is a realization of the well-known Birman–Schwinger principle for the operator $\mathcal{A}_\mu$ (see [8]).

Lemma 3.1. *For* $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu); \tau_{\max}(\mathcal{A}_\mu)]$ *the operator* $\widehat{T}_\mu(z)$ *is compact and continuous in* z *and*

$$N_{(-\infty;z)}(\mathcal{A}_\mu) = n(1, \widehat{T}_\mu(z)) \quad \textit{for} \quad z < \tau_{\min}(\mathcal{A}_\mu),$$
$$N_{(z;+\infty)}(\mathcal{A}_\mu) = n(1, \widehat{T}_\mu(z)) \quad \textit{for} \quad z > \tau_{\max}(\mathcal{A}_\mu).$$

This lemma can be proven quite similarly to the corresponding result of [8].

Let $\mathbb{S}^2$ being the unit sphere in $\mathbb{R}^3$ and

$$S_r : L_2((0, r), \sigma_0) \to L_2((0, r), \sigma_0), \quad r > 0, \quad \sigma_0 = L_2(\mathbb{S}^2)$$

be the integral operator with the kernel

$$S(t; y) = \frac{25}{8\pi^2\sqrt{6}} \frac{1}{5\cos(hy) + t},$$
$$y = x - x', \quad x, x' \in (0, r), \quad t = (\xi, \eta), \quad \xi, \eta \in \mathbb{S}^2.$$

For $\lambda > 0$, define

$$U(\lambda) = \frac{1}{2} \lim_{r \to \infty} r^{-1} n(\lambda, S_r).$$

The existence of the latter limit and the fact $U(1) > 0$ shown in [12].

From the definitions of the quantities $\mu_l^0(\gamma)$ and $\mu_r^0(\gamma)$, it is easy to see that $\mu_l^0(6) = \mu_r^0(6)$. We set $\mu_0 := \mu_l^0(6)$.

We can now formulate our last main result.

Theorem 3.2. *The following relations hold:*

$$\sharp(\sigma_{\rm disc}(\mathcal{A}_{\mu_0}) \cap (-\infty, 0)) = \sharp(\sigma_{\rm disc}(\mathcal{A}_{\mu_0}) \cap (18, \infty)) = \infty;$$
$$\lim_{z \nearrow 0} \frac{N_{(-\infty,\, z)}(\mathcal{A}_{\mu_0})}{|\log|z||} = \lim_{z \searrow 18} \frac{N_{(z,\, \infty)}(\mathcal{A}_{\mu_0})}{|\log|z - 18||} = U(1). \tag{3.1}$$

Clearly, by equality (3.1), the infinite cardinality of the parts of discrete spectrum of $\mathcal{A}_{\mu_0}$ in $(-\infty; 0)$ and $(18; +\infty)$ follows automatically from the positivity of $U(1)$.

Acknowledgements

The authors thank the anonymous referee for reading the manuscript carefully and for making valuable suggestions.

References

[1] Tretter C. *Spectral theory of block operator matrices and applications*. Imperial College Press, 2008.

[2] Muminov M., Neidhardt H., Rasulov T. On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case. *Journal of Mathematical Physics*, 2015, **56**, 053507.

[3] Leggett A.J., Chakravarty S., et al. Dynamics of the dissipative two-state system. *Rev. Mod. Phys.*, 1987, **59**, P. 1–85.

[4] Hübner M., Spohn H. Radiative decay: nonperturbative approaches. *Rev. Math. Phys.*, 1995, **7** (3), P. 363–387.

[5] Gérard C. Asymptotic completeness for the spin-boson model with a particle number cutoff. *Rev. in Math. Phys.*, 1996, **8** (4), P. 549–589.

[6] Hübner M., Spohn H. Spectral properties of spin-boson Hamiltonian. *Annl. Inst. Poincare*, 1995, **62** (3), P. 289–323.

[7] Zhukov Y.V., Minlos R.A. The spectrum and scattering in the 'spin-boson' model with at most three photons. *Theor. Math. Phys.*, 1995, **103** (1), P. 63–81.

[8] Albeverio S., Lakaev S.N., Rasulov T.H. On the spectrum of an Hamiltonian in Fock space. Discrete spectrum asymptotics. *J. Stat. Phys.*, 2007, **127** (2), P. 191–220.

[9] Albeverio S., Lakaev S.N., Rasulov T.H. The Efimov effect for a model operator associated with the Hamiltonian of a non conserved number of particles. *Methods Funct. Anal. Topology*, 2007, **13** (1), P. 1–16.

[10] Muminov M.I., Rasulov T.H. On the number of eigenvalues of the family of operator matrices. *Nanosystems: Physics, Chemistry, Mathematics*, 2014, **5** (5), P. 619–626.

[11] Rasulov T.H., Tosheva N.A. Analytic description of the essential spectrum of a family of 3×3 operator matrices. *Nanosystems: Physics, Chemistry, Mathematics*, 2019, **10** (5), P. 511–519.

[12] Sobolev A.V. The Efimov effect. Discrete spectrum asymptotics. *Comm. Math. Phys.*, 1993, **156**, P. 101–126.

[13] Rasulov T.Kh. Branches of the essential spectrum of the lattice spin-boson model with at most two photons. *Theoretical and Mathematical Physics*, 2016, **186** (2), P. 251–267.

[14] Abdullaev Zh.I., Lakaev S.N. Asymptotics of the discrete spectrum of the three-particle Schrödinger difference operator on a lattice. *Theor. Math. Phys.*, 2003, **136** (2), P. 1096–1109.

[15] Albeverio S., Lakaev S.N., Muminov Z.I. Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics. *Ann. Henri Poincaré*, 2004, **5**, P. 743–772.

[16] Lakaev S.N., Muminov M.É. Essential and discrete spectra of the three-particle Schrödinger operator on a lattices. *Theor. Math. Phys.*, 2003, **135** (3), P. 849–871.
[17] Mogilner A.I. Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: problems and results. *Advances in Sov. Math.*, 1991, **5**, P. 139–194.
[18] Reed M., Simon B. *Methods of modern mathematical physics. IV: Analysis of Operators*. Academic Press, New York, 1979.
[19] Malishev V.A., Minlos R.A. *Linear infinite-particle operators*. Translations of Mathematical Monographs. 143, AMS, Providence, RI, 1995.
[20] Minlos R.A., Spohn H. The three-body problem in radioactive decay: the case of one atom and at most two photons. *Topics in Statistical and Theoretical Physics. Amer. Math. Soc. Transl.*, 1996, **177** (2), AMS, Providence, RI, P. 159–193.
[21] Friedrichs K.O. *Perturbation of spectra in Hilbert space*. Amer. Math. Soc. Providence, Rhole Island, 1965.
[22] Rasulov T.H., Dilmurodov E.B. Threshold analysis for a 2×2 operator matrix. *Nanosystems: Physics, Chemistry, Mathematics*, 2019, **10** (6), P. 616–622.
[23] Rasulov T.H., Dilmurodov E.B. Eigenvalues and virtual levels of a family of 2×2 operator matrices. *Methods of Functional Analysis and Topology*, 2019, **25** (3), P. 273–281.