Non-compact perturbations of the spectrum of multipliers given with functions

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The change in the spectrum of the multipliers \( H_0 f(x, y) = x^\alpha + y^\beta f(x, y) \) and \( H_0 f(x, y) = x^\alpha y^\gamma f(x, y) \) for perturbation with partial integral operators in the spaces \( L^2_2[0, 1]^2 \) is studied. Precise description of the essential spectrum and the existence of simple eigenvalue is received. We prove that the number of eigenvalues located below the lower edge of the essential spectrum in the model is finite.

Keywords: essential spectrum, discrete spectrum, lower bound of the essential spectrum, partial integral operator.

1. Introduction

The first results on the finiteness of the discrete spectrum of \( N \) – particle Hamiltonians with \( N > 2 \) were obtained by Uchiyama in 1969 [1–3]. He found sufficient conditions for the finiteness of the number of discrete eigenvalues for energy operators in the space \( L^2_2(\mathbb{R}^N) \) for the system of two identically charged particles in the field of a fixed center with or without an external electromagnetic field. In 1971, Zhislin proved the finiteness of the discrete spectrum for energy operators in symmetry spaces of negative atomic ions with nuclei of any mass and of molecules with infinitely heavy nuclei under the assumption that the total charge of the system is less than \(-1\) [4].

Let \( \Omega_1 \) and \( \Omega_2 \) be closed bounded sets in \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively. In the space \( L^p_p(\Omega_1 \times \Omega_2) \), \( p \geq 1 \) partially integral operator (PIO) \( T \) of the Fredholm type in general is given by the equality [5]:

\[
T = T_0 + T_1 + T_2 + K,
\]

where the operators \( T_0, T_1, T_2, K \) have the following view:

\[
T_0 f(x, y) = k_0(x, y)f(x, y), \quad T_1 f(x, y) = \int_{\Omega_1} k_1(x, s, y)f(s, y)d\mu_1(s),
\]

\[
T_2 f(x, y) = \int_{\Omega_2} k_2(x, t, y)f(x, t)d\mu_2(t), \quad K f(x, y) = \int_{\Omega_1 \Omega_2} k(x, y; s, t)f(s, t)d\mu_1(s)d\mu_2(t).
\]

Here, the functions \( k_0, k_1, k_2, \) and \( k \) are given measurable functions in the concept of Lebesque on \( \Omega_1 \times \Omega_2, \Omega_1^2 \times \Omega_2, \Omega_1 \times \Omega_2^2 \) and \((\Omega_1 \times \Omega_2)^2\), respectively, and integration of functions is understood in the concept of Lebesgue, where \( \mu_k(\cdot) \) – Lebesque measure on \( \Omega_k, \ k = 1, 2 \).

In the Hilbert space \( L^2_2(\Omega \times \Omega) \), where \( \Omega = [a, b]^n \), consider the following model operator:

\[
H = H_0 - (\gamma T_1 + \mu T_2), \quad \gamma > 0, \quad \mu > 0.
\]

Here, the actions of the operators \( H_0, T_1 \) and \( T_2 \) are determined by formulas:

\[
H_0 f(x, y) = k_0(x, y)f(x, y), \quad T_1 f(x, y) = \int_{\Omega} \varphi_1(x)\varphi_1(s)f(s, y)ds, \quad T_2 f(x, y) = \int_{\Omega} \varphi_2(y)\varphi_2(t)f(x, t)dt,
\]

where \( k_0(x, y) \) is a nonnegative continuous function on \( \Omega \times \Omega, \varphi_j(\cdot) \) is a continuous function on \( \Omega \) and

\[
\int_{\Omega} \varphi_j^2(t)dt = 1, \quad j = 1, 2.
\]

Via \( \rho(\cdot), \rho(\cdot), \sigma_{ess}(\cdot) \) and \( \sigma_{disc}(\cdot) \) denote, respectively, the resolvent set, spectrum, essential spectrum and discrete spectrum self-adjoint operators [6].
In [7], sufficient conditions for finiteness and infinity were obtained in the discrete spectrum for \( \sigma_{\text{ess}}(H) = \sigma(H_0) \). In work [8] proved the existence of the Efimov effect in model (2) for given \( k_0(x, y) \). In [9], the essential spectrum and the number eigenvalues below the lower bound of the essential spectrum in model (2), when the function \( k_0(x, y) \) has the form: \( k_0(x, y) = u(x)u(y) \), where \( u(x) \) is a nonnegative continuous function on \( \Omega = \Omega_1 = \Omega_2 \) and \( \int_\Omega \frac{dx}{u(x)} < \infty \). In [10] studied the existence of an infinite number of eigenvalues (the existence of Efimov’s effect) for a selfadjoint partial integral operators.

2. The lower boundary of the essential spectrum of \( V \)

Consider the multiplier:

\[
V_0f(x, y) = (x^\alpha + y^\beta)f(x, y), \quad \alpha > 0, \quad \beta > 0.
\]

Let us define a partially integral operator (PIO) \( V \):

\[
V = V_0 - \gamma (T_1 + T_2), \quad \gamma > 0,
\]

where:

\[
T_1f(x, y) = \frac{1}{\alpha} \int_0^1 f(s, y)ds, \quad T_2f(x, y) = \frac{1}{\beta} \int_0^1 f(x, t)dt, \quad f \in L^2[0, 1]^2.
\]

In the space \( L^2[0, 1] \) we define the operators \( H_1 \) and \( H_2 \) in Friedrichs models:

\[
H_1\varphi(x) = x^\alpha \varphi(x) - \gamma \int_0^1 \varphi(s)ds, \quad H_2\psi(y) = y^\beta \psi(y) - \gamma \int_0^1 \psi(t)dt.
\]

**Lemma 1.** [11] The number \( \lambda \in \mathbb{R} \setminus [0, 1] \) is the eigenvalue of the operator \( H_1 \) (of the operator \( H_2 \)) if and only if

\[
\Delta_1(\lambda) = 1 - \gamma \int_0^1 \frac{ds}{s^\alpha - \lambda}, \quad \Delta_2(\lambda) = 1 - \gamma \int_0^1 \frac{ds}{s^\beta - \lambda}.
\]

**Lemma 2.**

A)

\[
\lim_{\lambda \to 0^-} \Delta_1(\lambda) = \begin{cases} 
1 - \gamma, & \text{if } 0 < \alpha < 1; \\
-\infty, & \text{if } \alpha \geq 1,
\end{cases}
\]

B)

\[
\lim_{\lambda \to 0^-} \Delta_2(\lambda) = \begin{cases} 
1 - \gamma, & \text{if } 0 < \beta < 1; \\
-\infty, & \text{if } \beta \geq 1.
\end{cases}
\]

**Proof.** First, we prove the statement A.

a) Let \( 0 < \alpha < 1 \). Consider an arbitrary increasing sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) of negative numbers approaching to zero, i.e \( \lambda_n \leq \lambda_{n+1} < 0 \) and \( \lim_{n \to \infty} \lambda_n = 0 \). Then:

\[
0 < \frac{1}{s^\alpha - \lambda_n} \leq \frac{1}{s^\alpha - \lambda_{n+1}}, \quad n \in \mathbb{N}
\]

and

\[
\frac{1}{s^\alpha - \lambda_n} \leq \frac{1}{s^\alpha} \text{ for almost all } s \in [0, 1].
\]

The function \( h_0(s) = \frac{1}{s^\alpha} \) is integrable by \( [0, 1] \) in the concept of Lebesgue and:

\[
\int_0^1 h_0(s)ds = \frac{1}{1 - \alpha}.
\]

Hence, due to Lebesgue theorem on limited transition under the sign of the Lebesgue integral it follows that:
the operator eigenvalue of the operator; 

Hence: 

\[ \lim_{\lambda \to 0-} \Delta_1(\lambda) = 1 - \frac{\gamma}{1 - \alpha}. \]

b) Let \( \alpha \geq 1 \). Suppose that \( \alpha = 1 \). Then: 

\[ \lim_{\lambda \to 0-} \Delta_1(\lambda) = 1 - \gamma \lim_{\lambda \to 0-} \frac{1}{s - \lambda} = 1 - \gamma \lim_{\lambda \to 0-} \ln \left(1 - \frac{1}{\lambda}\right) = -\infty. \]

If \( \alpha > 1 \), then we have: 

\[ \frac{1}{s^\alpha - \lambda} \geq \frac{1}{s - \lambda}, \quad s \in [0, 1], \]

Hence: 

\[ \lim_{\lambda \to 0-} \int_0^1 \frac{ds}{s^\alpha - \lambda} \geq \lim_{\lambda \to 0-} \int_0^1 \frac{ds}{s - \lambda} = +\infty, \]

i.e. 

\[ \lim_{\lambda \to 0-} \Delta_1(\lambda) = -\infty. \]

**Proposition 1.** I) Let \( 0 < \alpha < 1 \) (\( 0 < \beta < 1 \)). Then: 

a) if \( \alpha + \gamma \leq 1 \) (\( \beta + \gamma \leq 1 \)), then the operator \( H_1 \) (operator \( H_2 \)) outside the essential spectrum has the only eigenvalue of the operator; 

b) if \( \alpha + \gamma > 1 \) (\( \beta + \gamma > 1 \)), then the operator \( H_1 \) (operator \( H_2 \)) outside the essential spectrum has the only eigenvalue of the operator \( \xi_k \) (the eigenvalue value \( \xi_k \)), while \( \xi_k \) is a simple proper value \( H_k \xi_k < 0, k = 1, 2 \).

II) Let \( \alpha \geq 1 \) (\( \beta \geq 1 \)). Then the operator \( H_1 \) (operator \( H_2 \)) outside the essential spectrum has a unique eigenvalue \( \xi_1 \) (eigenvalue \( \xi_2 \)), for this \( \xi_k \) is a simple eigenvalue of the operator \( H_k \) and \( \xi_k < 0, k = 1, 2 \).

**Proof.** It is easy to note that the function \( \Delta_1(\lambda) \) by \((-\infty, 0)\) is strictly decreasing and \( \Delta_1(\lambda) > 0 \) to \((1, \infty)\), thus, the operator \( H_1 \) on the set \((1, \infty)\) has no eigenvalue. 

Let \( 0 < \alpha < 1 \). By Lemma 2 and monotonicity, the function \( \Delta_1(\lambda) \) to \((-\infty, 0)\) states a and b, since: 

\[ \text{Ran}(\Delta_1) = \left(1 - \frac{\gamma}{1 - \alpha}, 1\right). \]

Let \( \alpha \geq 1 \). Then from Lemma 2 we obtain: \( \text{Ran}(\Delta_1) = (-\infty, 1) \). Due to monotonicity functions \( \Delta_1(\lambda) \) \((-\infty, 0)\) equation \( \Delta_1(\lambda) = 0 \) \((-\infty, 0)\) has a unique solution \( \xi_1 < 0 \) \( \xi_1 \) is a simple eigenvalue operator \( H_1 \).

**Proposition 1** for the operator \( H_2 \) is proved similarly.

**Theorem 1.** Let \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). Then: 

a) if \( \alpha + \gamma \leq 1 \) and \( \alpha + \beta \leq 1 \), then: 

\[ \sigma(V) = \sigma_{ess}(V) = \sigma(V_0) = [0, 2]; \]

b) if \( \alpha + \gamma > 1 \) and \( \alpha + \beta \leq 1 \), then 

\[ \sigma(V) = \sigma_{ess}(V) = \sigma(V_0) \cup [\xi_1, 1 + \xi_1], \]

where \( \xi_1 \) – negative eigenvalue of operator \( H_1 \);

c) if \( \alpha + \gamma \leq 1 \) and \( \alpha + \beta > 1 \), then: 

\[ \sigma_{ess}(V) = \sigma(V_0) \cup [\xi_2, 1 + \xi_2], \]

where \( \xi_2 \) is a negative eigenvalue of operator \( H_2 \);

d) if \( \alpha + \gamma > 1 \) and \( \alpha + \beta > 1 \), then: 

\[ \sigma_{ess}(V) = \sigma(V_0) \cup [\xi_1, 1 + \xi_1] \cup [\xi_2, 1 + \xi_2] \quad \text{and} \quad \sigma_{disc}(V) = \{\omega_0\}, \]

where \( \omega_0 = \xi_1 + \xi_2 \) and \( \omega_0 \) is a simple eigenvalue of operator \( V \).

**Proof.** It is easy to note that the operator \( V \) will be unitarily equivalent to the operator \( H_1 \otimes E + E \otimes H_2 \) (see [12]).

Then \( \sigma(V) = \sigma(H_1) + \sigma(H_2) \) and for of multiplicity \( n_V(\omega) \) eigenvalue \( \omega \in \sigma(V) \setminus \sigma_{ess}(V) \) of the operator \( V \) the following equality takes place:

\[ \lim_{\lambda \to 0-} \int_0^1 \frac{ds}{s^\alpha - \lambda} = \frac{1}{1 - \alpha}. \]
\[ n_V(\omega) = \sum_{\substack{p+q=\omega, \\ (p,q)\in\sigma(H_1)\times\sigma(H_2)}} n_{H_1}(p) \cdot n_{H_2}(q), \]

where \( n_{H_1}(p) \) and \( n_{H_2}(q) \) — multiplicity of the eigenvalues \( p \) and \( q \) of the operators \( H_1 \) and \( H_2 \), respectively. This and Proposition 1 imply the proof the theorem.

**Theorem 2.** Let \( \alpha \geq 1 \) and \( 0 < \beta < 1 \). Then:

a) if \( \beta + \gamma \leq 1 \), then:

\[ \sigma(V) = \sigma_{ess}(V) = \sigma(V_0) \cup [\xi_1, 1 + \xi_1]; \]

b) if \( \beta + \gamma > 1 \), then:

\[ \sigma_{ess}(V) = \sigma(V_0) \cup [\xi_1, 1 + \xi_1] \cup [\xi_2, 1 + \xi_2] \quad \text{and} \quad \sigma_{disc}(V) = \{ \omega_0 \}, \]

where \( \omega_0 = \xi_1 + \xi_2 \) and \( \omega_0 \) is a simple eigenvalue of the operator \( V \).

**Theorem 3.** Let \( \alpha \geq 1 \) and \( \beta \geq 1 \). Then:

\[ \sigma_{ess}(V) = \sigma(V_0) \cup [\xi_1, 1 + \xi_1] \cup [\xi_2, 1 + \xi_2] \quad \text{and} \quad \sigma_{disc}(V) = \{ \omega_0 \}, \]

where \( \omega_0 = \xi_1 + \xi_2 \) is a simple eigenvalue of the operator \( V \).

**Corollary 1.** Let \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). Then:

\[ E_{\min}(V) = \begin{cases} 0, & \text{if } \alpha + \gamma \leq 1 \text{ and } \beta + \gamma \leq 1, \\ \xi_1, & \text{if } \alpha + \gamma > 1 \text{ and } \beta + \gamma \leq 1, \\ \xi_2, & \text{if } \alpha + \gamma \leq 1 \text{ and } \beta + \gamma \geq 1, \\ \min\{\xi_1, \xi_2\}, & \text{if } \alpha + \gamma > 1 \text{ and } \beta + \gamma > 1. \end{cases} \]

### 3. Discrete spectrum of partial integral operators

Let’s define the multiplier \( H_0 \):

\[ H_0 f(x, y) = x^\alpha y^\beta f(x, y), \quad \alpha > 0, \quad \beta > 0, \]

and the operators \( T_1, T_2 \):

\[ T_1 f(x, y) = \int_0^1 f(s, y) ds, \quad T_2 f(x, y) = \int_0^1 f(x, t) dt. \]

Let us define a self-conjugate PIO \( H \):

\[ H = H_0 - \gamma(T_1 + T_2), \quad \gamma > 0. \]

We have \( \sigma(H_0) = [0, 1] \). For each \( \lambda \in \mathbb{R} \setminus [0, 1] \) define the function \( \Delta_1(y; \lambda) \) on \([0, 1]\) \((\Delta_2(x; \lambda) \text{ on } [0, 1]) \) by formula:

\[ \Delta_1(y; \lambda) = 1 - \gamma \int_0^1 \frac{ds}{x^\alpha y^\beta - \lambda}, \quad \Delta_2(x; \lambda) = 1 - \gamma \int_0^1 \frac{ds}{x^\alpha s^\beta - \lambda}. \]

In the space \( L_2[0, 1] \) we define the family \( \{H_1(t)\}_{t \in [0, 1]} \) of the self-adjoint operators in the Friedrichs’ model:

\[ H_1(t) \varphi(x) = t^\beta x^\alpha \varphi(x) - \gamma \int_0^1 \varphi(s) ds. \]

Similarly, in the space \( L_2[0, 1] \) we define the family \( \{H_2(t)\}_{t \in [0, 1]} \):

\[ H_2(t) \psi(y) = t^\alpha y^\beta \psi(y) - \gamma \int_0^1 \psi(s) ds. \]

**Lemma 3.** Function:

\[ \pi_j(t) = \inf_{\|\varphi\|=1} (H_j(t) \varphi, \varphi), \quad t \in [0, 1] \quad (j = 1, 2) \quad (3) \]

is non-positive, continuous and increasing on \([0, 1]\).
Proof. In work [9], there is a proof of the continuity and non-positivity of the function \( \pi_j(t) \) on \([0, 1]\). We will show the monotonicity of the function \( \pi_j(t) \) on \([0, 1]\). We define the family of the \( \{H_0(t)\}_{t \in [0, 1]} \) multipliers:

\[
H_0(t) \varphi(x) = x^\alpha t^\beta \varphi(x), \quad \varphi \in L_2[0, 1].
\]

Then it follows from \( t_1 \leq t_2, \ t_1, t_2 \in [0, 1] \) that:

\[
H_0(t_1) \leq H_0(t_2).
\]

Therefore, we have:

\[
\pi_1(t_1) = \inf_{\|\varphi\|=1} (H_1(t_1) \varphi, \varphi) = \inf_{\|\varphi\|=1} [(H_0(t_1) \varphi, \varphi) - \gamma(K_1 \varphi, \varphi)] \leq
\]

\[
\inf_{\|\varphi\|=1} (H_1(t_2) \varphi, \varphi) = \inf_{\|\varphi\|=1} [(H_0(t_2) \varphi, \varphi) - \gamma(K_1 \varphi, \varphi)] = \pi_1(t_2),
\]

Where:

\[
K_1 \varphi(x) = \int_0^1 \varphi(s)ds.
\]

This means that the function \( \pi_1(t) \) is increasing on the set \([0, 1]\).

Obviously, for each \( y \in [0, 1] \) the function \( \Delta_1(\lambda) = \Delta_1(y; \lambda) \) is strictly decreasing on \((\infty, 0)\). Therefore, for each \( y \in [0, 1] \) there exists finite or infinite limit \( \lim_{\lambda \to 0^-} \Delta_1(y; \lambda) \). Moreover, there is:

**Lemma 4.** a) if \( 0 < \alpha < 1 \), then for each \( y \in (0, 1] \):

\[
\lim_{\lambda \to 0^-} \Delta_1(y; \lambda) = 1 - \frac{\gamma}{1 - \alpha} \cdot \frac{1}{y^\beta};
\]

b) if \( \alpha \geq 1 \), then for each \( y \in (0, 1] \):

\[
\lim_{\lambda \to 0^-} \Delta_1(y; \lambda) = -\infty.
\]

**Proof.** a) Let \( 0 < \alpha < 1 \). Then, for \( y \in (0, 1] \) we get

\[
\frac{1}{s^\alpha y^\beta - \lambda} \leq h_0(s, y) = \frac{1}{s^\alpha y^\beta}
\]

and for any ascending sequence \( \{\lambda_n\} \) negative numbers decreasing to zero we have:

\[
\frac{1}{s^\alpha y^\beta - \lambda_n} \leq \frac{1}{s^\alpha y^\beta - \lambda_{n+1}}, \quad n \in \mathbb{N}.
\]

On the other hand, for each \( y \in (0, 1] \) there exists a Lebesgue integral from function \( h_1(s, y) \) on \( s \in (0, 1] \):

\[
\int_0^1 h_0(s, y)ds = \frac{1}{1 - \alpha} \cdot \frac{1}{y^\beta},
\]

Then, by Lebesgue’s theorem on the limited transition under the sign of the Lebesgue integral, we obtain:

\[
\lim_{\lambda \to 0^-} \Delta_1(y; \lambda) = 1 - \frac{\gamma}{1 - \alpha} \cdot \frac{1}{y^\beta}, \quad y \in (0, 1];
\]

b) Let \( \alpha \geq 1 \) and assume that \( \alpha = 1 \). It is obvious that for \( y = 0 \) we have: \( \lim_{\lambda \to 0^-} \Delta_1(y; \lambda) = -\infty \). For each \( y \in (0, 1] \) we have:

\[
\int_0^1 \frac{ds}{s^\alpha y^\beta - \lambda} = \int_0^1 \frac{ds}{sy^\beta - \lambda} = \frac{1}{y^\beta} \ln \left(1 - \frac{y^\beta}{\lambda} \right).
\]

Therefore for \( y \in (0, 1] \) we get:

\[
\lim_{\lambda \to 0^-} \Delta_1(y; \lambda) = 1 - \frac{\gamma}{y^\beta} \lim_{\lambda \to 0^-} \ln \left(1 - \frac{y^\beta}{\lambda} \right) = -\infty;
\]

Suppose that \( \alpha > 1 \). Then from inequality:
we get that
\[
\lim_{\lambda \to 0-} \int_0^1 \frac{ds}{s^\alpha y^\beta - \lambda}, \quad y \in [0, 1]
\]
and accordingly, \( \lim_{\lambda \to 0-} \Delta_1(y; \lambda) = -\infty. \)

Obviously, the function:
\[
h_1(y) = \lim_{\lambda \to 0-} \Delta_1(y; \lambda) = 1 - \frac{\gamma}{1 - \alpha} \cdot \frac{1}{y^\beta}
\]
increases by \((0, 1]\) from \(-\infty\) to \(h_1^{\max} = h_1(1) = 1 - \frac{\gamma}{1 - \alpha}.\)

We put:
\[
\pi_j^{\max} = \max_{t \in [0, 1]} \pi_j(t), \quad j = 1, 2.
\]

Then \(\pi_j^{\max} = \pi_j(1).\)

**Lemma 5.** Let \(0 < \alpha < 1 (0 < \beta < 1).\) Then:

a) if \(\gamma + \alpha \leq 1 (\gamma + \beta \leq 1),\) then \(\pi_1^{\max} = 0 (\pi_2^{\max} = 0);\)

b) if \(\gamma + \alpha > 1 (\gamma + \beta > 1),\) then \(\pi_1^{\max} < 0 (\pi_2^{\max} < 0).\)

**Proof.** Let \(0 < \alpha < 1.\) a) Assume that: \(\gamma + \alpha = 1.\) We have:
\[
h_1^{\max} = h_1(1) = \lim_{\lambda \to 0-} \Delta_1(1; \lambda) = 1 - \frac{\gamma}{1 - \alpha} = 0.
\]

Hence, taking into account the monotonicity of the function \(\Delta_1(1; \lambda)\) by \(\lambda < 0\) we get that \(\Delta_1(1; \lambda) > 0\) for any \(\lambda < 0.\) Then, according to Proposition 1, the operator \(H_1(1)\) has no eigenvalue below the bottom edge the essential spectrum of the operator \(H_1(1).\) By the minmax principle and from equality (3) we obtain that \(\pi_1^{\max} = \pi_1(1) = E_{\min}(H_1(1)) = 0.\)

If \(\gamma + \alpha < 1.\) Then \(h_1^{\max} = h_1(1) > 0.\) On the other hand \(\Delta_1(1; \lambda) > h_1^{\max}.\) Then according to the proposition 1, the operator \(H_1(1)\) has no negative eigenvalue value. It follows that \(\pi_1^{\max} = 0.\)

b) Let \(\gamma + \alpha > 1.\) Then:
\[
h_1(y) \leq h_1^{\max} = 1 - \frac{\gamma}{1 - \alpha} < 0.
\]

Therefore, for each \(y \in (0, 1]\) we have \(h_1(y) = \lim_{\lambda \to 0-} \Delta_1(y; \lambda) < 0.\) Hence, since the function \(\Delta_1(y; \lambda)\) is monotonic with respect to \(\lambda < 0\) implies the existence of a unique number \(\lambda_0 = \lambda_0(y) < 0\) (for each \(y \in (0, 1]\)) such that \(\Delta_1(y; \lambda_0(y)) = 0.\) For \(y = 0\) we have \(\Delta_1(0; -\gamma) = 0, i.e. \lambda_0 = \lambda_0(0) = -\gamma\) is a solution to the equation \(\Delta_1(0; \lambda) = 0.\) Due to minmax principle [13] solution of \(\lambda_0(y)\) equation \(\Delta_1(y; \lambda) = 0\) is defined using continuous function \(\pi_1(t), i.e. \lambda_0(t) = \pi_1(t), t \in [0, 1].\) However \(\lambda_0(y) < 0, y \in [0, 1].\) Therefore \(\pi_1(1) = \pi_1^{\max} < 0.\)

**Lemma 6.** Let \(\alpha \geq 1 (\beta \geq 1).\) Then \(\pi_1^{\max} < 0 (\pi_2^{\max} < 0).\)

**Proof.** For \(y = 1\) we get:
\[
\Delta_1(\lambda) = \Delta_1(1; \lambda) = 1 - \int_0^1 \frac{ds}{s^\alpha - \lambda}.
\]

We have:
\[
\lim_{\lambda \to -\infty} \Delta_1(\lambda) = 1 \quad \text{and} \quad \lim_{\lambda \to 0-} \Delta_1(\lambda) = -\infty.
\]

Then, due to the monotonicity of the function \(\Delta_1(\lambda)\) for \(\lambda < 0\) we obtain the existence of unique number \(\lambda_0 < 0,\) such as \(\Delta_1(\lambda_0) = 0.\) Therefore, \(\lambda_0 = \pi_1^{\max} < 0.\)

By the theorem 3.3 [14] and Lemma 3.4 or the essential spectrum operator \(H\) we obtain the following statement.

**Theorem 4.** For the essential spectrum of the operator \(H\) there is a place for equality:
\[
\sigma_{\text{ess}}(H) = [-\gamma, \gamma_0] \cup [0, 1],
\]
where \(\gamma_0 = \max\{\pi_1^{\max}, \pi_2^{\max}\}.\)

From the positivity of the operators \(H_0, T_1\) and \(T_2\) for the operator \(H\) we have:
\[
\sigma(H) \subset (-\infty, 1],
\]
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i.e. above the upper edge of the essential spectrum \( \sigma_{\text{ess}}(H) \) of the operator \( H \) eigenvalues are missing. Then, by the theorem 4 the discrete spectrum of the operator \( H \) lies in the set of negative numbers.

We put:

\[
\xi_0 = \frac{1}{(1 + \alpha)(1 + \beta)}.
\]

**Theorem 5.** If \( \gamma > \xi_0 \), then the operator \( H \) has the negative eigenvalue, lying to the left of the bottom edge of the essential spectrum.

**Proof.** Assume the conditions \( \gamma > \xi_0 \). Put \( f_0(x, y) = 1 \). Then \( \|f_0\| = 1 \) and

\[
(Hf_0, f_0) = (H_0f_0, f_0) - \gamma((T_1f_0, f_0) + (T_2f_0, f_0)) = \xi_0 - 2\gamma.
\]

Then, by the theorem 4 we have \( E_{\text{min}}(H) = -\gamma \) and from \( \gamma > \xi_0 \) we get that

\[
\lambda_0 = (Hf_0, f_0) < -\gamma = E_{\text{min}}(H).
\]

Hereof and according to the minmax principle we get that \( \lambda_0 \in \sigma_{\text{dis}}(H) \), i.e. \( \lambda_0 = \xi_0 - 2\gamma \) – is the eigenvalue of the operator \( H \).

**Corollary 2.** Number of eigenvalues of the operator \( H \) is at most one and for \( \gamma > \xi_0 \) the discrete spectrum of the operator \( H \) is not empty.

**References**


