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Comments on the Chernoff estimate

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ABSTRACT The Chernoff $\sqrt{n}$-Lemma is revised. This concerns two aspects: a re-examination of the Chernoff estimate in the strong operator topology and the operator-norm estimate for quasi-sectorial contractions. Applications to the Lie-Trotter product formula approximation C_0-semigroups are also discussed.

KEYWORDS Chernoff lemma, Semigroup theory, Product formula, Convergence rate.

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1. Introduction

Recall that the Chernoff $\sqrt{n}$-Lemma [3, Lemma 2] is one of a key tool in the theory of semigroup approximations, see e.g. [4, Chapter III, Section 5]. For the reader's convenience and for motivation of the present comments, we show this lemma below.

Lemma 1.1. *Let bounded operator C on a Banach space $\mathfrak{X}$ ($C \in \mathcal{L}(\mathfrak{X})$) be a contraction, i.e., $\|C\| \leq 1$. Then, $\{e^{t(C-\mathbb{1})}\}_{t\geq 0}$ is a norm-continuous contraction semigroup on $\mathfrak{X}$ and one has the estimate:*

$$\|(C^n - e^{n(C-\mathbb{1})})x\| \leq \sqrt{n}\,\|(C-\mathbb{1})x\| \ , \tag{1.1}$$

for all $x \in \mathfrak{X}$ and $n \in \mathbb{N}$.

Proof. To prove the inequality (1.1) we use the representation:

$$C^n - e^{n(C-\mathbb{1})} = e^{-n}\sum_{m=0}^{\infty}\frac{n^m}{m!}(C^n - C^m) \ . \tag{1.2}$$

To proceed we insert:

$$\|(C^n - C^m)x\| \leq \left\|(C^{|n-m|} - \mathbb{1})x\right\| \leq |m-n|\|(C-\mathbb{1})x\| \ , \tag{1.3}$$

into (1.2) to obtain by the Cauchy-Schwarz inequality the estimate:

$$\|(C^n - e^{n(C-\mathbb{1})})x\| \leq \|(C-\mathbb{1})x\|\, e^{-n}\sum_{m=0}^{\infty}\frac{n^m}{m!}|m-n| \leq \{\sum_{m=0}^{\infty} e^{-n}\frac{n^m}{m!}|m-n|^2\}^{1/2}\|(C-\mathbb{1})x\| \ . \ x \in \mathfrak{X} \ , \tag{1.4}$$

Note that the sum in the right-hand side of (1.4) can be calculated explicitly. This gives the value n, which yields (1.1). □

The aim of the present comments is to revise the Chernoff $\sqrt{n}$-Lemma in two directions. First, we modify the $\sqrt{n}$-estimate (1.1) for contractions. Then, we apply two new estimates for the proof of the Chernoff product formula for *strongly continuous* semigroups (C_0-semigroups) in the *strong* operator topology, see Section 2 and Section 3.

Second, we use the idea of the *probabilistic approach* to the estimate in strong operator topology (Section 2) to uplift it to the *operator-norm* estimate for a special class of contractions: the *quasi-sectorial* contractions, see Section 4.

2. Revised $\sqrt{n}$-Lemma and Chernoff product formula

We start by a technical lemma. It is a *revised* version of the Chernoff $\sqrt{n}$-Lemma 1.1. Our *variational* estimate (2.1) in $\sqrt[3]{n}$-Lemma 2.1 and the *probabilistic approach* are, in a certain sense, more flexible than (1.1). Indeed, the scheme of the proof will be used later (Section 4) for uplifting the convergence of the Chernoff and the Lie-Trotter product formulae to the *operator-norm* topology.

Lemma 2.1. ($\sqrt[3]{n}$ -Lemma) *Let C be a contraction on a Banach space $\mathfrak{X}$. Then, $\{e^{t(C-\mathbb{1}}\}_{t\geq 0}$ is a norm-continuous contraction semigroup on $\mathfrak{X}$ and one has the estimate:*

$$\|(C^n - e^{n\,(C-\mathbb{1})})\,x\| \leq \frac{n}{\epsilon_n^2}\,2\,\|x\| + \epsilon_n\,\|(\mathbb{1}-C)\,x\,\|\,, \qquad n \in \mathbb{N}\setminus\{0\}\,, \tag{2.1}$$

for all $x \in \mathfrak{X}$ and $\epsilon_n > 0$. For the optimal value of the parameter ϵ_n :

$$\epsilon_n^* := \left(\frac{4\,n\,\|x\|}{\|(\mathbb{1}-C)\,x\,\|}\right)^{1/3}, \tag{2.2}$$

on the right-hand side of (2.1)*, we obtain the estimate:*

$$\|(C^n - e^{n\,(C-\mathbb{1})})\,\| \leq \frac{3}{2}\,\sqrt[3]{n}\,\|2\,(\mathbb{1}-C)\,\|^{2/3}\,, \tag{2.3}$$

which is the $\sqrt[3]{n}$ -Lemma.

Proof. Since operator C is bounded and $\|C\| \leq 1$, the operator $(\mathbb{1}-C)$ is the generator of a norm-continuous contraction semigroup:

$$\|\,e^{\,t(C-\mathbb{1})}\,\| \leq e^{-t}\left\|\sum_{m=0}^{\infty}\frac{t^m}{m!}C^m\right\| \leq 1\,. \tag{2.4}$$

In order to prove estimate (2.1), we use the representation:

$$C^n - e^{n(C-\mathbb{1})} = e^{-n}\sum_{m=0}^{\infty}\frac{n^m}{m!}(C^n - C^m)\,. \tag{2.5}$$

Then, we *split* the sum (2.5) into two parts: the *central* part for $|m-n| \leq \epsilon_n$ and the *tails* for $|m-n| > \epsilon_n$. Optimisation of the *splitting* parameter ϵ_n in (2.1) yields the best estimate and thus the optimal value of $\delta \in \mathbb{R}$.

For evaluation of the *tails*, we use the *Tchebychëv inequality*. Let $X_n \in \mathbb{N}_0$ be the *Poisson random variable* with the *rate* parameter n, that is, with the probability distribution $\mathbb{P}\{X_n = m\} = n^m e^{-n}/m!$. Then, one gets for the expectation: $\mathbb{E}(X_n) = n$, and for the variance: $\mathrm{Var}(X_n) := \mathbb{E}((X_n - \mathbb{E}(X_n))^2) = n$. That being so, the Tchebychëv inequality yields

$$\mathbb{P}\{|X_n - \mathbb{E}(X_n)| > \epsilon\} \leq \frac{\mathrm{Var}(X_n)}{\epsilon_n^2}\,, \ \text{ for any } \ \epsilon_n > 0. \tag{2.6}$$

Note that although for any $x \in \mathfrak{X}$ there is an evident bound: $\|(C^n - C^m)\,x\| \leq 2\,\|x\|$, for estimating (2.5) we shall also use below inequalities:

$$\begin{aligned}\|(C^{\,n} - C^{\,m})\,x\| &= \|C^{\,n-k}(C^k - C^{\,m-n+k})\,x\| \\ &\leq |m-n|\;\|C^{\,n-k}(\mathbb{1}-C)\,x\|, \quad k = 0,1,\ldots,n\,.\end{aligned} \tag{2.7}$$

Then by $\|C\| \leq 1$ and by the Tchebychëv inequality (2.6) we obtain the estimate for *tails*:

$$\begin{aligned}&e^{-n}\sum_{|m-n|>\epsilon_n}\frac{n^m}{m!}\|(C^n - C^m)\,x\| \leq e^{-n}\sum_{|m-n|>\epsilon_n}\frac{n^m}{m!}\cdot\,2\,\|x\| \\ &= \mathbb{P}\{|X_n - \mathbb{E}(X_n)| > \epsilon_n\}\cdot\,2\,\|x\| \leq \frac{n}{\epsilon_n^2}\,2\,\|x\|\,.\end{aligned} \tag{2.8}$$

To evaluate the *central* part of the sum (2.5), when $|m-n| \leq \epsilon_n$, note that by virtue of (2.7):

$$\begin{aligned}\|(C^n - C^m)x\| &\leq |m-n|\;\|C^{n-[\epsilon_n]}\,(\mathbb{1}-C)x\| \\ &\leq \epsilon_n\,\|(\mathbb{1}-C)\,x\|.\end{aligned} \tag{2.9}$$

Then we obtain:

$$e^{-n}\sum_{|m-n|\leq\epsilon_n}\frac{n^m}{m!}\,\|(C^n - C^m)\,x\| \leq \epsilon_n\,\|(\mathbb{1}-C)\,x\|\,, \quad x \in \mathfrak{X}\,, \tag{2.10}$$

for $n \in \mathbb{N}\setminus\{0\}$. Estimate (2.10), together with (2.8), yield (2.1) for all $u \in \mathfrak{X}$ and $\epsilon_n > 0$.

Minimising the estimate (2.1) with respect to parameter $\epsilon_n > 0$ one obtains the optimal value for ϵ_n^* (2.2) and

$$\frac{n}{{\epsilon_n^*}^2}\, 2\,\|x\| + \epsilon_n^*\, \|(\mathbb{1} - C)\,x\,\| = \frac{3}{2}\,\sqrt[3]{n}\,(4\,\|x\|)^{1/3}\,\|(\mathbb{1} - C)\,x\,\|^{2/3}\,, \tag{2.11}$$

for all $x \in \mathfrak{X}$ and $n \in \mathbb{N} \setminus \{0\}$. As a consequence, (2.1) and (2.11) yield (2.3), which is the $\sqrt[3]{n}$ -Lemma. □

Theorem 2.2. (Chernoff product formula) *Let $\Phi : t \mapsto \Phi(t)$ be a function from $\mathbb{R}_0^+$ to contractions on $\mathfrak{X}$ such that $\Phi(0) = \mathbb{1}$. Let $\{U_A(t)\}_{t\geq 0}$ be a contraction C_0-semigroup, and let $D \subset \operatorname{dom}(A)$ be a core of the generator A.*

If the function $\Phi(t)$ has a strong right-derivative $\Phi'(+0)$ at $t = 0$ (that is, $\Phi'(+0)x$ exists for any $x \in \operatorname{dom}(\Phi'(+0))$) and if

$$\Phi'(+0)\,x := \lim_{t\to+0} \frac{1}{t}(\Phi(t) - \mathbb{1})\,x = -\,A\,x\,,$$

for all $x \in D$, then

$$\lim_{n\to\infty} [\Phi(t/n)]^n\, x = U_A(t)\,x\,, \tag{2.12}$$

for all $t \in \mathbb{R}_0^+$ and $x \in \mathfrak{X}$.

Proof. Consider the bounded approximations $\{A_n(s)\}_{n\geq 1}$ of generator A:

$$A_n(s) := \frac{\mathbb{1} - \Phi(s/n)}{s/n}\,, \quad s \in \mathbb{R}^+\,, \quad n \in \mathbb{N}\,. \tag{2.13}$$

Note that these operators are m-accretive: $\|(A_n(s) + \zeta\mathbb{1})^{-1}\| \leq (\operatorname{Re}(\zeta))^{-1}$ for $\operatorname{Re}(\zeta) > 0$ and for any $n \in \mathbb{N}$. By $\|\Phi(t)\| \leq 1$ together with (2.13) we obtain $\|e^{-t\,A_n(s)}\| \leq 1$, but also

$$\lim_{n\to\infty} A_n(s)\,x = A\,x\,, \tag{2.14}$$

for all $x \in D$ and any $s \in \mathbb{R}^+$. Then, given that $D = \operatorname{core}(A)$, by virtue of the *Trotter-Neveu-Kato* generalised strong convergence theorem (see, e.g., [5, Theorem 3.17] or [4, Chapter III, Theorem 4.8]) one obtains

$$\lim_{n\to\infty} e^{-t\,A_n(s)}\,x = U_A(t)\,x\,, \quad x \in \mathfrak{X}\,, \quad t \in \mathbb{R}_0^+\,. \tag{2.15}$$

This is the strong and uniform in t and in s convergence (2.15) of contractive approximants $\{e^{-t\,A_n(s)}\}_{n\geq 1}$ for $t \in [0, \tau]$ and $s \in (0, s_0]$.

Now, by Lemma 2.1 for contraction $C := \Phi(t/n)$ we obtain owing to (2.3) that:

$$\begin{aligned} \|[\Phi(t/n)]^n\,x - e^{-t\,A_n(t)}\,x\| &= \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbb{1})})\,x\| \\ &\leq \frac{3}{2}\,\sqrt[3]{n}\,(4\,\|x\|)^{1/3}\,\|(\mathbb{1} - \Phi(t/n))\,x\,\|^{2/3}\,, \quad x \in \mathfrak{X}\,. \end{aligned} \tag{2.16}$$

Since by (2.14) one gets for any $x \in D$ and uniformly on $(0, t_0]$:

$$\lim_{n\to\infty} \sqrt[3]{n}\,\|\,(\mathbb{1} - \Phi(t/n))\,x\,\|^{2/3} = \lim_{n\to\infty} t^{2/3}\,n^{-1/3}\,\|A_n(t)\,x\|^{2/3} = 0\,, \tag{2.17}$$

equations (2.16) and (2.17) provide uniformly on $(0, t_0]$:

$$\lim_{n\to\infty} \|\,[\Phi(t/n)]^n\,x - e^{-t\,A_n(t)}\,x\| = 0, \quad x \in D\,. \tag{2.18}$$

Then, (2.15) and (2.18) yield uniformly in $t \in [0, t_0]$ limit:

$$\lim_{n\to\infty} [\Phi(t/n)]^n\,x = U_A(t)\,x\,, \quad x \in D\,. \tag{2.19}$$

Note that by *density* of D and by the *uniform* estimate $\|\,[\Phi(t/n)]^n\,x - e^{-t\,A_n(t)}x\| \leq 2\,\|x\|$ the convergence in (2.18) can be extended to all $x \in \mathfrak{X}$. Indeed, it is known that on the *bounded* subsets of $\mathcal{L}(\mathfrak{X})$ the topology of *point-wise* convergence on a *dense* subset $D \subset \mathfrak{X}$ coincides with the *strong* operator topology, see, e.g., [6, Chapter III, Lemma 3.5]. As a consequence, limit (2.18) being extended to $x \in \mathfrak{X}$ and limit (2.15) yield (2.12). □

The limit (2.12) is called the *Chernoff product formula* in the *strong* operator topology for contractive C_0-semigroup $\{U_A(t)\}_{t\geq 0}$.

Proposition 2.3. [3] (Lie-Trotter product formula) *Let A, B and C be generators of contraction C_0-semigroups on $\mathfrak{X}$. Suppose that algebraic sum:*

$$Cx = Ax + Bx\,, \tag{2.20}$$

is valid for all $x \in D$, where domain $D = \operatorname{core}(C)$. Then, the semigroup $\{U_C(t)\}_{t\geq 0}$ can be approximated on $\mathfrak{X}$ in the strong operator topology by the Lie-Trotter product formula:

$$e^{-tC}\,x = \lim_{n\to\infty} (e^{-tA/n}e^{-tB/n})^n\,x\,, \quad x \in \mathfrak{X}\,, \tag{2.21}$$

for all $t \in \mathbb{R}_0^+$ and $C := \overline{(A+B)}$, which is closure of the sum (2.20).

Proof. Let us define the contraction $\mathbb{R}_0^+ \ni t \mapsto \Phi(t)$, $\Phi(0) = \mathbb{1}$, by:

$$\Phi(t) := e^{-tA} e^{-tB} \ . \tag{2.22}$$

Note that if $x \in D$, then derivative

$$\Phi'(+0)x = \lim_{t \to +0} \frac{1}{t} (\Phi(t) - \mathbb{1})\, x = -(A+B)\, x \ . \tag{2.23}$$

Now, we are in position to apply Theorem 2.2. This yields (2.21) for $C := \overline{(A+B)}$. □

Corollary 2.4. *Extension of the strongly convergent Lie-Trotter product formula of Proposition 2.3 to quasi-bounded and holomorphic semigroups follows through verbatim.*

3. Revision of the Chernoff estimate

In this section, we show a one more Chernoff-type estimate (3.1), which is of a different nature than the variational estimate (2.1) ($\sqrt[3]{n}$-Lemma 2.1). In fact, it is a kind of *improvement* of the original Chernoff estimate (1.1) ($\sqrt{n}$-Lemma 1.1).

Lemma 3.1. *Let $C \in \mathcal{L}(\mathfrak{X})$ be contraction on a Banach space $\mathfrak{X}$. Then $\{e^{t(C-\mathbb{1}}\}_{t\geq 0}$ is a norm-continuous contraction semigroup on $\mathfrak{X}$ and the following estimate:*

$$\|(C^n - e^{n(C-\mathbb{1})})\, x\| \leq \ \frac{n}{2}\, (\|(C-\mathbb{1})^2\, x\| + \frac{e^2}{3}\, \|(C-\mathbb{1})^3\, x\|) \ , \tag{3.1}$$

holds for all $n \in \mathbb{N}$ and $x \in \mathfrak{X}$.

Proof. The first assertion is proven in Lemma 2.1, see (2.4).

To prove inequality (3.1) we use the *telescopic* representation:

$$C^n - e^{n(C-\mathbb{1})} = \sum_{k=0}^{n-1} C^{n-k-1}\, (C - e^{(C-\mathbb{1})})\, e^{k(C-\mathbb{1})} \ . \tag{3.2}$$

To proceed we exploit that operator $C \in \mathcal{L}(\mathfrak{X})$ is bounded and therefore:

$$C - e^{(C-\mathbb{1})} = -\,\frac{1}{2}\, (\mathbb{1} - C)^2 - (\mathbb{1} - C)^3 \sum_{m=3}^{\infty} \frac{(-1)^m}{m!}\, (\mathbb{1} - C)^{m-3} \ , \tag{3.3}$$

Owing to $\|C\| \leq 1$ one obtains the estimate:

$$\| \sum_{m=3}^{\infty} \frac{1}{m!}\, (\mathbb{1} - C)^{m-3} \| \leq \frac{1}{6}\, e^{\|\mathbb{1}-C\|} \leq \frac{e^2}{6} \, . \tag{3.4}$$

Then on account of (3.2) - (3.4) and (2.4) we obtain inequality (3.1). □

Corollary 3.2. (Chernoff product formula) *Let $\Phi : t \mapsto \Phi(t)$ be a function from $\mathbb{R}_0^+$ to contractions on $\mathfrak{X}$ such that $\Phi(0) = \mathbb{1}$, which satisfies conditions of Theorem* 2.2. *Then*

$$\lim_{n\to\infty} \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbb{1})})\, x\| = 0 \, , \quad x \in \mathfrak{X} \, , \tag{3.5}$$

and one gets the product formula (2.12).

Proof. On account of (3.1) we obtain estimate

$$\begin{aligned} &\|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbb{1})})\, x\| \leq \\ &\frac{t^2}{2n} \left(\left\| \frac{n^2}{t^2}\, (\mathbb{1} - \Phi(t/n))^2\, x \right\| + \frac{e^2}{3}\, \frac{t}{n} \left\| \frac{n^3}{t^3} (\mathbb{1} - \Phi(t/n))^3\, x \right\| \right) , \quad x \in \mathfrak{X} \, . \end{aligned} \tag{3.6}$$

Note that by (2.14) we have on the dense set $D = \operatorname{core}(A)$ for any $t \in \mathbb{R}^+$:

$$\lim_{n\to\infty} \frac{n}{t}\, (\mathbb{1} - \Phi(t/n))\, x = A\, x \ , \quad x \in D \, . \tag{3.7}$$

Given that generator A of contractive C_0-semigroup is *accretive*, for $\operatorname{Re}(\zeta) > 0$ the range of resolvent: $\operatorname{ran}((A + \zeta\mathbb{1})^{-1}) = \mathfrak{X}$. As a consequence (cf. [6, Chapter III, Problem 2.9]), domains $\operatorname{dom}(A^2) \supset \operatorname{dom}(A^3)$ are *dense* in $\mathfrak{X}$ and limit (3.7) provides:

$$\lim_{n\to\infty} (A_n(t))^2\, x = A^2\, x \ , \quad \lim_{n\to\infty} (A_n(t))^3\, x = A^3\, x \ , \quad x \in D \subset \operatorname{dom}(A^3), \tag{3.8}$$

where $A_n(t) = (t/n)^{-1}\, (\mathbb{1} - \Phi(t/n))$.

By virtue of estimate (3.6) and (3.8), we obtain:

$$\lim_{n\to\infty} \|([\Phi(t/n)]^n - e^{n(\Phi(t/n)-\mathbb{1})})\, x\| = 0\,, \quad x \in D\,. \tag{3.9}$$

Then similarly to concluding arguments in Theorem 2.2 (that on the *bounded* subsets of $\mathcal{L}(\mathfrak{X})$ the topology of *point-wise* convergence on a *dense* subset $D \subset \mathfrak{X}$ coincides with the *strong* operator topology) the limit (3.9) can be extended to $x \in \mathfrak{X}$.

Now, given that $D = \mathrm{core}(A)$, by virtue of the *Trotter-Neveu-Kato* theorem we obtain the limit (2.15), and owing to (3.9) for $x \in \mathfrak{X}$, we deduce *Chernoff product formula* (2.12). □

Resuming the Chernoff $\sqrt{n}$-inequality: (1.1), and its variety: (2.1) and (3.1), we conclude that due to the terms with $\|(C-\mathbb{1})\, x\|$ all of them control only the *strong* convergence of the product formulae. The *rates*: $R_n(t)$, of these converges *conditioned* to $x \in D$ have the following asymptotic form for $t > 0$ and large $n \in \mathbb{N}$:
(a) For (1.1): $R_n(t) = 1/\sqrt{n}\, \|A_n(t)\, x\|$.
(b) For (2.1): $R_n(t) = 1/\sqrt[3]{n}\, \|A_n(t)\, x\|^{2/3}$.
(c) For (3.1): $R_n(t) = 1/n\, \|A_n(t)^2\, x\|$.

Remark 3.3. None of these three methods has an evident straightforward extension that could ensure the *operator-norm* convergence of the Chernoff product formula. In the next Section 4 we show that a relatively sophisticated method (cf.(b)) based on the Tchebychëv inequality (Section 2) is *a fortiori* more accurate to allow *uplifting* the convergence of the Chernoff product formula to the operator-norm topology for *quasi-sectorial* contractions on a Hilbert space.

4. Quasi-sectorial contractions and $(\sqrt[3]{n})^{-1}$-Theorem

Definition 4.1. [7] A contraction C on the Hilbert space $\mathfrak{H}$ is called *quasi-sectorial* with semi-angle $\alpha \in [0, \pi/2)$ with respect to the vertex at $z = 1$, if its numerical range $W(C) \subseteq D_\alpha$. Here

$$D_\alpha := \{z \in \mathbb{C} : |z| \le \sin\alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \le \alpha \text{ and } |z-1| \le \cos\alpha\}. \tag{4.1}$$

We comment that $D_{\alpha=\pi/2} = \mathbb{D}$ (unit disc) and recall that a *genera*l contraction C satisfies condition: $W(C) \subseteq \mathbb{D}$.

Note that if operator C is a quasi-sectorial contraction, then $\mathbb{1} - C$ is an m-sectorial operator with vertex $z = 0$ and semi-angle α. Then for C the limits: $\alpha = 0$ and $\alpha = \pi/2$, correspond respectively to self-adjoint and to standard contractions whereas for $\mathbb{1} - C$ they give a non-negative self-adjoint and an m-accretive (bounded) operators.

For $\lambda > 0$ the resolvent $(A + \lambda\,\mathbb{1})^{-1}$ of an m-sectorial operator A, with semi-angle $\alpha \in [0, \alpha_0]$, $\alpha_0 < \pi/2$, and vertex at $z = 0$, gives an example of the quasi-sectorial contraction.

Proposition 4.2. [7,8] *If C is a quasi-sectorial contraction on a Hilbert space $\mathfrak{H}$ with semi-angle $0 \le \alpha < \pi/2$, then*

$$\|C^n(\mathbb{1} - C)\| \le \frac{K_\alpha}{n+1}\,,\ n \in \mathbb{N}\,. \tag{4.2}$$

The property (4.2) implies that the quasi-sectorial contractions belong to the class of so-called *Ritt's* operators [9]. This allows one to go beyond the $\sqrt[3]{n}$ -Lemma 2.1 to the $(\sqrt[3]{n})^{-1}$-Theorem and from estimates in the strong operator topology to the operator-norm topology.

Theorem 4.3. ($(\sqrt[3]{n})^{-1}$-Theorem) *Let C be a quasi-sectorial contraction on $\mathfrak{H}$ with numerical range $W(C) \subseteq D_\alpha$, $0 \le \alpha < \pi/2$. Then*

$$\left\|C^n - e^{n(C-\mathbb{1})}\right\| \le \frac{M_\alpha}{n^{1/3}}\,,\ \ n \in \mathbb{N}\,, \tag{4.3}$$

where $M_\alpha = 2K_\alpha + 2$ and K_α is defined by (4.2).

Proof. With help of inequality (4.2) we can improve the estimate of the *central* part of the sum (2.5) in Lemma 2.1. Note that on account of (2.7) we obtain by (4.2) and $\|C\| \le 1$:

$$\|C^n - C^m\| \le |m-n|\, \|C^{n-[\epsilon_n]}(\mathbb{1} - C)\| \le \epsilon_n\, \frac{K_\alpha}{n - [\epsilon_n] + 1}\,, \tag{4.4}$$

cf. (2.9). Here $\epsilon_n := n^{\delta+1/2}$ for $\delta < 1/2$, which makes sense for the estimate (2.8) of *tails*, and $[\epsilon_n]$ is the *integer* part of $\epsilon_n \ge |m-n|$. Then owing to (4.4) the *central* part has the estimate:

$$e^{-n} \sum_{|m-n|\le\epsilon_n} \frac{n^m}{m!}\, \|(C^n - C^m)\, x\| \le \epsilon_n\, \frac{K_\alpha}{n - [\epsilon_n] + 1}\, \|x\|\,, \quad x \in \mathfrak{X}\,, \quad n \in \mathbb{N}\,. \tag{4.5}$$

As a consequence, (2.8) and (4.5) yield instead of (2.3) (or (1.1)) the *operator-norm* estimate:

$$\left\|C^n - e^{n(C-\mathbb{1})}\right\| \le \frac{2}{n^{2\delta}} + \epsilon_n\, \frac{K_\alpha}{n - [\epsilon_n] + 1}\ \,,\ n \in \mathbb{N}\,. \tag{4.6}$$

Let $n_0 \in \mathbb{N}$ satisfies inequality: $2\,(\epsilon_{n_0} - 1) \leq n_0$. Then (4.6) gives

$$\left\| C^n - e^{n(C-\mathbb{1})} \right\| \leq \frac{2}{n^{2\delta}} + \frac{2K_\alpha}{n^{1/2-\delta}} \ , \ \ n > n_0 \ . \tag{4.7}$$

The estimate $M_\alpha/n^{1/3}$ of the Theorem 4.3 results from the *optimal* choice of the value: $\delta = 1/6$, in (4.7). □

Similar to $(\sqrt[3]{n})$-Lemma, the $(\sqrt[3]{n})^{-1}$-Theorem is the first step in developing the *operator-norm* approximation formula à la Chernoff. To this end one needs an operator-norm analogue of Theorem 2.2. Since the last includes the Trotter-Neveu-Kato strong convergence theorem, we need the *operator-norm* extension of this assertion for quasi-sectorial contractions.

Proposition 4.4. [7] *Let $\{X(s)\}_{s>0}$ be a family of m-sectorial operators in a Hilbert space $\mathfrak{H}$ such that for some $0 < \alpha < \pi/2$ and any $s > 0$ the numerical range $W(X(s)) \subseteq S_\alpha$. Let X_0 be an m-sectorial operator defined in a closed subspace $\mathfrak{H}_0 \subseteq \mathfrak{H}$, with $W(X_0) \subseteq S_\alpha$. Then the two following assertions are equivalent:*

$$(a) \qquad \lim_{s\to+0} \left\|(\zeta\mathbb{1} + X(s))^{-1} - (\zeta\mathbb{1} + X_0)^{-1}P_0\right\| = 0 \ , \ \ \textit{for} \ \ \zeta \in S_{\pi-\alpha} \ ,$$

$$(b) \qquad \lim_{s\to+0} \left\| e^{-tX(s)} - e^{-tX_0}P_0 \right\| = 0 \ , \ \ \textit{for} \ \ t > 0 \ .$$

Here P_0 denotes the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_0$ and $S_\alpha = \{z \in \mathbb{C} : |\arg(z)| \leq \alpha\}$ is a sector in complex plane $\mathbb{C}$ with semi-angle α and vertex at $z = 0$.

Now $(\sqrt[3]{n})^{-1}$-Theorem 4.3 and Proposition 4.4 yield a desired generalisation of the operator-norm approximation formula:

Proposition 4.5. [7] *Let $\{\Phi(s)\}_{s\geq 0}$ be a family of uniformly quasi-sectorial contractions on a Hilbert space $\mathfrak{H}$, i.e. such that there exists $0 \leq \alpha < \pi/2$ and $W(\Phi(s)) \subseteq D_\alpha$, for all $s \geq 0$. Let*

$$X(s) := (\mathbb{1} - \Phi(s))/s \ , \tag{4.8}$$

and let X_0 be a closed operator with non-empty resolvent set, defined in a subspace $\mathfrak{H}_0 \subseteq \mathfrak{H}$. Then, the family $\{X(s)\}_{s>0}$ converges, when $s \to +0$, in the uniform resolvent sense to the operator X_0 if and only if

$$\lim_{n\to\infty} \left\|\Phi(t/n)^n - e^{-tX_0}P_0\right\| = 0 \ , \quad \text{for} \ \ t > 0 \ . \tag{4.9}$$

Here, P_0 denotes the orthogonal projection onto the subspace $\mathfrak{H}_0$.

Let A be an m-sectorial operator with semi-angle $0 < \alpha < \pi/2$ and with vertex at $z = 0$, which means that numerical range $W(A) \subseteq S_\alpha = \{z \in \mathbb{C} : |\arg(z)| \leq \alpha\}$. Then, $\{\Phi(t) := (\mathbb{1} + tA)^{-1}\}_{t\geq 0}$ is the family of quasi-sectorial contractions, i.e., $W(\Phi(t)) \subseteq D_\alpha$. Let $X(s) := (\mathbb{1} - \Phi(s))/s$, $s > 0$, and $X_0 := A$. Then, $X(s)$ converges when $s \to +0$, to X_0 in the uniform resolvent sense with the asymptotic

$$\|(\zeta\mathbb{1} + X(s))^{-1} - (\zeta\mathbb{1} + X_0)^{-1}\| = s\left\|\frac{A}{\zeta\mathbb{1} + A + \zeta sA} \cdot \frac{A}{\zeta\mathbb{1} + A}\right\| = O(s),$$

for any $\zeta \in S_{\pi-\alpha}$, since we have the estimate:

$$\left\|\frac{A}{\zeta\mathbb{1} + A + \zeta sA} \cdot \frac{A}{\zeta\mathbb{1} + A}\right\| \leq \left(1 + \frac{|\zeta|}{\operatorname{dist}\left(\zeta(1+s\zeta)^{-1}, -S_\alpha\right)}\right)\left(1 + \frac{|\zeta|}{\operatorname{dist}(\zeta, -S_\alpha)}\right) \ .$$

Therefore, the family $\{\Phi(t)\}_{t\geq 0}$ satisfies the conditions of Proposition 4.5. This implies the operator-norm approximation of the exponential function , i.e. the semigroup for m-sectorial generator, by the powers of resolvent (*the Euler approximation formula*):

Corollary 4.6. [8,10] *If A is an m-sectorial operator in a Hilbert space $\mathfrak{H}$, with semi-angle $\alpha \in (0, \pi/2)$ and with vertex at 0, then*

$$\left\|(\mathbb{1} + tA/n)^{-n} - e^{-tA}\right\| \leq \frac{L_\alpha}{n} \ , \quad t \in S_{\pi/2-\alpha} \ , \tag{4.10}$$

for $n \in \mathbb{N}$.

5. Conclusion

Summarising we note that for the quasi-sectorial contractions instead of *divergent* (for $n \to \infty$) Chernoff's estimate (1.1), we find the estimate (4.7) which converges for $n \to \infty$ to zero in the *operator-norm* topology. Note that the rate $O(1/n^{1/3})$ of this convergence is obtained with help of the Poisson representation and the Tchebychëv inequality in the spirit of the proof of Lemma 2.1, and which is not optimal.

The estimate $M/n^{1/3}$ in the $(\sqrt[3]{n})^{-1}$-Theorem 4.3 can be improved by a more refined lines of reasoning. For example, by scrutinising our probabilistic arguments one can find a more precise Tchebychëv-type bound for probability of *tails*. This improves the estimate (4.7) to the rate $O(\sqrt{\ln(n)/n})$, see [11], but again only for *quasi-sectorial* contractions providing due to Proposition 4.2 the *operator-norm* contrôl (4.5) of the *central* part.

On the other hand, a careful analysis of localisation the *numerical range* of quasi-sectorial contractions [8, 10], generated in a Hilbert space $\mathfrak{H}$ by m-sectorial operators with semi-angle $\alpha \in (0, \pi/2)$, permits one to uplift the operator-norm estimate in Corollary 4.6 to the ultimate optimal α-dependent rate $O(1/n)$, [10, Theorem 4.1].

We note that with help of the *spectral representation*, one can easily obtain in (4.7) the optimal rate $O(1/n)$ of the operator-norm convergence for *self-adjoint* contractions C. This is a particular case of the quasi-sectorial contraction for $\alpha = 0$, cf. [7, Remark 3.2]. This also concerns the optimal rate of convergence $O(1/n)$ for the self-adjoint Euler approximation formula (4.10) for $A = A^* \geq 0$, which is m(sectorial operator for $\alpha = 0$.

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