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# Positive fixed points of Lyapunov operator

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In this paper, fixed points of Lyapunov integral equation are found and considered the connections between Gibbs measures for four competing interactions of models with uncountable (i.e. $[0,1]$) set of spin values on the Cayley tree of order two.

**Keywords:** Lyapunov integral operator, fixed points, Cayley tree, Gibbs measure.

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## 1. Introduction

Spin models on a graph or in continuous spaces form a large class of systems considered in mechanics, biology, nanoscience, etc. Some of them have a real physical meaning, others have been proposed as suitably simplified models of more complicated systems. The geometric structure of the graph or a physical space plays an important role in such investigations. For example, in order to study the phase transition problem on a cubic lattice $Z^d$ or in space one uses, essentially, the Pirogov-Sinai theory; see [1–3]. A general methodology of phase transitions in $\mathbb{Z}^d$ or $\mathbb{R}^d$ was developed in [4]; some recent results in this direction have been established in [5,6] (see also the bibliography therein).

During last years, an increasing attention was given to models with a *uncountable* many spin values on a Cayley tree. Until now, one considered nearest-neighbor interactions $(J_3 = J = \alpha = 0,\ J_1 \neq 0)$ with the set of spin values $[0,1]$ (for example, [7–12]).

In [13] it is described that splitting Gibbs measures on $\Gamma_2$ by solutions to a nonlinear integral equation for the case $J_3^2 + J_1^2 + J^2 + \alpha^2 \neq 0$ which a generalization of the case $J_3 = J = \alpha = 0,\ J_1 \neq 0$. Also, it is proven that periodic Gibbs measure for Hamiltonian (1) with four competing interactions is either $translation\text{-}invariant$ or $G_k^{(2)} - periodic$.

In this paper, we consider Lyapunov's operator with degenerate kernel. In [11], Fixed points of Lyapunov's operator with special degenerate kernel are studied. The present paper is a continuation of the paper [11], i.e., we give full description of fixed points of Lyapunov's operator with another special degenerate kernel.

A Cayley tree $\Gamma^k = (V, L)$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertices. Here $V$ is the set of vertices and $L$ that of edges (arcs). The distance $d(x,y), x, y \in V$ is the number of edges of the path from $x$ to $y$. Let $x^0 \in V$ be a fixed and we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\},$$

$$L_n = \{l =< x, y >\in L \mid x, y \in V_n\},$$

If the distance from $x$ to $y$ equals one then we say $x$ and $y$ are nearest neighbors and use the notation $l = \langle x, y\rangle$. The set of the direct successors of $x$ is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1} |\ d(x,y) = 1\},\ x \in W_n.$$

We observe that for any vertex $x \neq x^0,\ x$ has $k$ direct successors and $x^0$ has $k+1$. The vertices $x$ and $y$ are called second neighbor which is denoted by $> x, y <$, if there exist a vertex $z \in V$ such that $x$, $z$ and $y$, $z$ are nearest neighbors. We will consider only second neighbors $> x, y <$, for which there exist $n$ such that $x, y \in W_n$. Three vertices $x,\ y$ and $z$ are called a triplet of neighbors and they are denoted by $< x, y, z >$, if $< x, y >,\ < y, z >$ are nearest neighbors and $x,\ z \in W_n,\ y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Now, we consider models with four competing interactions where the spin takes values in the set $[0,1]$. For some set $A \subset V$ an arbitrary function $\sigma_A : A \to [0,1]$ is called a configuration and the set of all configurations on $A$ we denote by $\Omega_A = [0,1]^A$. Let $\sigma(\cdot)$ belong to $\Omega_V = \Omega$ and $\xi_1 : (t,u,v) \in [0,1]^3 \to \xi_1(t,u,v) \in R$, $\xi_i : (u,v) \in [0,1]^2 \to \xi_i(u,v) \in R,\ i \in \{2,3\}$ are given bounded, measurable functions.

We consider models with four competing interactions where the spin takes values in the unit interval $[0,1]$. Given a set $\Lambda \subset V$ a configuration on $\Lambda$ is an arbitrary function $\sigma_\Lambda : \Lambda \to [0,1]$, with values $\sigma(x),\ x \in \Lambda$. The set of all configurations on $\Lambda$ is denoted by $\Omega_\Lambda = [0,1]^\Lambda = \Omega$ and denote by $\mathcal{B}$ the sigma-algebra generated by measurable cylinder subsets of $\Omega$.

Fix bounded, measurable functions $\xi_1 : (t,u,v) \in [0,1]^3 \to \xi_1(t,u,v) \in R$ and $\xi_i : (u,v) \in [0,1]^2 \to \xi_i(u,v) \in R,\ i = 2,3$. We consider a model with four competing interactions on the Cayley tree which is defined by a formal Hamiltonian

$$H(\sigma) = -J_3 \sum_{\langle x,y,z\rangle} \xi_1(\sigma(x),\sigma(y),\sigma(z)) - J \sum_{\rangle x,y\langle} \xi_2(\sigma(x),\sigma(z))$$

$$-J_1 \sum_{\langle x,y\rangle} \xi_3(\sigma(x),\sigma(y)) - \alpha \sum_x \sigma(x), \tag{1}$$

where the sum in the first term ranges all triples of neighbors, the second sum ranges all second neighbors, the third sum ranges all nearest neighbors, and $J, J_1, J_3, \alpha \in R\setminus\{0\}$.

Let $h : [0,1] \times V \setminus \{x^0\} \to \mathbb{R}$ and $|h(t,x)| = |h_{t,x}| < C$ where $x_0$ is a root of Cayley tree and $C$ is a constant which does not depend on $t$. For some $n \in \mathbb{N}$, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function we consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by:

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right), \tag{2}$$

$$Z_n = \int\limits_{\Omega^{(p)}_{V_{n-1}}}\!\!\!...\!\int \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda^{(p)}_{V_{n-1}}(d\widetilde{\sigma}_n), \tag{3}$$

where

$$\underbrace{\Omega_{W_n} \times \Omega_{W_n} \times ... \times \Omega_{W_n}}_{3\cdot 2^{p-1}} = \Omega^{(p)}_{W_n}, \quad \underbrace{\lambda_{W_n} \times \lambda_{W_n} \times ... \times \lambda_{W_n}}_{3\cdot 2^{p-1}} = \lambda^{(p)}_{W_n},\ n,p \in \mathbb{N},$$

Let $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. For $n \in \mathbb{N}$ we say that the probability distributions $\mu^{(n)}$ are compatible if $\mu^{(n)}$ satisfies the following condition:

$$\int\!\!\int\limits_{\Omega_{W_n}\times\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)(\lambda_{W_n} \times \lambda_{W_n})(d\omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{4}$$

By Kolmogorov's extension theorem, there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$. The measure $\mu$ is called *splitting Gibbs measure* corresponding to Hamiltonian (1) and function $x \mapsto h_x$, $x \neq x^0$ (see [7, 8, 14, 15]).

We denote:

$$K(u,t,v) = \exp\{J_3\beta\xi_1(t,u,v) + J\beta\xi_2(u,v) + J_1\beta(\xi_3(t,u) + \xi_3(t,v)) + \alpha\beta(u+v)\}, \tag{5}$$

and

$$f(t,x) = \exp(h_{t,x} - h_{0,x}),\ \ (t,u,v) \in [0,1]^3,\ x \in V \setminus \{x^0\}.$$

The following statement describes conditions on $h_x$ guaranteeing the compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

**Proposition 1** [16] The measure $\mu^{(n)}(\sigma_n)$, $n = 1,2,\dots$ satisfies the consistency condition (4) iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$f(t,x) = \prod_{>y,z<\in S(x)} \frac{\int_0^1\int_0^1 K(t,u,v)f(u,y)f(v,z)dudv}{\int_0^1\int_0^1 K(0,u,v)f(u,y)f(v,z)dudv}, \tag{6}$$

where $S(x) = \{y,z\},\ < y,x,z >$ is a ternary neighbor.

## 2. Lyapunov operator with degenerate kernel

Let $\varphi_1(t)$, $\varphi_2(s)$ and $\varphi_3(u)$ are positive functions from $C_0^+[0,1]$. We consider Lyapunov's operator $A$ (see [9,17]):

$$(Af)(t) = \int_0^1 \int_0^1 (\varphi_1(t) + \varphi_2(s) + \varphi_3(u))\, f(s)f(u)dsdu.$$

and quadratic operator $P$ on $\mathbb{R}^3$ by the rule

$$P(x,y,z) = (\alpha_{11}x^2 + xy + xz,\ \alpha_{21}x^2 + \alpha_{22}xy + \alpha_{22}xz,\ \alpha_{31}x^2 + \alpha_{33}xy + \alpha_{33}xz).$$

Here,

$$\alpha_{11} = \int_0^1 \varphi_1(s)ds > 0;$$

$$\alpha_{22} = \int_0^1 \varphi_2(s)ds > 0,\ \ \alpha_{21} = \int_0^1 \varphi_1(s)\varphi_2(s)ds > 0;$$

$$\alpha_{33} = \int_0^1 \varphi_3(s)ds > 0,\ \ \alpha_{31} = \int_0^1 \varphi_1(s)\varphi_3(s)ds > 0.$$

The existence of fixed points of Lyapunov's operator $A$ is proved in [16]. A sufficient condition of uniqueness of fixed points of Lyapunov operator $A$ s given (see [8]).

**Lemma 2.1** *Lyapunov's operator $A$ has a nontrivial positive fixed point iff the quadratic operator $P$ has a nontrivial positive fixed point, moreover, $N^+_{fix}(A) = N^+_{fix}(P)$.*

*Proof* (a) Put

$$\mathbb{R}_3^+ = \left\{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\right\},$$
$$\mathbb{R}_3^> = \left\{(x,y,z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\right\}.$$

Let Lyapunov's operator $A$ has a nontrivial positive fixed point $f(t) \in C_0^+[0,1]$. Let

$$x_1 = \int_0^1 f(u)du, \tag{7}$$

$$x_2 = \int_0^1 \varphi_2(u)f(u)du, \tag{8}$$

and

$$x_3 = \int_0^1 \varphi_3(u)f(u)du, \tag{9}$$

Clearly, $x_1 > 0$, $x_2 > 0$,$x_3 > 0$, i.e. $(x_1, x_2, x_3) \in \mathbb{R}_3^>$. Then, for the function $f(t)$, the equality

$$f(t) = \varphi_1(t)x_1^2 + x_1x_2 + x_1x_3 \tag{10}$$

holds.

Consequently, for parametrs $c_1$, $c_2$, $c_3$ from the equality (7), (8) and (9), we have the three identities:

$$x_1 = x_1(\alpha_{11}x_1 + x_2 + x_3),$$

$$x_2 = x_1(\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{22}x_3),$$

$$x_3 = x_1(\alpha_{31}x_1 + \alpha_{33}x_2 + \alpha_{33}x_3).$$

Therefore, the point $(c_1, c_2)$ is fixed point of the quadratic operator $P$.

(b) Assume, that the fixed point $x_0, y_0, z_0$ is a nontrivial positive fixed point of the quadratic operator $P$, i.e. $(x_0, y_0, z_0) \in \mathbb{R}_3^>$ and number $x_0$, $y_0$, $z_0$ satisfies the following equalities

$$x_0(\alpha_{11}x_0 + y_0 + z_0) = x_0,$$

$$x_0(\alpha_{21}x_0 + \alpha_{22}y_0 + \alpha_{22}z_0) = y_0,$$

$$x_0(\alpha_{31}x_0 + \alpha_{33}y_0 + \alpha_{33}z_0) = z_0.$$

Similary, we can prove that the function $f_0(t) = \varphi_1(t)x_0^2 + x_0y_0 + x_0z_0$ is fixed point of Lyapunov's operator $A$ and $f_0(t) \in C_0^+[0,1]$. This completes the proof.

## 3. Positive fixed points of the quadratic operators in cone $\mathbb{R}_3^+$

We define quadratic operator (QO) $\mathcal{Q}$ in cone $\mathbb{R}_3$ by the rule

$$\mathcal{Q}(x,y,z) = (a_{11}x^2 + xy + xz,\ a_{21}x^2 + a_{22}xy + a_{22}xz,\ a_{31}x^2 + a_{33}xy + a_{33}xz).$$

**3.1-lemma** *If the point $\omega = (x_0, y_0, z_0) \in \mathbb{R}_2^+$ is fixed point of QO $\mathcal{Q}$, then $x_0$ is a root of the quadratic algebraic equation*

$$(a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33})x^2 + (a_{11} + a_{22} + a_{33})x - 1 = 0 \tag{11}$$

*Proof* Let the point $\omega = (x_0, y_0, z_0) \in \mathbb{R}_3^+$ be a fixed point of $QO\ \mathcal{Q}$. Then

$$a_{11}x_0^2 + x_0y_0 + x_0z_0, \quad a_{21}x_0^2 + a_{22}x_0y_0 + a_{22}x_0z_0,$$
$$a_{31}x_0^2 + a_{33}x_0y_0 + a_{33}x_0z_0.$$

Using the bellowing equalities, we obtain:

$$y_0 + z_0 = 1 - a_{11}x_0$$
$$y_0 = x_0(a_{21}x_0 + a_{22}(1 - a_{11}x_0))$$
$$z_0 = x_0(a_{31}x_0 + a_{33}(1 - a_{11}x_0))$$
$$y_0 + z_0 = x_0(a_{21}x_0 + a_{22}(1 - a_{11}x_0)) + x_0(a_{31}x_0 + a_{33}(1 - a_{11}x_0)) =$$
$$= (a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33})x_0^2 + (a_{22} + a_{33})x_0 = a_{11}x_0$$

By the last equality, we get:

$$(a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33})x_0^2 + (a_{11} + a_{22} + a_{33})x_0 - 1 = 0.$$

This completes the proof.

**3.2-lemma** *If the positive number $x_0$ is root of the quadratic algebraic Eq.(11), then the point $\omega_0 = (x_0,\ x_0(a_{21}x_0 + a_{22}(1 - a_{11}x$ is fixed point of QO $\mathcal{Q}$.*

*Proof* Let $x_0$ be a root of the quadratic Eq.(11), i.e.,

$$(a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33})x_0^2 + (a_{11} + a_{22} + a_{33})x_0 - 1 = 0.$$
$$x_0(a_{11}x_0 + y_0 + z_0) =$$
$$= x_0(a_{11}x_0 + x_0(a_{21}x_0 + a_{22}(1 - a_{11}x_0)) + x_0(a_{31}x_0 + a_{33}(1 - a_{11}x_0))) =$$
$$= x_0(a_{11}x_0 + (a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33})x_0^2 + (a_{22} + a_{33})x_0) =$$
$$= x_0((a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33})x_0^2 + (a_{11} + a_{22} + a_{33})x_0 - 1 + 1) = x_0(0 + 1) = x_0$$

Then

$$y_0 + z_0 = 1 - a_{11}x_0.$$

From the last equality, we get:

$$a_{21}x_0^2 + a_{22}x_0y_0 + a_{22}x_0z_0 =$$
$$= x_0(a_{21}x_0 + a_{22}(y_0 + z_0)) = x_0(a_{21}x_0 + a_{22}(1 - a_{11}x_0)),$$
$$a_{31}x_0^2 + a_{33}x_0y_0 + a_{33}x_0z_0 =$$
$$= x_0(a_{31}x_0 + a_{33}(y_0 + z_0)) = x_0(a_{31}x_0 + a_{33}(1 - a_{11}x_0)).$$

This completes the proof.

We put

$$\mu_0 = a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33}, \quad \mu_1 = a_{11} + a_{22} + a_{33}$$

and define polynomial $P_2(x)$:

$$P_2(x) = \mu_0x^2 + \mu_1x_1 - 1. \tag{12}$$

**Theorem 3.3** *QO $\mathcal{Q}$ has a unique nontrivial positive fixed point.*

*Proof* To prove the Theorem, we use properties of the polynomial $P_2(x)$. It is known that there are two roots of the polynomial. They are:

$$x_1 = \frac{-\mu_1 + \sqrt{\mu_1^2 + 4\mu_0}}{2\mu_0}$$
$$x_2 = \frac{-\mu_1 - \sqrt{\mu_1^2 + 4\mu_0}}{2\mu_0}$$

**I** Let $\mu_0 > 0$. In this case, $x_1 > 0$ and $x_2 < 0$.

$$
\begin{aligned}
1 - a_{11}x_1 = 1 - \frac{-\mu_1 + \sqrt{\mu_1^2 + 4\mu_0}}{2\mu_0} a_{11} = \\
= \frac{2\mu_0 + a_{11}\mu_1 - \sqrt{(\mu_1^2 + 4\mu_0)a_{11}^2}}{2\mu_0} > \\
> \frac{2\mu_0 + a_{11}\mu_1 - \sqrt{\mu_1^2 a_{11}^2 + 4\mu_0\mu_1 a_{11}}}{2\mu_0} > \\
> \frac{2\mu_0 + a_{11}\mu_1 - \sqrt{\mu_1^2 a_{11}^2 + a\mu_0\mu_1 a_{11} + 4\mu_0\mu_1 a_{11}}}{2\mu_0} = 0
\end{aligned}
$$

i.e., $1 - a_{11}x_1 > 0$. It means:

$$
\begin{aligned}
y_1 = x_1(a_{21}x_1 + a_{22}(1 - a_{11}x_1)) > 0, \\
z_1 = x_1(a_{31}x_1 + a_{33}(1 - a_{11}x_1)) > 0.
\end{aligned}
$$

**II** Let $\mu_0 < 0$. In this case, $x_1 > 0$ and $x_2 > 0$.

Clearly,

$$
(P_2(x))' = 2\mu_0 x + \mu_1 \tag{13}
$$

and $P_2'\left(\frac{-\mu_1}{2\mu_0}\right) = 0$. Moreover, the function $P_2(x)$ is an increasing function on $\left(-\infty, \frac{-\mu_1}{2\mu_0}\right)$ and it is a decreasing function on $\left(\frac{-\mu_1}{2\mu_0}, \infty\right)$.

If we put $x' = \frac{-\mu_1}{2\mu_0}$, then

$$
x_1 < x' < x_2.
$$

**II.I** Let $x' = \frac{-\mu_1}{2\mu_0} < \frac{1}{a_{11}}$.

$$
a_{11}\mu_1 < -2\mu_0 \tag{14}
$$

Then $x_1 < \frac{1}{a_{11}}$ and from $1 - a_{11}x_1 > 0$. Moreover,

$$
(x_1, y_1, z_1) \in \mathbb{R}_3^+
$$

By other hand, we have the following identity:

$$
1 - a_{11}x_2 = \frac{2\mu_0 + a_{11}\mu_1 + a_{11}\sqrt{\mu_1^2 + 4\mu_0}}{2\mu_0}
$$

By (14):

$$
\begin{aligned}
2\mu_0 + a_{11}\mu_1 + a_{11}\sqrt{\mu_1^2 + 4\mu_0} > \\
> 2\mu_0 + (-2\mu_0) + a_{11}\sqrt{\mu_1^2 + 4\mu_0} = a_{11}\sqrt{\mu_1^2 + 4\mu_0} > 0.
\end{aligned}
$$

From the last inequality,

$$
1 - a_{11}x_2 < 0
$$

and

$$
(x_2, y_2, z_2) \notin \mathbb{R}_3^+.
$$

**II.II** Let $x' = \frac{-\mu_1}{2\mu_0} > \frac{1}{a_{11}}$. We have:

$$
1 - a_{11}x_1 = \frac{2\mu_0 + a_{11}\mu_1 - a_{11}\sqrt{\mu_1^2 + 4\mu_0}}{2\mu_0}
$$

Consequently,

$$
\begin{aligned}
a_{21} + a_{31} > 0, \\
a_{21} + a_{31} - a_{11}a_{22} - a_{11}a_{33} + a_{11}^2 + a_{11}a_{22} + a_{11}a_{33} > a_{11}, \\
\mu_0 + a_{11}\mu_1 > a_{11}^2, \\
4\mu_0(\mu_0 + a_{11}\mu_1) < 4\mu_0(a_{11}^2), \\
a\mu_0^2 + 4a_{11}\mu_0\mu_1 + a_{11}^2\mu_1^2 < a_{11}^2\mu_1^2 + 4\mu_0 a_{11}^2, \\
(2\mu_0 + a_{11}\mu_1)^2 < (a_{11}\sqrt{\mu_1^2 + 4\mu_0})^2,
\end{aligned}
$$

$$2\mu_0 + a_{11}\mu_1 < a_{11}\sqrt{\mu_1^2 + 4\mu_0},$$

$$2\mu_0 + a_{11}\mu_1 - a_{11}\sqrt{\mu_1^2 + 4\mu_0} < 0.$$

From the last identity:

$$1 - a_{11}x_1 > 0,$$

and

$$(x_1, y_1, z_1) \in \mathbb{R}_3^+.$$

By the other hand, $x_2 > x' > \dfrac{1}{a_{11}}$. So, $1 - a_{11}x_2 < 0$ and $(x_2, y_2, z_2) \notin \mathbb{R}_3^+$.

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