

## On a nonlinear impulsive system of integro-differential equations with degenerate kernel and maxima

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**ABSTRACT** A nonlocal boundary value problem for a system of ordinary integro-differential equations with impulsive effects, degenerate kernel and maxima is investigated. The boundary value problem is given by the integral condition. The method of successive approximations in combination with the method of compressing mapping is used. The existence and uniqueness of the solution of the boundary value problem are proved. The continuous dependence of the solution on the right-hand side of the boundary value condition is shown.

**KEYWORDS** impulsive integro-differential equations, nonlocal condition, successive approximations, existence and uniqueness, continuous dependence of solution

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### 1. Introduction

Many problems in modern sciences, technology and economics are described by differential equations, the solutions of which are functions with first kind discontinuities at fixed or non-fixed times. Such differential equations are called differential equations with impulse effects [1–8]. As is known, in recent years the interest in the study of differential equations with nonlocal boundary conditions has increased (see, for example, [9–17]). In particular, in [15] a physical situation in which a non-metallic conductor is in contact with a perfect conductor is studied. In [16], the problems of mathematical models in reaction-diffusion systems are considered. In [17], the nonlocal conditions are used in the theory of phase transitions.

In [18–24] the problems of solvability for some type of integro-differential equations with degenerate kernel were considered. Also, a lot of publications of studying on differential equations with impulsive effects, describing many natural and technical processes, are appearing [25–35].

In this paper, we investigate a nonlocal boundary value problem for a system of first order Fredholm integro-differential equations with impulsive effects, degenerate kernel and nonlinear maxima. The questions of the existence and uniqueness of the solution to the boundary value problem, as well as the continuous dependence of the solution on the right-hand side of the boundary condition, are investigated. In [36], it is justified that the theoretical study of differential equations with maxima is relevant.

We consider the following system of Fredholm integro-differential equations:

$$x'(t) = \lambda \int_0^T H(t, s)x(s)ds + f(t, x(t), \max\{x(\tau) | \tau \in [h_1; h_2]\}), \quad (1)$$

for  $t \in [0, T]$ ,  $t \neq t_i$ ,  $i = 1, 2, \dots, p$  with nonlocal boundary value conditions:

$$Ax(0) + \int_0^T K(t)x(t)dt = B, \quad (2)$$

and impulsive effect:

$$x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i = 1, 2, \dots, p, \quad (3)$$

where  $H(t, s) = \sum_{k=1}^m a_k(t)b_k(s)$ ,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $K(t) \in \mathbb{R}^{n \times n}$  are given matrix and

$\det Q \neq 0$ ,  $Q = A + \int_0^T K(t) dt$ ,  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $I_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are given functions;  $0 < h_1 < h_2 < t$ ,

$h_j = h_j(t, x(t)), j = 1, 2$ ,  $\lambda$  is real nonzero parameter,  $x(t_i^+) = \lim_{h \rightarrow 0^+} (x_i + h)$ ,  $x(t_i^-) = \lim_{h \rightarrow 0^-} (t_i - h)$  are right-sided and left-sided limits of function  $x(t)$  at the point  $t = t_i$ , respectively. Every system of functions  $\{a_k(t)\}_{k=1}^m$  and  $\{b_k(s)\}_{k=1}^m$  are linearly independent.

## 2. Reduction to an integral equation

Here are some notations that will be used below. We denote by  $C([0, T], \mathbb{R}^n)$  the Banach space, which consists of continuous functions  $x(t) \in \mathbb{R}^n$  on the segment  $[0, T]$  with the norm:

$$\|x\| = \sqrt{\sum_{j=1}^n \max_{t \in [0, T]} |x_j(t)|}.$$

Since we consider the integro-differential equation (1) with impulsive effect at the points  $t_i, i = 1, 2, \dots, p$ , use the following linear space:

$$PC([0, T], \mathbb{R}^n) = \{x : [0, T] \rightarrow \mathbb{R}^n; x(t) \in C((t_i, t_{i+1}], \mathbb{R}^n), i = 1, \dots, p\},$$

where  $x(t_i^+)$  and  $x(t_i^-)$  ( $i = 0, 1, \dots, p$ ) exist and bounded;  $x(t_i^-) = x(t_i)$ .

It is obvious, that the linear space  $PC([0, T], \mathbb{R}^n)$  is Banach space with the following norm:

$$\|x\|_{PC} = \max \left\{ \|x\|_{C((t_i, t_{i+1}])}, \quad i = 1, 2, \dots, p \right\}.$$

**Formulation of problem.** To find the function  $x(t) \in PC([0, T], \mathbb{R}^n)$ , which for all  $t \in [0, T], t \neq t_i, i = 1, 2, \dots, p$  satisfies the integro-differential equation (1), nonlocal integral condition (2) and for  $t = t_i, i = 1, 2, \dots, p, 0 < t_1 < t_2 < \dots < t_p < T$  satisfies the limit condition (3).

Let the function  $x(t) \in PC([0, T], \mathbb{R}^n)$  is a solution of the problem (1)–(3). Then we rewrite the Fredholm integro-differential equation (1) as:

$$x'(t) = \lambda \int_0^T \sum_{k=1}^m a_k(t) b_k(s) x(s) ds + f(t, x(t), \max \{x(\tau) | \tau \in [h_1; h_2]\}).$$

By the designation:

$$c_k = \int_0^T b_k(s) x(s) ds,$$

the last integro-differential equation we rewrite in the following form:

$$x'(t) = \lambda \sum_{k=1}^m a_k(t) c_k + f(t, x(t), \max \{x(\tau) | \tau \in [h_1; h_2]\}).$$

Then, by integration of the last equation on the interval  $t \in (0, t_{i+1}]$ , we obtain:

$$\begin{aligned} & \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds = \\ & = \int_0^t x'(s) ds = [x(t_1) - x(0^+)] + [x(t_2) - x(t_1^+)] + \dots + [x(t) - x(t_i^+)] = \\ & = -x(0) - [x(t_1^+) - x(t_1)] - [x(t_2^+) - x(t_2)] - \dots - [x(t_i^+) - x(t_i)] + x(t). \end{aligned}$$

Taking into account the integral condition (2) in the last equality, we obtain:

$$x(t) = x(0) + \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds + \sum_{0 < t_i < t} I_i(x(t_i)). \quad (4)$$

Let the function  $x(t) \in PC([0, T], \mathbb{R}^n)$  in (4), satisfies the boundary value conditions (2):

$$\begin{aligned} \left[ A + \int_0^T K(t) dt \right] x(0) = \\ = B - \int_0^T K(t) \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds dt - \int_0^T K(t) \sum_{0 < t_i < t} I_i(x(t_i)) dt. \end{aligned} \quad (5)$$

Since  $\det Q \neq 0$ , from the equality (5) we have:

$$x(0) = Q^{-1} \left[ B - \int_0^T K(t) \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds dt - \int_0^T K(t) \sum_{0 < t_i < t} I_i(x(t_i)) dt \right]. \quad (6)$$

Substituting the equality (6) into representation (4), we obtain:

$$\begin{aligned} x(t) = Q^{-1} \left[ B - \int_0^T K(t) \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds dt - \int_0^T K(t) \sum_{0 < t_i < t} dt \right] + \\ + \int_0^t f(s, x(s), \cdot) ds + \sum_{0 < t_i < t} I_i(x(t_i)). \end{aligned} \quad (7)$$

Since the following equalities hold:

$$\begin{aligned} \int_0^T K(t) \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds dt = \int_0^T \int_t^T K(s) ds \left[ \lambda \sum_{k=1}^m a_k(t) c_k + f(t, x(t), \cdot) \right] dt, \\ \int_0^T K(t) \sum_{0 < t_i < t} I_i(x(t_i)) dt = \sum_{0 < t_i < T} \int_{t_i}^T K(t) dt I_i(x(t_i)), \end{aligned}$$

from presentation (7) we obtain:

$$\begin{aligned} x(t) = Q^{-1} B - Q^{-1} \int_0^T \int_t^T K(s) ds \left[ \lambda \sum_{k=1}^m a_k(t) c_k + f(t, x(t), \cdot) \right] dt - \\ - Q^{-1} \sum_{0 < t_i < t} \int_{t_i}^T K(t) dt I_i(x(t_i)) + \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds + \sum_{0 < t_i < t} I_i(x(t_i)). \end{aligned} \quad (8)$$

Let us make some simplifications in representation (8). Then the following equalities hold:

$$\begin{aligned} \int_0^t \left[ \lambda \sum_{k=1}^m a_k(s) c_k + f(s, x(s), \cdot) \right] ds - Q^{-1} \int_0^T \int_t^T K(s) ds \left[ \lambda \sum_{k=1}^m a_k(t) c_k + f(t, x(t), \cdot) \right] dt = \\ = Q^{-1} \int_0^t \left( A + \int_0^\theta K(s) ds \right) \left[ \lambda \sum_{k=1}^m a_k(\theta) c_k + f(\theta, x(\theta), \cdot) \right] d\theta - \\ - Q^{-1} \int_0^T \int_t^T K(s) ds \left[ \lambda \sum_{k=1}^m a_k(\theta) c_k + f(\theta, x(\theta), \cdot) \right] d\theta; \end{aligned} \quad (9)$$

$$\begin{aligned} \sum_{0 < t_i < t} I_i(x(t_i)) - Q^{-1} \sum_{0 < t_i < T} \int_{t_i}^T K(t) dt I_i(x(t_i)) = \\ = Q^{-1} \sum_{0 < t_i < t} \left( A + \int_0^{t_i} K(t) dt \right) I_i(x(t_i)) - \sum_{t < t_{i+1} < T} Q^{-1} \int_{t_i}^T K(t) dt I_i(x(t_i)). \end{aligned} \quad (10)$$

Taking into account (9) and (10), from the presentation (8) we obtain the following integral equation:

$$x(t) = Q^{-1}B + \sum_{0 < t_i < t} G(t_i) I_i(x(t_i)) + \lambda \sum_{k=1}^m \int_0^T G(s) a_k(s) c_k ds + \int_0^T G(t, s) f(s, x(s), \cdot) ds, \quad (11)$$

for  $t \in (t_i, t_{i+1}]$ ,  $i = 0, 1, \dots, p$ , where:

$$G(t) = \begin{cases} Q^{-1} \left( A + \int_0^t K(s) ds \right), & 0 \leq s \leq t, \\ -Q^{-1} \int_t^T K(s) ds, & t < s \leq T. \end{cases}$$

Substituting the equation (11) into designation:

$$c_k = \int_0^T b_k(s) x(s) ds,$$

we obtain the following linear system of algebraic equations (LSAE):

$$c_k + \lambda \sum_{j=1}^m c_{kj} \Phi_{kj} = \Psi_{1k} + \Psi_{2k}(f, I_i), \quad k = \overline{1, m}, \quad (12)$$

where:

$$\Phi_{kj}(f, I_i) = \int_0^T b_k(s) \int_0^T G(\theta) a_j(\theta) d\theta ds, \quad \Psi_{1k} = Q^{-1}B \int_0^T b_k(s) ds,$$

$$\Psi_{2k}(f, I_i) = \int_0^T b_k(s) \left[ \int_0^T G(\theta) f(\theta, x(\theta), \max\{x(\tau) | \tau \in [h_1; h_2]\}) d\theta + \sum_{0 < t_i < t} G(t_i) I_i(x(t_i)) \right] ds, \quad k = \overline{1, m}, \quad h_l = h_l(t, x(\theta)), \quad l = 1, 2. \quad (13)$$

The LSAE (12) is uniquely solvable for any finite right-hand side, if the following Fredholm condition is satisfied:

$$\Delta_k(\lambda) = \begin{vmatrix} 1 + \lambda\Phi_{k11} & \lambda\Phi_{k12} & \dots & \lambda\Phi_{k1m} \\ \lambda\Phi_{k21} & 1 + \lambda\Phi_{k22} & \dots & \lambda\Phi_{k2m} \\ \dots & \dots & \dots & \dots \\ \lambda\Phi_{km1} & \lambda\Phi_{km2} & \dots & 1 + \lambda\Phi_{kmm} \end{vmatrix} \neq 0. \quad (14)$$

Consider such regular values of parameter  $\lambda$ , for which condition (14) is satisfied. Then, solving LSAE (12), we obtain:

$$c_k = \frac{\Delta_{1k}(\lambda)}{\Delta(\lambda)} + \frac{\Delta_{2k}(\lambda, f, I_i)}{\Delta(\lambda)}, \quad (15)$$

where:

$$\Delta_{lk}(\lambda) = \begin{vmatrix} 1 + \lambda\Phi_{11} & \dots & \lambda\Phi_{1(i-1)} & \Psi_{l1} & \lambda\Phi_{1(i+1)} & \dots & \lambda\Phi_{1m} \\ \lambda\Phi_{21} & \dots & \lambda\Phi_{2(i-1)} & \Psi_{l2} & \lambda\Phi_{2(i+1)} & \dots & \lambda\Phi_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda\Phi_{m1} & \dots & \lambda\Phi_{m(i-1)} & \Psi_{lm} & \lambda\Phi_{m(i+1)} & \dots & 1 + \lambda\Phi_{mm} \end{vmatrix}, \quad l = 1, 2. \quad (16)$$

Substituting equality (15) into representation (11), we obtain the following new presentation of solution:

$$x(t) = \Theta(t; x) \equiv \chi_0 + \lambda \sum_{k=1}^m \left[ \frac{\Delta_{1k}(\lambda)}{\Delta(\lambda)} + \frac{\Delta_{2k}(\lambda, f, I_i)}{\Delta(\lambda)} \right] \chi_{1k} + \int_0^T G(s) f(s, x(s), \max\{x(\tau) | \tau \in [h_1; h_2]\}) ds + \sum_{0 < t_i < t} G(t_i) I_i(x(t_i)), \quad (17)$$

where:

$$\chi_0 = Q^{-1}B, \quad \chi_{1k} = \int_0^T G(s)a_k(s)ds, \quad k = \overline{1, m}, \quad h_l = h_l(t, x(s)), \quad l = 1, 2.$$

### 3. The questions of one value solvability

**Theorem.** Suppose the following conditions are fulfilled:

1) For all  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^n$  holds:

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq M_1(t) |x_1 - x_2| + M_2(t) |y_1 - y_2|;$$

2) For all  $t \in [0, T]$ ,  $x \in \mathbb{R}^n$  holds:

$$|h_j(t, x_1) - h_j(t, x_2)| \leq M_{3j}(t) |x_1 - x_2|, \quad j = 1, 2;$$

3) For all  $x \in \mathbb{R}^n$ ,  $i = 0, 1, \dots, p$  holds:

$$|I_i(x_1) - I_i(x_2)| \leq m_i |x_1 - x_2|;$$

4)  $\rho = S_1 + S_2 < 1$ , where:

$$S_1 = |\lambda| \sum_{k=1}^m |\chi_{1k}| \cdot |\bar{\Delta}_{2k}(\lambda)| \int_0^T |G(s)| [M_1(s) + M_2(s) (1 + M_f (M_{31}(s) + M_{32}(s)))] ds,$$

$$S_2 = |\lambda| \sum_{k=1}^m |\chi_{1k}| \cdot |\bar{\Delta}_{2k}(\lambda)| \sum_{i=1}^p |G(t_i)| m_i,$$

$$\bar{\Delta}_{2k}(\lambda) = \begin{pmatrix} 1 + \lambda\Phi_{11} & \dots & \lambda\Phi_{1(i-1)} & \bar{\Psi}_{21} & \lambda\Phi_{1(i+1)} & \dots & \lambda\Phi_{1m} \\ \lambda\Phi_{21} & \dots & \lambda\Phi_{2(i-1)} & \bar{\Psi}_{22} & \lambda\Phi_{2(i+1)} & \dots & \lambda\Phi_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda\Phi_{m1} & \dots & \lambda\Phi_{m(i-1)} & \bar{\Psi}_{2m} & \lambda\Phi_{m(i+1)} & \dots & 1 + \lambda\Phi_{mm} \end{pmatrix},$$

$$\bar{\Psi}_{2k} = \int_0^T b_k(s) ds.$$

Then, the nonlocal boundary value problem (1)–(3) has a unique solution  $x(t) \in PC([0, T], \mathbb{R}^n)$  for the regular values of parameter  $\lambda$ . This solution can be found from the following iterative process:

$$\begin{cases} x^j(t) = \Theta(t; x^{j-1}), & j = 1, 2, 3, \dots \\ x^0(t) = \chi_0 = Q^{-1}B, & t \in (t_i, t_{i+1}), i = 0, 1, 2, \dots, p. \end{cases} \quad (18)$$

Moreover, for this solution the following estimate is true:

$$\|x_1(t) - x_2(t)\|_{PC} \leq (1 - \rho)^{-1} \|Q^{-1}\| \cdot \|B_1 - B_2\|.$$

**Proof.** We consider the following operator:

$$\Theta : PC([0, T]; \mathbb{R}^n) \rightarrow PC([0, T] \times \mathbb{R}^n),$$

defined by the right-hand side of integral equation (11). Obviously, the fixed point of the operator  $\Theta$  is the unique solution to the boundary value problem (1)–(3). Using the principle of contracting operators, we show that the operator  $\Theta$  defined by equation (17), has a unique fixed point.

For the zero approximation from (18) we easily obtain that:

$$\|x^0(t)\| \leq \|Q^{-1}B\| < \infty. \quad (19)$$

For the first difference from approximation (18) we have the estimate:

$$\begin{aligned} \|x^1(t) - x^0(t)\| &\leq |\lambda| \sum_{k=1}^m \left[ \left| \frac{\Delta_{1k}(\lambda)}{\Delta(\lambda)} \right| + \left| \frac{\Delta_{2k}(\lambda, f, I_i)}{\Delta(\lambda)} \right| \right] |\chi_{1k}| + \\ &+ \int_0^T |G(s)| \cdot |f(s, x^0(s), \max \{x^0(\tau) | \tau \in [h_1^0; h_2^0]\})| ds + \sum_{i=1}^p |G(t_i)| \cdot |I_i(x^0(t_i))| \leq \\ &\leq |\lambda| \sum_{k=1}^m \left[ \left| \frac{\Delta_{1k}(\lambda)}{\Delta(\lambda)} \right| + \left| \frac{\Delta_{2k}(\lambda, f^0, I_i^0)}{\Delta(\lambda)} \right| \right] |\chi_{1k}| + S(M_f + m_I) < \infty, \quad (20) \end{aligned}$$

where:

$$f^0 = \int_0^T G(t) f(t, x^0(t), \max \{x^0(\tau) | \tau \in [h_1^0; h_2^0]\}) dt,$$

$$I_i^0 = I_i(x^0(t_i)), \quad S = \int_0^T |G(s)| ds + \sum_{i=1}^p |G(t_i)|,$$

$$M_f = \max_{t \in [0, T]} |f(t, Q^{-1}B, Q^{-1}B)|, \quad m_I = \max_{i \in \{1, 2, \dots, p\}} |I_i(Q^{-1}B)|.$$

Then, by virtue of the conditions of the theorem and (13), (16), for arbitrary  $t \in (t_i, t_{i+1}]$  we have:

$$\begin{aligned} |x^j(t) - x^{j-1}(t)| &\leq |\lambda| \sum_{k=1}^m \left| \frac{\Delta_{2k}(\lambda, f^{j-1}, I_i^{j-1}) - \Delta_{2k}(\lambda, f^{j-2}, I_i^{j-2})}{\Delta(\lambda)} \right| |\chi_{1k}| + \\ &+ \int_0^T |G(s)| \cdot |f(s, x^{j-1}(s), \max \{x^{j-1}(\tau) | \tau \in [h_1^{j-1}; h_2^{j-1}]\}) - f(s, x^{j-2}(s), \max \{x^{j-2}(\tau) | \tau \in [h_1^{j-2}; h_2^{j-2}]\})| ds + \\ &+ \sum_{i=1}^p |G(t_i)| \cdot |I_i(x^{j-1}(t_i)) - I_i(x^{j-2}(t_i))| \leq \\ &\leq |\lambda| \sum_{k=1}^m |\chi_{1k}| \cdot |\bar{\Delta}_{2k}(\lambda)| \int_0^T |G(s)| \cdot [M_1(s) \cdot |x^{j-1}(s) - x^{j-2}(s)| + \\ &+ M_2(s) \cdot |\max \{x^{j-1}(\tau) | \tau \in [h_1^{j-1}; h_2^{j-1}]\} - \max \{x^{j-2}(\tau) | \tau \in [h_1^{j-2}; h_2^{j-2}]\}|] ds + \\ &+ |\lambda| \sum_{k=1}^m |\chi_{1k}| \cdot |\bar{\Delta}_{2k}(\lambda)| \cdot \sum_{i=1}^p |G(t_i)| \cdot m_i \cdot |x^{j-1}(t_i) - x^{j-2}(t_i)| + \int_0^T |G(s)| \cdot [M_1(s) \cdot |x^{j-1}(s) - x^{j-2}(s)| + \\ &+ M_2(s) \cdot |\max \{x^{j-1}(\tau) | \tau \in [h_1^{j-1}; h_2^{j-1}]\} - \max \{x^{j-2}(\tau) | \tau \in [h_1^{j-2}; h_2^{j-2}]\}|] ds + \\ &+ \sum_{i=1}^p |G(t_i)| \cdot m_i \cdot |x^{j-1}(t_i) - x^{j-2}(t_i)|, \quad (21) \end{aligned}$$

where  $h_l^j = h_l(t, x^j(t))$ ,  $l = 1, 2$  and:

$$\bar{\Delta}_{2k}(\lambda) = \begin{pmatrix} 1 + \lambda\Phi_{11} & \dots & \lambda\Phi_{1(i-1)} & \bar{\Psi}_{21} & \lambda\Phi_{1(i+1)} & \dots & \lambda\Phi_{1m} \\ \lambda\Phi_{21} & \dots & \lambda\Phi_{2(i-1)} & \bar{\Psi}_{22} & \lambda\Phi_{2(i+1)} & \dots & \lambda\Phi_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda\Phi_{m1} & \dots & \lambda\Phi_{m(i-1)} & \bar{\Psi}_{2m} & \lambda\Phi_{m(i+1)} & \dots & 1 + \lambda\Phi_{mm} \end{pmatrix},$$

$$\bar{\Psi}_{2k} = \int_0^T b_k(s) ds.$$

By virtue of third condition of the theorem, we have:

$$\begin{aligned}
& \left| \max \left\{ x^{j-1}(\tau) \mid \tau \in [h_1^{j-1}; h_2^{j-1}] \right\} - \max \left\{ x^{j-2}(\tau) \mid \tau \in [h_1^{j-2}; h_2^{j-2}] \right\} \right| \leq \\
& \leq \left| \max \left\{ x^{j-1}(\tau) \mid \tau \in [h_1^{j-1}; h_2^{j-1}] \right\} - \max \left\{ x^{j-2}(\tau) \mid \tau \in [h_1^{j-1}; h_2^{j-1}] \right\} \right| + \\
& + \left| \max \left\{ x^{j-2}(\tau) \mid \tau \in [h_1^{j-1}; h_2^{j-1}] \right\} - \max \left\{ x^{j-2}(\tau) \mid \tau \in [h_1^{j-2}; h_2^{j-2}] \right\} \right| \leq \\
& \leq |x^{j-1}(t) - x^{j-2}(t)| + M_f \left[ |h_1(t, x^{j-1}(t)) - h_1(t, x^{j-2}(t))| + |h_2(t, x^{j-1}(t)) - h_2(t, x^{j-2}(t))| \right] \leq \\
& \leq (1 + M_f (M_{31}(t) + M_{32}(t))) |x^{j-1}(t) - x^{j-2}(t)|. \quad (22)
\end{aligned}$$

Substituting the estimate (22) into (21), we obtain:

$$\|x^j(t) - x^{j-1}(t)\|_{PC} \leq \rho \cdot \|x^{j-1}(t) - x^{j-2}(t)\|_{PC}, \quad (23)$$

where  $\rho = S_1 + S_2$  and:

$$S_1 = |\lambda| \sum_{k=1}^m |\chi_{1k}| \cdot |\bar{\Delta}_{2k}(\lambda)| \int_0^T |G(s)| \cdot [M_1(s) + M_2(s) (1 + M_f (M_{31}(s) + M_{32}(s)))] ds,$$

$$S_2 = |\lambda| \sum_{k=1}^m |\chi_{1k}| \cdot |\bar{\Delta}_{2k}(\lambda)| \sum_{i=1}^p |G(t_i)| \cdot m_i.$$

According to the last condition of the theorem,  $\rho < 1$ . Therefore, from the estimate (23) we have:

$$\|x^j(t) - x^{j-1}(t)\|_{PC} < \|x^{j-1}(t) - x^{j-2}(t)\|_{PC}. \quad (24)$$

It follows from (24) that the operator  $\Theta$  on the right-hand side of (17) is contracting. According to fixed point principle, taking into account estimates (19), (20) and (24), we conclude that the operator  $\Theta$  has a unique fixed point. Consequently, the nonlocal boundary value problem (1)–(3) has a unique solution  $x(t) \in PC([0, T], \mathbb{R}^n)$ .

Now, let us show the continuous dependence of the solution to the boundary value problem (1)–(3) on the right-hand side of condition (2). Let  $B_1, B_2 \in \mathbb{R}^n$  are two different constants and  $x_1(t), x_2(t) \in PC([0, T], \mathbb{R}^n)$  are corresponding solutions of the problem (1)–(3). Then, we have:

$$\begin{aligned}
x_1(t) - x_2(t) &= Q^{-1} [B_1 - B_2] + \lambda \sum_{k=1}^m \frac{\Delta_{2k}(\lambda, f_1, I_{1i}) - \Delta_{2k}(\lambda, f_2, I_{2i})}{\Delta(\lambda)} \chi_{1k} + \\
&+ \int_0^T G(s) \cdot |f(s, x_1(s), \max \{x_1(\tau) \mid \tau \in [h_1^1; h_2^1]\}) - f(s, x_2(s), \max \{x_2(\tau) \mid \tau \in [h_1^2; h_2^2]\})| ds + \\
&+ \sum_{i=1}^p G(t_i) [I_i(x_1(t_i)) - I_i(x_2(t_i))], \quad (25)
\end{aligned}$$

where  $h_j^k = h_j(t, x_k(t))$ ,  $j, k = 1, 2$ . Now, using the first two conditions of the theorem, from (25) we obtain:

$$\begin{aligned}
|x_1(t) - x_2(t)| &\leq Q^{-1} \|B_1 - B_2\| + |\lambda| \sum_{k=1}^m \left| \frac{\Delta_{2k}(\lambda, f_1, I_{1i}) - \Delta_{2k}(\lambda, f_2, I_{2i})}{\Delta(\lambda)} \right| |\chi_{1k}| + \\
&+ \int_0^T |G(s)| \cdot [M_1(s) \cdot |x_1(s) - x_2(s)| + M_2(s) \cdot |\max \{x_1(\tau) \mid \tau \in [h_1^1; h_2^1]\} - \max \{x_2(\tau) \mid \tau \in [h_1^2; h_2^2]\}|] ds + \\
&+ \sum_{i=1}^p |G(t_i)| \cdot m_i \cdot |x_1(t_i) - x_2(t_i)|.
\end{aligned}$$

Hence, as in the case of estimation process for (23), we obtain:

$$\|x_1(t) - x_2(t)\|_{PC} \leq \|Q^{-1}\| \|B_1 - B_2\| + \rho \cdot \|x_1(t) - x_2(t)\|_{PC}.$$

Since  $\rho < 1$ , from the last inequality, it follows that:

$$\|x_1(t) - x_2(t)\|_{PC} \leq (1 - \rho)^{-1} \|Q^{-1}\| \cdot \|B_1 - B_2\|.$$

If we put  $\|B_1 - B_2\| < \delta$  and  $\varepsilon = (1 - \rho)^{-1} \|Q^{-1}\| \cdot \delta$ , then, from the last inequality, we obtain  $\|x_1(t) - x_2(t)\|_{PC} < \varepsilon$ . The theorem is proved.

#### 4. Conclusion

The theory of differential equations plays an important role in solving applied problems. Especially, nonlocal boundary value problems for differential equations with impulsive actions have many applications in mathematical physics, mechanics and technology, in particular in nanotechnology.

In this paper, we investigated the system of first order Fredholm integro-differential equations (1) with nonlocal boundary value condition (2) and with condition (3) of impulsive effects for  $t = t_i$   $i = 1, 2, \dots, p$ ,  $0 < t_1 < t_2 < \dots < t_p < T$ . The kernel of integro-differential equation (1) is degenerate. The nonlinear right-hand side of this equation consists of the construction of nonlinear maxima. The questions of the existence and uniqueness of the solution of the boundary value problem (1)–(3) are studied. The continuous dependence of the solution on the right-hand side of the boundary condition was proved.

The results obtained in this work will allow us in the future to investigate nonlocal boundary value problems for the heat equation and the wave equation with impulsive actions. We hope that our work will stimulate the study of various boundary value problems for partial differential and integro-differential equations with impulsive actions.

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