

## On the construction of de Branges spaces for dynamical systems associated with finite Jacobi matrices

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**ABSTRACT** We consider dynamical systems with boundary control associated with finite Jacobi matrices. Using the method previously developed by the authors, we associate with these systems special Hilbert spaces of analytic functions (de Branges spaces).

**KEYWORDS** Boundary control method, Krein equations, Jacobi matrices, de Branges spaces

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### 1. Introduction

For a given sequence of positive numbers  $\{a_0, a_1, \dots, a_{N-1}\}$  (in what follows we assume  $a_0 = 1$ ) and real numbers  $\{b_1, b_2, \dots, b_N\}$ , we denote by  $A$  the finite Jacobi matrix given by:

$$A = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_{N-2} & b_{N-1} & a_{N-1} \\ \dots & \dots & \dots & \dots & a_{N-1} & b_N \end{pmatrix}. \quad (1)$$

Let  $u = (u_1, \dots, u_N) \in \mathbb{R}^N$  and  $T > 0$  be fixed. With the matrix  $A$  we associate the dynamical system:

$$\begin{cases} u_{tt}(t) - Au(t) = F(t), & t > 0, \\ u(0) = u_t(0) = 0, \end{cases} \quad (2)$$

where the vector function  $F(t) = (f(t), 0, \dots, 0)$ ,  $f \in L_2(0, T)$  is interpreted as a *boundary control*. The solution of (2) is denoted by  $u^f$ . With the system (2), we associate the *response operator* acting by the rule:

$$(R^T f)(t) = u_1^f(t), \quad 0 < t < T. \quad (3)$$

The forward and inverse problems for the system (2) and for the special case of this system, the finite Krein-Stieltjes string, were the subjects of [1, 2], where, as a main tool, we used the Boundary control method [3, 4]. In this paper, we would like to demonstrate one more application of the Boundary control method, namely the construction of the de Branges space associated with (2).

De Branges spaces play an important role in the inverse spectral theory of first order canonical systems, see for example [5–7]. In [8, 9], the authors shows how to use the Boundary control method to associate de Branges spaces with different dynamical systems. Note that our approach differs from the classical one and potentially admits theallows generalization to multidimensional systems. The algorithm proposed in [8, 9] is as follows: fixing some finite time  $t = T$ , one denotes by  $\mathcal{F}^T$  the set of controls acting on the time interval  $(0, T)$  and introduces the *reachable set* of the dynamical system at this time:

$$U^T := \{u^f(T) \mid f \in \mathcal{F}^T\}.$$

Then, one applies the Fourier transform  $\mathbb{F}$  associated with the operator  $A$  to elements from  $U^T$  and get a linear manifold  $\mathbb{F}U^T$ . Then, this linear manifold is equipped with the norm defined by the *connecting operator*, which resulted in the de

Branges space associated with the initial dynamical system. In the models considered in [8,9], models the dynamical systems have different properties in with respect to the *boundary controllability*: the system associated with the Schrödinger operator is exactly controllable from the boundary; the system associated with the one-dimensional Dirac operator is not controllable, but the controllability restores after some trick associated with doubling the state of the system, ; the discrete system associated with a finite Jacobi matrix is boundary controllable, but the time in the model considered was discrete, see also [10] for the case of semi-infinite matrix. The peculiarity of the system (2) is the lack of the boundary controllability, and in opposite to all systems considered in [9, 10], the speed of the wave propagation in (2) is infinite. Nevertheless we will show that the method from [8, 8] works, and using it, one can construct de Branges space associated with  $A$ .

In the second section, we provide the necessary information on for the solution of the forward and inverse problems for (2) from [2]. In the third section, we remind the reader of some useful definitions and construct the de Branges space associated with (2).

## 2. Dynamical system, forward problem, Krein equations

The following Cauchy problem for the difference equation:

$$\begin{cases} a_1\phi_2 + b_1\phi_1 = \lambda\phi_1, \\ a_n\phi_{n+1} + a_{n-1}\phi_{n-1} + b_n\phi_n = \lambda\phi_n, & n = 2, \dots, N, \\ \phi_1 = 1, \end{cases} \quad (4)$$

determines the set of polynomials  $\{1, \phi_2(\lambda), \dots, \phi_N(\lambda), \phi_{N+1}(\lambda)\}$ . Let  $\lambda_1, \dots, \lambda_N$  be the roots of the equation  $\phi_{N+1}(\lambda) = 0$ , it is known [11] that they are real and distinct. We denote by  $(\cdot, \cdot)$  the scalar product in  $\mathbb{R}^N$  and introduce the vectors and the coefficients by the rules:

$$\varphi(\lambda) = \begin{pmatrix} \phi_1(\lambda) \\ \vdots \\ \phi_N(\lambda) \end{pmatrix}, \quad \varphi_k = \begin{pmatrix} \phi_1(\lambda_k) \\ \vdots \\ \phi_N(\lambda_k) \end{pmatrix}, \quad \rho_k = (\varphi_k, \varphi_k), \quad k = 1, \dots, N.$$

Thus,  $\varphi_k$  are non-normalized eigenvectors of  $A$ , corresponding to eigenvalues  $\lambda_k$ :

$$A\varphi_k = \lambda_k\varphi_k, \quad k = 1, \dots, N.$$

We call by *spectral data* and *the spectral function*  $\rho$  the following objects:

$$\{\lambda_i, \rho_i\}_{i=1}^N, \quad \rho(\lambda) = \sum_{\{k: \lambda_k < \lambda\}} \frac{1}{\rho_k}.$$

The standard application of the Fourier method yields:

**Lemma 1.** *The solution to (2) admits the spectral representation:*

$$u^f(t) = \sum_{k=1}^N h_k(t)\varphi_k, \quad u^f(t) = \int_{-\infty}^{\infty} \int_0^t S(t-\tau, \lambda) f(\tau) d\tau \varphi(\lambda) d\rho(\lambda), \quad (5)$$

where:

$$h_k(t) = \frac{1}{\rho_k} \int_0^t f(\tau) S_k(t-\tau) d\tau,$$

$$S(t, \lambda) = \begin{cases} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}}, & \lambda > 0, \\ \frac{\text{sh} \sqrt{|\lambda|} t}{\sqrt{|\lambda|}}, & \lambda < 0, \\ t, & \lambda = 0, \end{cases} \quad S_k(t) = S(t, \lambda_k).$$

We introduce the *outer space* of the system (2), the space of controls:  $\mathcal{F}^T := L_2(0, T; \mathbb{C})$  with the scalar product  $f, g \in \mathcal{F}^T$ ,  $(f, g)_{\mathcal{F}^T} = \int_0^T f(t)\overline{g(t)} dt$ . The *response operator*  $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  is introduced by the formula (3).

Making use of (5) implies the representation formula for  $R^T$ :

$$(R^T f)(t) = u_1^f(t) = \sum_{k=1}^N h_k(t) = \int_0^t r(t-s)f(s) ds,$$

where:

$$r(t) = \sum_{k=1}^N \frac{1}{\rho_k} S_k(t),$$

is called a *response function*. Note that the operator  $R^T$  is a natural analog of a dynamic Dirichlet-to-Neumann operator [4] in continuous, and [9, 12, 13] in discrete cases.

The *inner space* of (2), i.e. the space of states is  $\mathcal{H}^N := \mathbb{C}^N$ , indeed for any  $T > 0$  and  $f \in \mathcal{F}^T$ , we have that  $u^f(T) \in \mathcal{H}^N$ . The scalar product in  $\mathcal{H}^N$  is given by:

$$(a, b)_{\mathcal{H}^N} = \sum_{k=1}^N a_k \bar{b}_k.$$

The *control operator*  $W^T : \mathcal{F}^T \mapsto \mathcal{H}^N$  is introduced by the rule:

$$W^T f = u^f(T).$$

Due to (5), we have that  $W^T f = \sum_{k=1}^N h_k(T) \varphi_k$ . In [1, 2], the authors used real inner and outer spaces, but in the complex case all the results are valid as well.

We introduce the subspace:

$$\mathcal{F}_1^T = \text{Lin} \{S_k(T-t)\}_{k=1}^N,$$

where we assume complex coefficients in the span. The following lemma establishes the boundary controllability of (2):

**Lemma 2.** *The operator  $W^T$  maps  $\mathcal{F}_1^T$  onto  $\mathcal{H}^N$  isomorphically.*

The *connecting operator*  $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$  is defined by the rule  $C^T := (W^T)^* W^T$ , so by the definition for  $f, g \in \mathcal{F}^T$ , one has:

$$(C^T f, g)_{\mathcal{F}^T} = (u^f(T), u^g(T))_{\mathcal{H}^N} = (W^T f, W^T g)_{\mathcal{H}^N}. \quad (6)$$

It is crucial in the Boundary control method that  $C^T$  can be expressed in terms of inverse data:

**Theorem 1.** *The connecting operator admits the representation in terms of dynamic inverse data:*

$$(C^T f)(t) = \frac{1}{2} \int_0^T \int_{|t-s|}^{2T-s-t} r(\tau) d\tau f(s) ds,$$

and in terms of spectral inverse data:

$$(C^T f)(t) = \int_0^T \sum_{k=1}^N \frac{1}{\rho_k} S_k(T-t) S_k(T-s) f(s) ds. \quad (7)$$

**Remark 1.** *The formula (7) implies that  $\mathcal{F}_1^T = C^T \mathcal{F}^T$ , so  $\mathcal{F}_1^T$  is completely determined by inverse data.*

### 2.1. Krein equations

By  $f_k^T \in \mathcal{F}_1^T$ , we denote the controls, driving the system (2) to prescribed *special states*:

$$d_k \in \mathcal{H}^N, \quad d_k = (0, \dots, 1, \dots, 0), \quad k = 1, \dots, N.$$

It is important that such a controls can be found as the solutions to the Krein equations:

**Theorem 2.** *The control  $f_1^T$  can be found as the solution to the following equation:*

$$(C^T f_1^T)(t) = r(T-t), \quad 0 < t < T. \quad (8)$$

The controls  $f_k^T$ ,  $k = 2, \dots, N$  satisfy the system:

$$\begin{cases} -(C^T f_1^T)'' = b_1 C^T f_1^T + a_1 C^T f_2^T, \\ -(C^T f_k^T)'' = a_{k-1} C^T f_{k-1}^T + b_k C^T f_k^T + a_k C^T f_{k+1}^T, & k = 2, \dots, N-1, \\ -(C^T f_N^T)'' = a_{N-1} C^T f_{N-1}^T + b_N C^T f_N^T. \end{cases} \quad (9)$$

### 3. De Branges space for $A$

Here, we provide the information on de Branges spaces in accordance with [5, 7]. The entire function  $E : \mathbb{C} \mapsto \mathbb{C}$  is called a *Hermite-Biehler function* if  $|E(z)| > |E(\bar{z})|$  for  $z \in \mathbb{C}_+$ . We use the notation  $F^\#(z) = \overline{F(\bar{z})}$ . The *Hardy space*  $H_2$  is defined by:  $f \in H_2$  if  $f$  is holomorphic in  $\mathbb{C}_+$  and  $\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx < \infty$ . Then, the *de Branges space*  $B(E)$  consists of entire functions such that:

$$B(E) := \left\{ F : \mathbb{C} \mapsto \mathbb{C}, F \text{ entire}, \frac{F}{E}, \frac{F^\#}{E} \in H_2 \right\}.$$

The space  $B(E)$  with the scalar product:

$$[F, G]_{B(E)} = \frac{1}{\pi} \int_{\mathbb{R}} F(\lambda) \overline{G(\lambda)} \frac{d\lambda}{|E(\lambda)|^2},$$

is a Hilbert space. For any  $z \in \mathbb{C}$ , the *reproducing kernel* is introduced by the relation:

$$J_z(\xi) := \frac{\overline{E(z)}E(\xi) - E(\bar{z})\overline{E(\xi)}}{2i(\bar{z} - \xi)}. \quad (10)$$

Then,

$$F(z) = [J_z, F]_{B(E)} = \frac{1}{\pi} \int_{\mathbb{R}} J_z(\lambda) \overline{F(\lambda)} \frac{d\lambda}{|E(\lambda)|^2}.$$

We observe that a Hermite-Biehler function  $E(\lambda)$  defines  $J_z$  by (10). The converse is also true [5, 6]:

**Theorem 3.** *Let  $X$  be a Hilbert space of entire functions with reproducing kernel such that:*

- 1) *For any  $\omega \in \mathbb{C}$  the point evaluation is a bounded functional, i.e.  $|f(\omega)| \leq C_\omega \|f\|_X$ ,*
- 2) *if  $f \in X$  then  $f^\# \in X$  and  $\|f\|_X = \|f^\#\|_X$ ,*
- 3) *if  $f \in X$  and  $\omega \in \mathbb{C}$  such that  $f(\omega) = 0$ , then  $\frac{z - \bar{\omega}}{z - \omega} f(z) \in X$  and  $\left\| \frac{z - \bar{\omega}}{z - \omega} f(z) \right\|_X = \|f\|_X$ ,*

then  $X$  is a de Branges space based on the function:

$$E(z) = \sqrt{\pi}(1 - iz) J_i(z) \|J_i\|_X^{-1},$$

where  $J_z$  is a reproducing kernel.

In the space  $L_{2,\rho}(\mathbb{R})$  we take the subspace spanned on the first  $N$  polynomials generated by (4):

$$L_N := \text{Lin}\{\phi_1(\lambda), \dots, \phi_N(\lambda)\}.$$

Note that  $\phi_1(\lambda), \dots, \phi_N(\lambda)$  are mutually orthogonal in  $L_{2,\rho}(\mathbb{R})$ , see [11]. By  $P_N : L_{2,\rho}(\mathbb{R}) \rightarrow L_{2,\rho}(\mathbb{R})$  we denote the orthogonal projector in  $L_{2,\rho}(\mathbb{R})$  onto  $L_N$  acting by the rule:

$$P_N a = \sum_{k=1}^N (a, \phi_k)_{L_{2,\rho}(\mathbb{R})} \phi_k(\lambda), \quad a \in L_{2,\rho}(\mathbb{R}).$$

We introduce the Fourier transformation  $\mathbb{R}^N \mapsto L_{2,\rho}(\mathbb{R})$  by the formula:

$$(Fb)(\lambda) = \sum_{k=1}^N b_k \phi_k(\lambda), \quad b = (b_1, \dots, b_N) \in \mathbb{R}^N.$$

Note that  $F$  is an unitary map between  $\mathbb{R}^N$  and  $L_N$ , and:

$$b_k = (Fb(\lambda), \phi_k(\lambda))_{L_{2,\rho}(\mathbb{R})}.$$

In accordance with the general approach proposed in [8, 9], we consider the *reachable set* of the dynamical system (2):

$$U^T := W^T \mathcal{F}^T = \{u^f(T) \mid f \in \mathcal{F}^T\}.$$

By the Lemma 2 we know that:

$$U^T = W^T \mathcal{F}_1^T.$$

Then, for any  $f \in \mathcal{F}^T$  we can evaluate:

$$(F u^f(T))(\lambda) = \sum_{k=1}^N \int_{\mathbb{R}} \int_0^T S_k(T - \tau, \beta) f(\tau) d\tau \phi_k(\beta) d\rho(\beta) \phi_k(\lambda) = P_N \int_0^T S(T - \tau, \cdot) f(\tau) d\tau.$$

We introduce the linear manifold of Fourier images of the reachable set:

$$B_N := FU^T = \text{Lin}\{\phi_1, \dots, \phi_N\},$$

thus,  $B_N$  is a set of polynomials with complex coefficients of the degree not greater than  $N - 1$ .

The metric in  $B_N$  is introduced by the following rule: for  $H, G \in B_N$ , such that  $H = P_N \int_0^T S(T - \tau, \cdot)h(\tau) d\tau$ ,  $G = P_N \int_0^T S(T - \tau, \cdot)g(\tau) d\tau$ , where  $h, g \in \mathcal{F}_1^T$  we set:

$$(H, G)_{B_N} := (C^T h, g)_{\mathcal{F}^T}.$$

On the other hand, for  $h, g \in \mathcal{F}_1^T$  we can evaluate using the definition of  $C^T$  and Fourier transformation:

$$\begin{aligned} (H, G)_{B_N} &= (C^T h, g)_{\mathcal{F}^T} = (u^h(T), u^g(T))_{\mathcal{H}^N} = \int_{\mathbb{R}} (Fu^h(T))(\lambda) (Fu^g(T))(\lambda) d\rho(\lambda) \\ &= \int_{\mathbb{R}} \left( P_N \int_0^T S(T - \tau, \cdot)h(\tau) d\tau \right) (\lambda) \left( P_N \int_0^T S(T - \tau, \cdot)g(\tau) d\tau \right) (\lambda) d\rho(\lambda) \\ &= \int_{\mathbb{R}} H(\lambda)G(\lambda) d\rho(\lambda), \end{aligned}$$

We note that for the systems considered in [8, 9] it was a certain option in the choosing of the measure  $d\rho(\lambda)$  in the above calculations. Due to the infinite speed of wave propagation in (2), we do not have this option here.

We set *the special control problem* for the system (2): to find a control  $j_z \in \mathcal{F}_1^T$  which drives (2) to the prescribed state:

$$u_k^{j_z}(T) = \overline{\phi_k(z)}, \quad k = 1, \dots, N.,$$

at time  $t = T$ . Due to Theorem 2, such a control exists and is unique in  $\mathcal{F}_1^T$ . Then, for such a control, we can evaluate:

$$(C^T j_z, g)_{\mathcal{F}^T} = (u^g(T), u^{j_z}(T))_{\mathcal{H}^N} = \sum_{k=1}^N u_k^g(T) \phi_k(z) = (Fu^g(T))(z).$$

Thus, for:

$$J_z(\lambda) := (Fu^{j_z}(T))(\lambda)$$

and  $G(\lambda) = (Fu^g(T))(\lambda)$ , we have that:

$$(J_z, G)_{B_N} = (C^T j_z, g)_{\mathcal{F}^T} = G(z).$$

In other words,  $J_z(\lambda)$  is a reproducing kernel in  $B_N$ .

To show that  $B_N$  is a de Branges space, we use the Theorem 3, all three conditions of which are trivially satisfied:

indeed, for  $G \in B_N$  such that  $G = P_N \int_0^T S(T - \tau, \cdot)g(\tau) d\tau$ , where  $g \in \mathcal{F}_1^T$  we can evaluate:

$$\begin{aligned} |G(z)| &= |(J_z, G)_{B_N}| = |(C^N j_z, g)_{\mathcal{F}^N}| \\ &\leq \| (C^T)^{\frac{1}{2}} j_z \|_{\mathcal{F}^N} \| (C^N)^{\frac{1}{2}} g \|_{\mathcal{F}^N} = \| (C^N)^{\frac{1}{2}} j_z \|_{\mathcal{F}^N} \| G \|_{B_N}. \end{aligned}$$

Clearly  $G^\#$ , being a polynomial is entire and:

$$\|G^\#\|_{B_N} = \left( \int_{-\infty}^{\infty} \overline{G(\overline{\mu})} G(\overline{\mu}) d\rho(\mu) \right)^{1/2} = \|G\|_{B_N}.$$

When  $\omega \in \mathbb{C}$  such that  $F(\omega) = 0$ , then  $\frac{z - \overline{\omega}}{z - \omega} F(z)$  is an entire function and:

$$\left\| \frac{z - \overline{\omega}}{z - \omega} G(z) \right\|_{B_N} = \left( \int_{-\infty}^{\infty} \overline{\frac{z - \overline{\omega}}{z - \omega} G(z)} \frac{z - \overline{\omega}}{z - \omega} G(z) d\rho(z) \right)^{1/2} = \|G\|_{B_N}.$$

Thus,  $B_N$  is a de Branges space.

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