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Green's function method for time-fractional diffusion equation on the star graph with equal bonds

Z. A. Sobirov[1,2], K. U. Rakhimov[2], R. E. Ergashov[2]

[1]University of Geological Sciences, Olimlar str., 49, 100041, Tashkent, Uzbekistan

[2]National University of Uzbekistan, Universitet str., 4, 100174, Tashkent, Uzbekistan

z.sobirov@nuu.uz, kamoliddin_ru@inbox.ru

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This work devoted to construction of the matrix-Green's functions of initial-boundary value problems for the time-fractional diffusion equation on the metric star graph with equal bonds. In the case of Dirichlet and mixed boundary conditions we constructed Green's functions explicitly. The uniqueness of the solutions of the considered problems were proved by the method of energy integrals. Some possible applications in branched nanostructures were discussed.

Keywords: Time-fractional diffusion equation, IBVP, PDE on metric graphs, Green's function.

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1. Introduction

In this work we consider initial-boundary value problem (IBVP) for time fractional diffusion equation

$$D_{0,t}^{\alpha}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - f(x,t),$$

on the bounded star graph. Here $D_{0,t}^{\alpha}g(t)$ is the fractional derivative defined by:

$$D_{\eta,t}^{\alpha}g(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_{\eta}^{t}\frac{g(\xi)}{|t-\xi|^{\alpha}}d\xi, \quad 0<\alpha<1,$$

where $\Gamma(x)$ is the Gamma function.

It is known that the diffusion equation is widely used in many fields of science including physics, biology, mechanics, chemistry and others. In recent years, the theory of fractional calculus has been studied with great interest. In particular, in the work [1] Bartlomiej Dybiec and Ewa Gudowska-Nowak considered a long-time, scaling limit for the anomalous diffusion composed of the subordinated Levy-Wiener process. In [2] T. A. M. Langlands and B. I. Henry introduced mesoscopic and macroscopic model equations of chemotaxis with anomalous subdiffusion for modeling chemically directed transport of biological organisms. In [3], the fractional Fokker-Planck equation for subdiffusion in a general space and time-dependent force field was studied. In [4], the authors solved an asympotic boundary value problem without initial conditions for diffusion-wave equation with time-fractional derivative and gave some applications in fractional electrodynamics. Igor Goychuk in [5], considered an alternative continuous time random walk and fractional Fokker-Planck equation description.

Naturally, such a powerful tool as a fractional derivative cannot stay away from nanoscience. In particular, this can be seen from the results in [6,7]. In [6] authors obtained solutions for the magnetohydrodynamic mixed convection problem of Maxwell fractional nanofluid. In [7] authors studied models with non-integer order derivatives to describe dynamics in nanofluids. In particular, it is noticed that the rate of heat transfer increases with increasing nanoparticle volume fraction and order of the time-fractional derivative.

It is known that the Green's function method is a powerful technique for solving boundary value problems. The Green's function method for fractional order equations was investigated by A. V. Pskhu in [8]. In [9] Green's function method was used to find numerical solution of boundary value problems for stationary and non-stationary differential equations in different dimensions.

Nowadays, differential equations are being investigated with great interest on the metric graphs. It is mostly due to the fact, that majority of physical systems demonstrate flows, e.g., flows through the branched nanotubes [10]. The Schrödinger equation on the metric graph are well studied (see [11–13] and references in them). In [14] the Schrödinger operator on the graph with varying edges was investigated. The Schrödinger equation on the metric graphs was also explored with Fokas unified transformation method in [15]. In [11] it was considered as scattering problem on the quantum graph, which consists ring with two attached semi-infinite leads. In [16] authors proposed the model of time-dependent geometric graph for description of the dynamical Casimir effect. In [17], the authors used metric graph approximation for investigation of strong variation of the viscosity and density in cylindrical domains of the small radius. And paper [18] devoted to an explicitly solvable model for periodic chain of coupled disks. In [19–21] IBVP for time-fractional Airy equation on the star graphs are considered via method of potentials.

2. Formulation of the problems

In this paper we construct Green's function of two IBVPs on the metric star graph for the time-fractional subdiffusion equation. Motivation for the study of the fractional diffusion equations on metric graphs comes from such practically important problems as anomalous heat transport in mesoscopic networks, subdiffusion processes in nanoscale network structures, molecular wires, different lattices and discrete structures. In such discrete structures one has memory effect which leads, in particular, to time fractional equations. These problems, in particular, describe transport of nanofluids in branched structures [6, 7].

We consider the star graph which has $m = k + l$ bonds connected in one point O (Fig. 1). We define coordinates in the graph's edges by isometric mapping this bonds to the interval from 0 to L, such that in each edge the coordinate 0 corresponds to the vertex O. The bonds of the graph are denoted by $B_j, j = \overline{1,m}$.

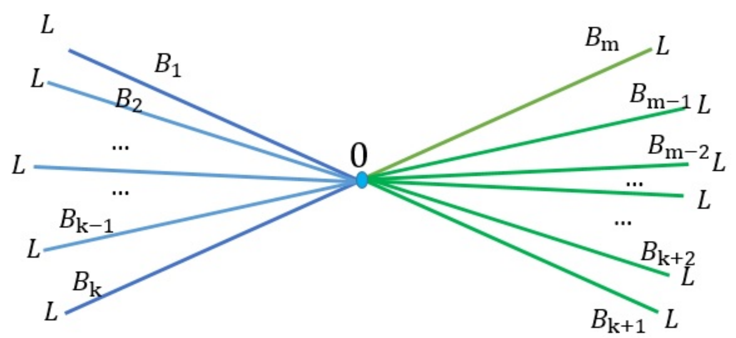

FIG. 1. Star graph with m bonds. We considered two cases of the boundary condition (BC): (a) Dirichlet BC at the all boundary vertices; (b) Dirichlet BC at the end points of the blue bonds and Neumann BC at the end points of the green bonds

On each bond B_j of the graph, we consider the time-fractional subdiffusion equation

$$D_{0,t}^{\alpha} u_j(x,t) = \frac{\partial^2}{\partial x^2} u_j(x,t) - f_j(x,t), \quad 0 < x < L,\ 0 < t < T,\ \ j = \overline{1,m}, \tag{1}$$

with the following initial conditions

$$\lim_{t\to 0} D_{0,t}^{\alpha-1} u_j(x,t) = \varphi_j(x), \quad 0 \le x \le L, \quad j = \overline{1,m}. \tag{2}$$

At the inner vertex of the graph, we use the following gluing (Kirchhoff) conditions

$$u_1(0,t) = u_2(0,t) = ... = u_m(0,t), \tag{3}$$

$$\lim_{x\to 0} \left(\sum_{i=1}^{m} \frac{\partial}{\partial x} u_i(x,t) \right) = 0, \tag{4}$$

for all $t \in [0,T]$. These conditions ensure the local flow conservation at the branching point of the graph.

At the boundary vertices, we will use Dirichlet or Neumann boundary conditions (BC) given by

$$u_i(L,t) = \psi_i(t), \qquad i = \overline{1,m}, \tag{5}$$

$$u_i(L,t) = \gamma_i(t), \quad i = \overline{1,k}, \tag{6}$$

$$\frac{\partial}{\partial x} u_j(L,t) = \gamma_j(t), \quad j = \overline{k+1,m}. \tag{7}$$

We suppose that the functions $f_j(x,t)$, $j = \overline{1,m}$, initial and boundary data are smooth enough on the closure of their domains and the compatibility conditions on the boundary and branching points hold.

Problem 1. *Find regular solutions of equations (1) which satisfy conditions (2) — (5).*
Problem 2. *Find regular solutions of equations (1) which satisfy conditions (2) — (4), (6), (7).*

3. Uniqueness of solution

We put $u = (u_1, u_2, ..., u_m)^T$, $f = (f_1, ..., f_m)^T$, $\varphi = (\varphi_1, ..., \varphi_m)^T$. We define the norm of a vector function $v(x) = (v_1(x), ..., v_m(x))$ by $\|v\|^2 = \sum_{i=1}^{m} \int_0^L v_i^2(x)dx$.

$$D_{0,t}^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(\xi)}{|t-\xi|^{1-\alpha}} d\xi, \quad 0 < \alpha < 1,$$

is the time-fractional integral [22].

Lemma 1. *The problem 1 has at most one solution. Furthermore, if $\psi_i(t) \equiv 0, i = \overline{1,m}$, then the solutions satisfy the following a-priori estimate*

$$||D_{0,t}^{\alpha-1} u||^2 \leq E_\alpha(t^\alpha) \cdot ||\varphi|| + \Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) D_{0,t}^{-\alpha} ||D_{0,t}^{\alpha-1} f||^2, \tag{8}$$

where $E_\alpha(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\alpha n + 1)$ and $E_{\alpha,\mu}(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\alpha n + \mu)$ are the Mittag-Leffler functions.

Proof. Let $\psi_i(t) \equiv 0, i = \overline{1,m}$. We put $w(x,t) = D_{0,t}^{\alpha-1} u(x,t)$. It is clear, that the vector function $w(x,t) = (w_1(x,t), w_2(x,t), ..., w_m(x,t))^T$ satisfies the following equation:

$$\partial_{0,t}^{\alpha-1} w = w_{xx} + D_{0,t}^{\alpha-1} f(x,t), 0 < x < L, 0 < t < T, \tag{9}$$

with Caputo time-fractional derivative:

$$\partial_{\eta,t}^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_\eta^t \frac{g'(\xi)}{|t-\xi|^{\alpha}} d\xi, \quad 0 < \alpha < 1.$$

From [22], we have:

$$\int_0^L w_j D_{0,t}^{\alpha} w_j dx \geq \frac{1}{2} D_{0,t}^{\alpha} \int_0^L w_j^2 dx, \ j = 1, 2, ..., m. \tag{10}$$

Taking into account inequality (10), we multiply (9) by w^T from the left side, integrate the resulting relation with respect to x from 0 to L and use Cauchy's inequality:

$$\partial_{0,t}^{\alpha} ||w||^2 + 2||w_x||^2 \leq ||w||^2 + ||D_{0,t}^{\alpha-1} f||^2.$$

Hence,

$$D_{0,t}^{\alpha} ||w||^2 \leq ||w||^2 + ||D_{0,t}^{\alpha-1} f||^2. \tag{11}$$

From the analog of Gronwall-Bellman lemma (see [22]), in the case of Caputo type fractional derivative and the relation (11), we get:

$$\|w(\cdot,t)\|^2 \leq E_\alpha(t^\alpha) \|w(\cdot,0)\|^2 \ + \Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) D^{-\alpha} \|D_{0,t}^{\alpha-1} f(\cdot,t)\|.$$

So, we proved a-priory estimate (8). Uniqueness of the solution follows from this estimate.

Lemma 2. *The problem 2 has at most one solution. Furthermore, if $\gamma_i(t) \equiv 0, i = \overline{1,m}$, then the solutions satisfy the following a-priori estimate*

$$||D_{0,t}^{\alpha-1} u||^2 \leq E_\alpha(t^\alpha) \cdot ||\varphi|| + \Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) D_{0,t}^{-\alpha} ||D_{0,t}^{\alpha-1} f||^2, \tag{12}$$

Proof. The proof is similar with the proof of the Lemma 1.

4. Green's function of Problem 1

The following theorem directly follows from the theorem **4.3.1.** (see [8]). Let $E = \{(x,t)\colon 0 < x < L, 0 < t < T\}$ (where T is positive real number) and $E_t = \{(\xi,\tau)\colon 0 < \xi < L, 0 < \tau < t\}$.

Theorem 1. Let the $m \times m$ matrix function $V = V(x,t;\xi,\tau)$ satisfy following conditions:

1. The matrix function V is the solution for equation

$$V_{\xi\xi}(x,t;\xi,\tau) - D^{\alpha}_{t,\tau}V(x,t;\xi,\tau) = 0$$

on the fixed point $(x,t) \in E$;

2. For every $m \times 1$ vector function $h(x) \in C[0,L]$ holds

$$\lim_{\tau\to t}\int_0^L D^{\alpha-1}_{t,\tau}V(x,t;\xi,\tau)h(\xi)d\xi = h(x). \tag{13}$$

3. The matrix functions $V, V_\xi, V_{\xi\xi}, D^{\alpha-1}_{0,t}V$ are continuous on the $\overline{E}\times\overline{E_t}\backslash\{t=0\}$ and for all points $(x,t)\in E$ and $(\xi,\tau)\in E_t$ holds inequality

$$|V_{ij}(x,t;\xi,\tau)| < (t-\tau)^{\alpha/2-1}.$$

If the function $u(x,t)$ is the solution of the equation (1) and satisfies the condition (2), then for all (x,t) is hold, that:

$$u(x,t) = \int_0^t (V(x,t;L,\tau)u_\xi(L,\tau) - V(x,t;0,\tau)u_\xi(0,\tau) - V_\xi(x,t;L,\tau)u(L,\tau) +$$
$$V_\xi(x,t;0,\tau)u(0,\tau))d\tau - \int_0^L V(x,t;\xi,0)\varphi(\xi)d\xi + \int_0^t\int_0^L V(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau.$$

Proof. The proof is similar to the proof of the theorem **4.3.1.** [8].

Taking into account the above theorem, we look for solution of **Problem 1** in the following form

$$u(x,t) = \int_0^t (G(x,t;L,\tau)u_\xi(L,\tau) - G(x,t;0,\tau)u_\xi(0,\tau) - G_\xi(x,t;L,\tau)u(L,\tau) +$$
$$G_\xi(x,t;0,\tau)u(0,\tau))d\tau - \int_0^L \varphi(t)G(x,t;\xi,\tau)d\xi - \int_0^t\int_0^L G(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau, \tag{14}$$

where:

$$G = \begin{pmatrix} G^{11} & G^{12} & \dots & G^{1m} \\ G^{21} & G^{22} & \dots & G^{2m} \\ \dots & \dots & \dots & \dots \\ G^{m1} & G^{m2} & \dots & G^{mm} \end{pmatrix}$$

is a matrix-Green's function. Green's function satisfies the equation:

$$G_{\xi\xi} - D^{\alpha}_{t,\tau}G = 0,$$

for all $\xi \neq x, 0 < \tau < t$.

We look for Green's function in the following form

$$G = \sum_{n=-\infty}^{+\infty}\big(A_n\Gamma(x-\xi+2nL,t-\tau) + B_n\Gamma(x+\xi+2nL,t-\tau)\big), \tag{15}$$

where A_n and B_n are constant matrices of dimension $m\times m$ and $\Gamma(s,t)$ is:

$$\Gamma(s,t) = \frac{1}{2}t^{\alpha/2-1}e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|s|}{t^{\alpha/2}}\right).$$

We have to find the matrices A_n and B_n.

Taking into account (14) and the conditions (2)—(4) we get the following conditions for matrix-Green's function:

$$G|_{\xi=L} = 0, \tag{16}$$

$$G^{i1}|_{\xi=0} = G^{i2}|_{\xi=0} = ... = G^{im}|_{\xi=0}, \quad i = \overline{1,m}, \tag{17}$$

$$G^{i1}_{\xi}|_{\xi=0} + G^{i2}_{\xi}|_{\xi=0} + ... + G^{in}_{\xi}|_{\xi=0} = 0, \quad i = \overline{1,m}. \tag{18}$$

Let

$$A_n = \begin{pmatrix} A_n^{11} & A_n^{12} & ... & A_n^{1m} \\ A_n^{21} & A_n^{22} & ... & A_n^{2m} \\ ... & ... & ... & ... \\ A_n^{m1} & A_n^{m2} & ... & A_n^{mm} \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} B_n^{11} & B_n^{12} & ... & B_n^{1m} \\ B_n^{21} & B_n^{22} & ... & B_n^{2m} \\ ... & ... & ... & ... \\ B_n^{m1} & B_n^{m2} & ... & B_n^{mm} \end{pmatrix}.$$

From (16), we have:

$$A_n = -B_{n-1}. \tag{19}$$

From (17), we get:

$$A_n^{i1} - A_n^{im} = B_n^{im} - B_n^{i1}. \tag{20}$$

From (18), we obtain:

$$\sum_{j=1}^{m} A_n^{ij} = \sum_{j=1}^{m} B_n^{ij}, \quad i = \overline{1,m}. \tag{21}$$

We rewrite the relations (20) and (21) in the following matrix form $B_n = A_n M$, where:

$$M = \frac{1}{m} \begin{pmatrix} 2-m & 2 & ... & 2 \\ 2 & 2-m & ... & 2 \\ ... & ... & ... & ... \\ 2 & 2 & ... & 2-m \end{pmatrix}. \tag{22}$$

Combining (19), (20) and (21), we get:

$$\begin{cases} B_n = A_n M \\ A_{n+1} = -B_n \end{cases}.$$

And we find

$$A_{n+1} = -A_n M.$$

From the condition (13) follows that $A_0 = I$. So, we have

$$A_n = (-1)^n M^n, \quad B_n = (-1)^n M^{n+1}. \tag{23}$$

Substituting (23) into (15) we get matrix Green's function of the **Problem 1**:

$$G = \sum_{n=-\infty}^{+\infty} (-1)^n M^n \left(\Gamma(x - \xi + 2nL, t - \tau) + M\Gamma(x + \xi + 2nL, t - \tau) \right). \tag{24}$$

Taking into account **Lemma 1** we get following theorem.

Theorem 2. Let $\phi_i(t), \varphi_i(t) \in C[0,T]$ $(i = \overline{1,m}, T > 0)$, and $f(x,t) \in C^{0,1}\{(x,t) : 0 \le x \le L, 0 < t < T\}$. Then the **Problem 1** has unique solution in the form of:

$$u(x,t) = -\int_0^t G_\xi(x,t;L,\tau)u(L,\tau)d\tau - \int_0^L \varphi(\xi)G(x,t;\xi,\tau)d\xi - \int_0^t \int_0^L G(x,t;\xi,\tau)f(\xi,\tau)d\xi d\tau,$$

where $G(x,t;\xi,\tau)$ is given by (24).

5. Green's function of Problem 2

In this case, the Green's function has a form of:

$$G_1 = \sum_{n=-\infty}^{+\infty} \big(C_n\Gamma(x-\xi+2nL,t-\tau) + D_n\Gamma(x+\xi+2nL,t-\tau)\big), \tag{25}$$

where C_n and D_n are unknown constant matrices of dimension $m\times m$. We have to find the matrices C_n and D_n.

Taking into account (13) and conditions (2)—(4), (6), (7), we get following conditions for Green's function:

$$\begin{cases} G_1^{ij}|_{\xi=L} = 0, & i=\overline{1,k},\quad j=\overline{1,m}, \\ G_{1\xi}^{ij}|_{\xi=L} = 0, & i=\overline{k+1,m},\quad j=\overline{1,m} \end{cases}, \tag{26}$$

$$G_1^{i1}|_{\xi=0} = G_1^{i2}|_{\xi=0} = \ldots = G_1^{in}|_{\xi=0}, \quad i=\overline{1,m}, \tag{27}$$

$$G_{1\xi}^{i1}|_{\xi=0} + G_{1\xi}^{i2}|_{\xi=0} + \ldots + G_{1\xi}^{im}|_{\xi=0}, \quad i=\overline{1,m}. \tag{28}$$

Let

$$C_n = \begin{pmatrix} C_n^{11} & C_n^{12} & \ldots & C_n^{1m} \\ C_n^{21} & C_n^{22} & \ldots & C_n^{2m} \\ \ldots & \ldots & \ldots & \ldots \\ C_n^{m1} & C_n^{m2} & \ldots & C_n^{mm} \end{pmatrix} \quad \text{and} \quad D_n = \begin{pmatrix} D_n^{11} & D_i^{12} & \ldots & D_n^{1m} \\ D_n^{21} & D_i^{22} & \ldots & D_n^{2m} \\ \ldots & \ldots & \ldots & \ldots \\ D_n^{m1} & D_n^{m2} & \ldots & D_n^{mm} \end{pmatrix}.$$

Combining (26), (27) and (28), respectively, we get:

$$\begin{cases} C_n^{ij} = -D_{n-1}^{ij} \quad \text{for} \quad i=\overline{1,k}, j=\overline{1,m}, \\ C_n^{ij} = D_{n-1}^{ij} \quad \text{for} \quad i=\overline{k+1,m}, j=\overline{1,m}. \end{cases} \tag{29}$$

$$C_n^{i1} - C_n^{im} = D_n^{im} - D_n^{i1}, \quad i=\overline{1,m}. \tag{30}$$

$$\sum_{j=1}^{m} C_n^{ij} = \sum_{j=1}^{m} D_n^{ij}, \quad i=\overline{1,m}. \tag{31}$$

And we rewrite the relations (30) and (31) on the following matrix form:

$$C_n \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ -1 & 0 & 0 & \ldots & 0 & 1 \\ 0 & -1 & 0 & \ldots & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix} = D_n \begin{pmatrix} -1 & -1 & -1 & \ldots & -1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix}.$$

From this relation, we find $D_n = C_n M$ where: M given at the form (22). We find

$$C_{n+1} = \Phi D_n,$$

where

$$\Phi = \begin{pmatrix} -I_k & 0 \\ 0 & I_l \end{pmatrix}.$$

From the condition (13), we have that $C_0 = I$. (29) can be written as:

$$C_n = \Phi^n M^n, D_n = \Phi^n M^{n+1}, \tag{32}$$

where:

$$\Phi^n = \begin{pmatrix} (-1)^n I_k & 0 \\ 0 & I_l \end{pmatrix}.$$

Substituting (32) into (25) we get matrix Green's function of the **Problem 2**:

$$G_1 = \sum_{n=-\infty}^{+\infty} \Phi^n M^n \big(\Gamma(x-\xi+2nL,t-\tau) + M\Gamma(x+\xi+2nL,t-\tau)\big), \tag{33}$$

As a result, we get following theorem.

Theorem 3. Let $\beta_i(x) \in C[0,L], \gamma_i(t) \in C[0,T]$ and $f_i(x,t) \in C^{0,1}\{(x,t) : 0 \le x \le L, 0 < t < T,\}(i = \overline{1,m}, T > 0)$. Then the **Problem 2** has a unique solution in the form of:

$$u(x,t) = \int_0^t \left(G_1^{(N)}(x,t;L,\tau) \frac{\partial u_N(\xi,\tau)}{\partial \xi}\bigg|_{\xi=L} - G_{1\xi}^{(D)}(x,L;t,\tau) u_D(L,\tau) \right) d\tau -$$

$$\int_0^L \varphi(\xi) G_1(x,t;\xi,\tau) d\xi - \int_0^t \int_0^L G_1(x,t;\xi,\tau) f(\xi,\tau) d\xi d\tau,$$

where

$$G_1^{(D)} = \begin{pmatrix} G_1^{11} & G_1^{12} & ... & G_1^{1m} \\ ... & ... & ... & ... \\ G_1^{k1} & G_1^{k2} & ... & G_1^{km} \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}, \quad G_1^{(N)} = \begin{pmatrix} 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \\ G_1^{k+1,1} & G_1^{k+1,2} & ... & G_1^{k+1,m} \\ ... & ... & ... & ... \\ G_1^{m1} & G_1^{m2} & ... & G_1^{mm} \end{pmatrix},$$

$u_D = (u_1, ..., u_k, 0, ..., 0)^T$, $u_N = (0, ..., 0, u_{k+1}, ..., u_m)^T$ and G_1 is on the form (33).

Conclusion

In this paper, we gave Green's function approach for IBVP to time-fractional diffusion equation with Neumann and Dirichlet boundary conditions. We constructed Green's functions of the considered problems in the form of matrix series. We notice that Green's functions for IBVP for time-fractional diffusive equation on metric graphs were firstly constructed in the present paper. Green's function on the case of line-interval for different IBVPs are constructed in [4].

It is well-known that Green's functions on metric graphs are closely related to scattering problem at the branching points [23]. In our case, the component $G_{ij}, (i \ne j)$ describes, for example, heat flow conduction from i-th bond to j-th bond, while G_{ii} describes heat flow reflection (thermal reflection) on i-th bond. Therefore, one can conclude that Green's functions constructed in the present paper can be considered as a powerful tool to investigate the conductivity (scattering) properties in sub-diffusive processes in nano-sized thin branched structures.

References

[1] Dybiec B., Gudowska-Nowak E. Anomalous diffusion and generalized Sparre-Andersen scaling. *Europhysics Letters Association*, 2009, **88**, P. 10003.

[2] Langlands T.A.M., Henry B.I. Fractional Chemotaxis Diffusion Equations. *Physics Review E*, 2010, **81**(5), P. 051102.

[3] Henry B.I., Langlands T.A.M., Starka P. Fractional Fokker-Planck Equations for Subdiffusion with Space-and-Time-Dependent Forces. *Physical Review Letters*, 2010, **105**, P. 170602.

[4] Pskhu A., Rekhviashvili S. Fractional Diffusion-Wave Equation with Application in Electrodynamics. *Mathematics*, 2020, **8**(2086), P. 1–13.

[5] Goychuk I. Fractional time random walk subdiffusion and anomalous transport with finite mean residence times: faster, not slower. *Physics Review E*, 2012, **86**(2), P. 021113.

[6] Aman S., Khan I., Ismail Z., Salleh M.Z., Tlili I. A new Caputo time fractional model for heat transfer enhancement of water based graphene nanofluid: An application to solar energy. *Results in Physics*, 2018, **9**, P. 1352–1362.

[7] Aman S., Khan I., Ismail Z., Salleh M.Z. Applications of fractional derivatives to nanofluids: exact and numerical solutions. *Mathematical Modelling of Natural Phenomena*, 2018, **13**(1), P. 1–12.

[8] Pskhu A.V. *Fractional partial differential equations*. Moscow, Nauka, 2005, 200 p.

[9] Mehdi D., Mohammad D., Mohsen J. Introduction to Green's Function and its Numerical Solution. *Middle-East Journal of Scientific Research*, 2012, **11**(7), P. 974–981.

[10] Chivilikhin S.A., Gusarov V.V., Popov I.Y. Charge pumping in nanotube filled with electrolyte. *Chinese Journal of Physics*, 2018, **56**(5), P. 2531–2537.

[11] Gerasimov D., Popov I., Blinova I., Popov A. Incompleteness of resonance states for quantum ring with two semi-infinite edges. *Analysis and Mathematical Physics*, 2019, **9**, P. 1287–1302.

[12] Kottos T., Smilansky U. Periodic Orbit Theory and Spectral Statistics for Quantum Graphs. *Annals of Physics*, 1999, **274**(1), P. 76–124.

[13] Gnutzmann S., Smilansky U. Quantum graphs: Applications to quantum chaos and universal spectral statistics. *Advances in Physics*, 2006, **55**(5-6), P. 527–625.

[14] Nikiforov D.S., Blinova I.V., Popov I.Y. Schrödinger and Dirac dynamics on time-dependent quantum graph. *Indian Journal of Physics*, 2019, **93**(7), P. 913–920.

[15] Khudayberganov G., Sobirov Z., Eshimbetov M. Unified transform method for the Schrodinger equation on a simple metric graph. *Journal of Siberian Federal University, Mathematics and Physics*, 2019, **12**(4), P. 412–420.

[16] Lobanov I.S., Nikiforov D.S., Popov I.Y., Trifanov A.I., Trifanova E.S. Model of time-dependent geometric graph for dynamical Casimir effect. *Indian Journal of Physics*, 2020, DOI: 10.1007/s12648-020-01866-5.

[17] Smolkina M.O., Popov I.Y., Blinova I.V., Milakis E. On the metric graph model for flows in tubular nanostructures. *Nanosystems: Physics, Chemistry, Mathematics*, 2019, **10**(1), P. 6–11.

[18] Grishanov E.N., Eremin D. A., Ivanov D.A., Popov I.Yu., Smirnov P.I. Periodic chain of disks in a magnetic field: bulk states and edge states. *Nanosystems: Physics, Chemistry, Mathematics*, 2015, **6**(5), P. 637–643.

[19] Sobirov Z.A., Rakhimov K.U. Cauchy problem for the Airy equation with fractional time-fractional on a star-shaped graph. *Institute of Mathematics Bulletin, Uzbekistan*, 2019, **5**, P. 40–49.

[20] Rakhimov K.U. The method of potentials for the Airy equation of fractional order. *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences*, 2020, **3**(2), P. 222–235.

[21] Rakhimov K.U., Sobirov Z.A. Jabborov N.M. The time-fractional Airy equation on the metric graph. *Journal of Siberian Federal University, Mathematics and Physics*, 2021, **14**(3), P. 376–388.

[22] Alikhanov A.A. A Priori Estimates for Solutions of Boundary Value Problems for Fractional-Order Equations. *Differential equations*, 2010, **46**(5), P. 658–664.

[23] Andrade F.M., Schmidt A.G.M., Vicentini E., Cheng B.K., M.G.E. da Luz. Green's function approach for quantum graphs: An overview. *Physics Reports*, 2016, **647**, P. 1–46.