

## Networks with point-like nonlinearities

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**ABSTRACT** We study static nonlinear waves in networks described by a nonlinear Schrödinger equation with point-like nonlinearities on metric graphs. Explicit solutions fulfilling vertex boundary conditions are obtained. Spontaneous symmetry breaking caused by bifurcations is found.

**KEYWORDS** metric graphs, point-like nonlinearity, NLSE, spontaneous symmetry breaking bifurcations

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### 1. Introduction

Modeling wave and particle transports in branched structures is an important problem with applications in many subjects of contemporary physics, such as optics, condensed matters, complex molecules, polymers and fluid dynamics. Mathematical treatment of such problems is reduced to solving different partial differential equations (PDEs) on so-called metric graphs. These are set of one-dimensional bonds, with assigned lengths. The connection rule of the bonds is called topology of the graph and is described in terms of the adjacency matrix [1, 2]. Linear and nonlinear wave equations on metric graphs attracted much attention recently and they are becoming a hot topic [3–29].

Solving wave equations on metric graphs requires imposing boundary conditions at the branching points (graph vertices). In the case of linear evolution equations, e.g., for the linear Schrödinger equation, the main requirement for such boundary conditions is that they should keep self-adjointness of the problem [30], while for nonlinear PDEs, one needs to use other fundamental conservation laws (e.g., energy, norm, momentum, charge, etc) for obtaining vertex boundary conditions [5, 15, 18]. Energy and norm conservation were used to derive vertex boundary conditions for the nonlinear Schrödinger equation (NLSE) on metric graphs in [5], where exact solutions were obtained and the integrability of the problem was shown under certain constraints. In [15], a similar study was done for the sine-Gordon equation on metric graphs. Soliton solutions of a nonlinear Dirac equation on metric graphs have been obtained in [18]. Static solitons in networks were studied in the Refs. [7–11] by solving stationary nonlinear Schrödinger equations on metric graphs. A model for transparent nonlinear networks was proposed recently in [22]. Earlier, transparent quantum graphs were studied in [20, 21, 23]. Modeling nonlinear waves and solitons in branched structures and networks provides a powerful tool for tunable wave, particle, heat and energy transport in different practically important systems, such as branched optical fibers, carbon nanotube networks, branched polymers and low-dimensional functional materials.

Here, we consider a Schrödinger equation with a point-like nonlinearity on metric graphs. NLSE with point-like nonlinearity can be implemented in dual-core fiber Bragg gratings as well as in the ordinary fibers [33] and Bose-Einstein condensates confined in double-well traps [34, 35]. A similar problem on a line with double-delta type nonlinearity was considered earlier in [31, 32], where explicit solutions were derived. On the basis of numerical analysis, it was shown that symmetric states are stable up to a spontaneous symmetry breaking bifurcation point. From a fundamental viewpoint, it would be interesting to see the difference between solutions of the problem on the line and networks, as the topology of a network may cause additional effects. Here, we use the methods of [31] to obtain explicit solutions of our problem. Degenerate spontaneous symmetry breaking bifurcations are also obtained.

The paper is organized as follows: in the next section formulation of the problem for metric star graph and the derivation of the vertex boundary conditions are presented. In Section 3 we obtain exact analytical solutions of the problem and formulate constraints for integrability. Numerical results and analysis of bifurcation are also presented in this section. Finally, Section 4 presents some concluding remarks.

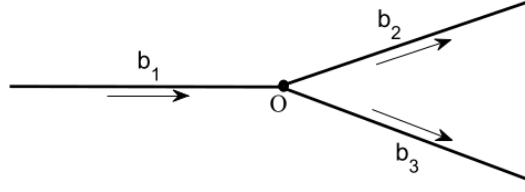


FIG. 1. Metric star graph

## 2. Vertex boundary conditions

Consider a metric star graph consisting of three semi-infinite bonds,  $b_1 \sim (-\infty; 0)$ ,  $b_2 \sim (0; +\infty)$ ,  $b_3 \sim (0; +\infty)$  (see, Fig. 1). On each bond of this graph the NLSE with variable nonlinearity coefficient,  $g_j(x)$  can be written as

$$i \frac{\partial \psi_j}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi_j}{\partial x^2} + g_j(x) |\psi_j|^2 \psi_j, \quad (1)$$

where  $j = 1, 2, 3$  denotes the bond number.

To solve Eq. (1), one needs to impose vertex boundary conditions (VBC), which can be derived, e.g., from norm and energy conservations laws. The norm and energy are given respectively by:

$$N = \sum_{j=1}^3 \int_{b_j} |\psi_j(x)|^2 dx, \quad (2)$$

and:

$$H = \frac{1}{2} \sum_{j=1}^3 \int_{b_j} \left( \left| \frac{\partial \psi_j}{\partial x} \right|^2 + g_j(x) |\psi_j|^4 \right) dx. \quad (3)$$

From  $\dot{N} = 0$  and  $\dot{H} = 0$  and using  $\psi_1, \frac{\partial \psi_1}{\partial x} \rightarrow 0$  as  $x \rightarrow -\infty$  and  $\psi_{2,3}, \frac{\partial \psi_{2,3}}{\partial x} \rightarrow 0$  as  $x \rightarrow +\infty$ , we obtain the following vertex boundary conditions (at  $x = 0$ ):

$$\text{Im} \left( \psi_1 \frac{\partial \psi_1^*}{\partial x} \right) = \text{Im} \left( \psi_2 \frac{\partial \psi_2^*}{\partial x} \right) + \text{Im} \left( \psi_3 \frac{\partial \psi_3^*}{\partial x} \right), \quad (4)$$

$$\text{Re} \left( \frac{\partial \psi_1}{\partial t} \frac{\partial \psi_1^*}{\partial x} \right) = \text{Re} \left( \frac{\partial \psi_2}{\partial t} \frac{\partial \psi_2^*}{\partial x} \right) + \text{Re} \left( \frac{\partial \psi_3}{\partial t} \frac{\partial \psi_3^*}{\partial x} \right). \quad (5)$$

Thus, both energy and current conservation give rise to nonlinear vertex boundary conditions. However, the VBC given by Eqs. (4) and (5) can be fulfilled if the following two types of the linear relations at the vertices are imposed:

Type I:

$$\begin{cases} \alpha_1 \psi_1|_{x=0} = \alpha_2 \psi_2|_{x=0} = \alpha_3 \psi_3|_{x=0}, \\ \frac{1}{\alpha_1} \frac{\partial \psi_1}{\partial x} \Big|_{x=0} = \frac{1}{\alpha_2} \frac{\partial \psi_2}{\partial x} \Big|_{x=0} + \frac{1}{\alpha_3} \frac{\partial \psi_3}{\partial x} \Big|_{x=0}, \end{cases} \quad (6)$$

and

Type II:

$$\begin{cases} \frac{1}{\alpha_1} \psi_1|_{x=0} = \frac{1}{\alpha_2} \psi_2|_{x=0} + \frac{1}{\alpha_3} \psi_3|_{x=0}, \\ \alpha_1 \frac{\partial \psi_1}{\partial x} \Big|_{x=0} = \alpha_2 \frac{\partial \psi_2}{\partial x} \Big|_{x=0} = \alpha_3 \frac{\partial \psi_3}{\partial x} \Big|_{x=0}, \end{cases} \quad (7)$$

where  $\alpha_1, \alpha_2, \alpha_3$  are real constants, which will be determined below. In the following we will focus on VBC of type I, as it looks more physical. In the next section we obtain exact analytical solutions of Eq. (1) for the VBCs given by Eq. (6) and derive a constraint, which provides integrability of the problem.

## 3. Exact solutions and bifurcations

Detailed treatment of Eq. (1) on a line was done in [31], where an exact solution was obtained for the localized nonlinearity given by:

$$g(x) = -\frac{1}{a\sqrt{\pi}} \left[ e^{-\frac{(x+1)^2}{a^2}} + e^{-\frac{(x-1)^2}{a^2}} \right].$$

Here, we will consider the same type of nonlinearity given on each bond of the star graph presented in Fig. 1. Our prescription for solving Eq. (1) for the vertex boundary conditions (6) and (7) was developed in our previous works (see, e.g., the Refs. [5, 6, 9, 17–20]). Briefly, it can be described as follows: Having known solution of a given evolution equation on a line, we require that it should fulfill the vertex boundary conditions given on a graph. Of course, this cannot be achieved in the general case. Therefore, one needs to find constraints that ensure fulfilling vertex boundary conditions by the solution of the evolution equation (Eq. (1) in our case) on a line. Usually, such constraints are given in terms of the parameters appearing in the evolution equation and vertex boundary conditions [5, 6, 9]. Here, for solving the problem given by Eqs. (1) and (6), we find a solution on each bond and fulfill the vertex boundary conditions.

Consider the star graph presented in Fig. 1, whose bonds are assigned localized nonlinearities given by the following expressions:

$$g_1(x) = -\frac{\beta_1}{a\sqrt{\pi}} e^{-\frac{(x+c_1)^2}{a^2}},$$

$$g_j(x) = -\frac{\beta_j}{a\sqrt{\pi}} e^{-\frac{(x-c_j)^2}{a^2}}, \quad j = 2, 3$$

where  $c_j > 0$  for  $j = 1, 2, 3$ . For this specific form of  $g_j(x)$ , the space and time variables in Eq. (1) can be separated, that yields (at  $a \rightarrow 0$ ):

$$-\mu\phi_1 + \frac{1}{2}\phi_1'' + \beta_1\delta(x+c_1)\phi_1^3 = 0,$$

$$-\mu\phi_j + \frac{1}{2}\phi_j'' + \beta_j\delta(x-c_j)\phi_j^3 = 0, \quad j = 2, 3. \quad (8)$$

The solution of Eqs. (8) without the vertex boundary conditions can be written as:

$$\phi_1(x) = \begin{cases} \frac{A_1}{\sqrt{\beta_1}} e^{\sqrt{2\mu}(x+c_1)}, & x < -c_1, \\ \frac{B_{11}}{\sqrt{\beta_1}} e^{\sqrt{2\mu}(x+c_1)} + \frac{B_{12}}{\sqrt{\beta_1}} e^{-\sqrt{2\mu}(x+c_1)}, & -c_1 < x \leq 0, \end{cases}$$

$$\phi_j(x) = \begin{cases} \frac{A_{j1}}{\sqrt{\beta_j}} e^{\sqrt{2\mu}(x-c_j)} + \frac{A_{j2}}{\sqrt{\beta_j}} e^{-\sqrt{2\mu}(x-c_j)}, & 0 \leq x < c_j, \\ \frac{B_j}{\sqrt{\beta_j}} e^{-\sqrt{2\mu}(x-c_j)}, & x > c_j. \end{cases}, \quad j = 2, 3. \quad (9)$$

Fulfilling the VBCs (6) by solutions (9) leads to

$$\frac{\alpha_1}{\sqrt{\beta_1}} (B_{11}e^{\sqrt{2\mu}c_1} + B_{12}e^{-\sqrt{2\mu}c_1}) = \frac{\alpha_2}{\sqrt{\beta_2}} (A_{21}e^{-\sqrt{2\mu}c_2} + A_{22}e^{\sqrt{2\mu}c_2}) = \frac{\alpha_3}{\sqrt{\beta_3}} (A_{31}e^{-\sqrt{2\mu}c_3} + A_{32}e^{\sqrt{2\mu}c_3}),$$

$$\frac{1}{\alpha_1\sqrt{\beta_1}} (B_{11}e^{\sqrt{2\mu}c_1} - B_{12}e^{-\sqrt{2\mu}c_1}) = \frac{1}{\alpha_2\sqrt{\beta_2}} (A_{21}e^{-\sqrt{2\mu}c_2} - A_{22}e^{\sqrt{2\mu}c_2}) + \frac{1}{\alpha_3\sqrt{\beta_3}} (A_{31}e^{-\sqrt{2\mu}c_3} - A_{32}e^{\sqrt{2\mu}c_3}). \quad (10)$$

Furthermore, for the sake of simplicity we consider the case when  $c_1 = c_2 = c_3 = c$ . Choosing parameters  $A$  and  $B$  to fulfill the relations will yield:

$$A_{j1} = B_{11}e^{2\sqrt{2\mu}c}, \quad B_{12} = B \cdot B_{11}e^{2\sqrt{2\mu}c}, \quad A_{j2} = B \cdot B_{11}, \quad B \neq \pm 1, \quad j = 2, 3.$$

From the first equation of (10), we get:

$$\frac{\alpha_1}{\alpha_{2,3}} = \frac{\sqrt{\beta_1}}{\sqrt{\beta_{2,3}}}, \quad \frac{1}{\alpha_1\sqrt{\beta_1}} = \frac{1}{\alpha_2\sqrt{\beta_2}} + \frac{1}{\alpha_3\sqrt{\beta_3}}.$$

These equations lead to the constraint given by:

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3}. \quad (11)$$

Eq. (11) presents a constraint that provides fulfilling the vertex boundary conditions (6) by the solution (9). Furthermore, from the continuity of the solution  $\phi_j(x)$  we have

$$A_1 = B_{11} (1 + Be^{2\sqrt{2\mu}c}),$$

$$B_j = B_{11} (e^{2\sqrt{2\mu}c} + B), \quad j = 2, 3.$$

For the jump  $\Delta(\phi_1')|_{x=-c} = -2\beta_1(\phi_1|_{x=-c})^3$  we can find:

$$B_{11} = \pm \sqrt{\frac{\sqrt{2\mu}Be^{2\sqrt{2\mu}c}}{(1 + Be^{2\sqrt{2\mu}c})^3}}. \quad (12)$$

For the jump  $\Delta(\phi'_j)|_{x=c} = -2\beta_j(\phi_j|_{x=c})^3$ ,  $j = 2, 3$  we can find:

$$B_{11} = \pm \sqrt{\frac{\sqrt{2\mu}e^{2\sqrt{2\mu}c}}{(e^{2\sqrt{2\mu}c} + B)^3}}. \quad (13)$$

Equating (12) and (13) we obtain:

$$B = \frac{(e^{2\sqrt{2\mu}c} \pm \sqrt{e^{4\sqrt{2\mu}c} - 4})^3}{8}, \quad (14)$$

where  $\mu \geq \frac{\ln^2 2}{8c^2}$ .

Now we consider the cases, when the nonlinearities appear on two and one bonds only. Similarly as the above, one can obtain solutions for these cases. For the star graph with nonlinearities appearing in two bonds we have the following stationary NLSE on each bond  $b_j$  ( $b_1 \sim (-\infty; 0]$ ,  $b_{2,3} \sim [0; +\infty)$ ) of the star graph

$$\begin{aligned} -\mu\phi_1 + \frac{1}{2}\phi_1'' + \beta_1\delta(x+c_1)\phi_1^3 &= 0, \quad c_1 > 0 \\ -\mu\phi_2 + \frac{1}{2}\phi_2'' + \beta_2\delta(x-c_2)\phi_2^3 &= 0, \quad c_2 > 0 \\ -\mu\phi_3 + \frac{1}{2}\phi_3'' &= 0. \end{aligned} \quad (15)$$

The solution of this equations without the vertex boundary conditions can be written as:

$$\begin{aligned} \phi_1(x) &= \begin{cases} \frac{A_1}{\sqrt{\beta_1}}e^{\sqrt{2\mu}(x+c_1)}, & x < -c_1, \\ \frac{B_{11}}{\sqrt{\beta_1}}e^{\sqrt{2\mu}(x+c_1)} + \frac{B_{12}}{\sqrt{\beta_1}}e^{-\sqrt{2\mu}(x+c_1)}, & -c_1 < x \leq 0, \end{cases} \\ \phi_2(x) &= \begin{cases} \frac{A_{21}}{\sqrt{\beta_2}}e^{\sqrt{2\mu}(x-c_2)} + \frac{A_{22}}{\sqrt{\beta_2}}e^{-\sqrt{2\mu}(x-c_2)}, & 0 \leq x < c_2, \\ \frac{B_2}{\sqrt{\beta_2}}e^{-\sqrt{2\mu}(x-c_2)}, & x > c_2, \end{cases} \\ \phi_3(x) &= \frac{B_3}{\sqrt{\beta_3}}e^{-\sqrt{2\mu}x}, \quad x \geq 0. \end{aligned} \quad (16)$$

Similarly, for the case when the nonlinearity appears in a single bond of the star graph, we have the solution:

$$\begin{aligned} \phi_1(x) &= \begin{cases} \frac{A_1}{\sqrt{\beta_1}}e^{\sqrt{2\mu}(x+c_1)}, & x < -c_1, \\ \frac{B_{11}}{\sqrt{\beta_1}}e^{\sqrt{2\mu}(x+c_1)} + \frac{B_{12}}{\sqrt{\beta_1}}e^{-\sqrt{2\mu}(x+c_1)}, & -c_1 < x \leq 0, \end{cases} \\ \phi_j(x) &= \frac{A_j}{\sqrt{\beta_j}}e^{-\sqrt{2\mu}x}, \quad j = 2, 3, \quad x \geq 0. \end{aligned} \quad (17)$$

Again, requiring fulfilling the vertex boundary conditions (6) by the solutions (16) and (17), one can obtain constraints (in terms of parameters  $\beta_j$ ) ensuring that (16) and (17) are the solution of the problem on graph.

The above analytical solution (9) is obtained under the assumption that the constraint (11) is satisfied. In the following, we solve Eq. (8) numerically both for the case when the constraint in Eq. (11) is fulfilled and broken. The point nonlinearity in Eq. (8) is represented by the Gaussian function with  $c = 3$  and  $a = 0.1$ . In the following, we only limit ourselves with positive solutions.

In Fig. 2, we plot two possible solutions for the case when the sum rule given by Eq. (11) is fulfilled. The first panel shows a solution when all the bonds are excited by the delta nonlinearity, while in the second one (b), only the first and the second bonds have point-like excitations. Using the configuration in the limit  $\mu \rightarrow \infty$  as our code, we represent solutions in panels (a) and (b) as  $[1, 1, 1]$  and  $[1, 1, 0]$ , respectively.

In Fig. 3(a), we present bifurcation diagrams of the solutions in Fig. 2. We obtain that the two configurations in Fig. 2 are connected with each other, with  $[1, 1, 1]$  as the main branch and  $[1, 1, 0]$  as a bifurcating solution through a pitchfork bifurcation with the configuration  $[0, 0, 1]$ . We therefore observe a spontaneous symmetry breaking bifurcation. It is particularly interesting to note that the bifurcation is quite degenerate in the sense that we obtain both a subcritical as well as a supercritical bifurcation emerging from the same point. We also obtain several other solutions bifurcating from the same bifurcation point, which are all indicated in Fig. 3(a).

We have considered a different case when all the nonlinearity coefficients are the same, i.e., without loss of nonlinearity  $\beta_j = 1$ . In this case, the condition (11) is not satisfied. We plot the bifurcation diagram of the positive solutions in Fig. 3(b), where now we obtain that all the asymmetric solutions merge into two branches only, which bifurcate from the same point.

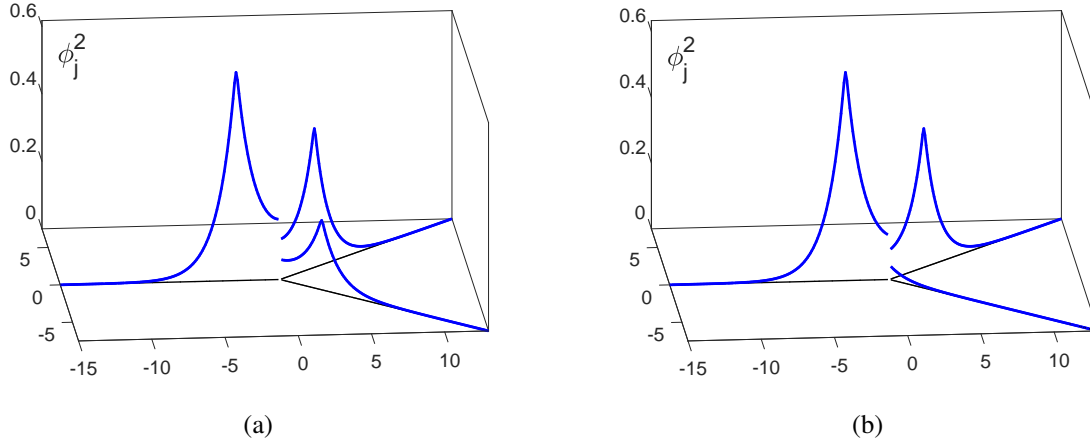


FIG. 2. Two possible solutions are plotted. Here,  $\beta_1 = 2/3$ ,  $\beta_2 = 1$ , and  $\beta_3 = 2$  satisfying the condition for conserved norm and energy, and  $\mu = 0.1$ . The solution in panel (a) is denoted by configuration  $[1, 1, 1]$  and in panel (b) by  $[1, 1, 0]$ .

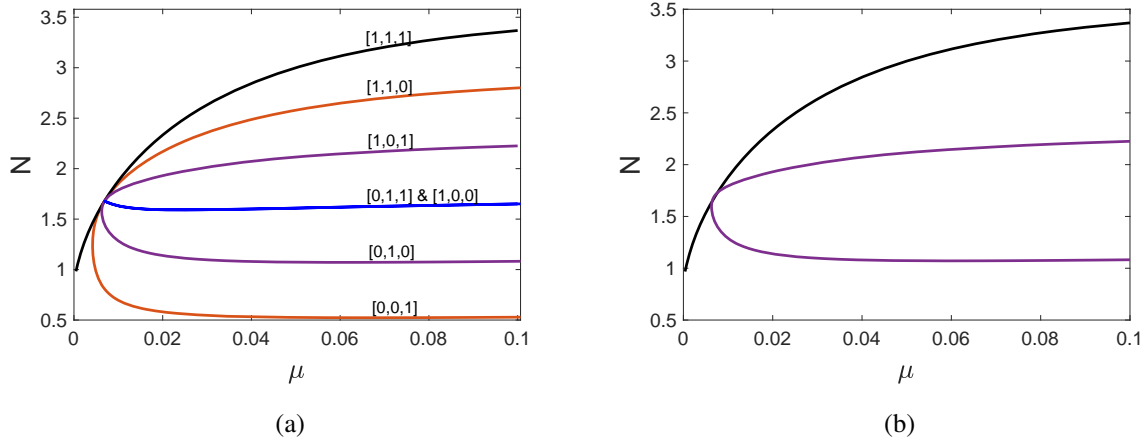


FIG. 3. (a) Bifurcation diagram of solutions in Fig. 2. (b) The same diagram, but for  $\beta_j = 1$ ,  $j = 1, 2, 3$ .

#### 4. Conclusions

In this paper we studied nonlinear Schrödinger equation with localized nonlinearities on metric graphs. Exact solutions are obtained and their stability is analyzed by exploring the bifurcations for the case of point-like (varying) nonlinearity that has the form of a delta-well. Exact analytical solutions of the problem were obtained for different cases of point excitations, determined by the presence of a delta-well on different bonds. The constraint providing existence of such analytical solutions are derived in the form of simple sum rule written in terms of the bond nonlinearity coefficients. Numerical solutions of the problem are also obtained both for integrable and non-integrable cases. Bifurcations of the solutions are studied in terms of chemical potential  $\mu$  using the numerical solutions. The model considered in this paper is relevant for different practically important problems such as BEC in branched traps, Bragg gratings in branched fibers, etc. Extension of the treatment to other graph topologies is rather straightforward, provided the graph contains arbitrary subgraph, which is connected to three or more outgoing semi-infinite bonds.

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