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Original article

Periodic solutions for an impulsive system of nonlinear differential equations with maxima

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ABSTRACT In this paper, a periodical boundary value problem for a first order system of ordinary differential equations with impulsive effects and maxima is investigated. We define a nonlinear functional-integral system, the set of periodic solutions of which consides with the set of periodic solutions of the given problem. In the proof of the existence and uniqueness of the periodic solution of the obtained system, the method of compressing mapping is used.

KEYWORDS impulsive differential equations, periodical boundary value condition, successive approximations, existence and uniqueness of periodic solution.

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1. Introduction

The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses. Mathematically, this leads to an impulsive dynamical system. So, differential equations, the solutions of which are functions with first kind "discontinuities" at fixed or non-fixed times, have applications in biological, chemical and physical sciences, ecology, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. [1–5]. In particular, such kind of problems appear in biophysics at micro- and nano-scales [6–10]. Such differential equations with "discontinuities" at fixed or non-fixed times are called differential equations with impulsive effects. A lot of publications of studying on differential equations with impulsive effects, describing many natural and technical processes, are appearing [11–21].

As is known, in recent years the interest in the study of differential equations with periodical boundary conditions has increased. In the works [22–26] periodic solutions of the differential equations with impulsive effects are studied. In [27] the problems of bifurcation of positive periodic solutions of first-order impulsive differential equations are studied. In [28] the problems of stability of periodic impulsive systems are studied.

In this paper, in contrast to works [22–26], we study a periodical boundary value problem for a system of first order differential equations with impulsive effects and "maxima". The questions of the existence and uniqueness of the solution to the periodical boundary value problem are studied. In addition, the technique used in our work is constructive (see [5]) and allows the practical calculation of periodic solutions of nonlinear dynamical systems with deviations, including those with maxima.

2. Problem statement

On the interval $[0,T]$ for $t \neq t_i,\ \ i = 1,2,...,p$ we consider the questions of existence and constructive methods of finding the periodic solutions of the system of nonlinear ordinary first order differential equations with impulsive effects and maxima

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau)\,|\tau \in [t-h,t]\right\}\right),\quad 0 < h = \text{const}. \tag{1}$$

We study equation (1) with periodic condition

$$\begin{cases} x(t) = \varphi(t), \quad t \in [-h,0], \\ x(0) = x(T) \end{cases} \tag{2}$$

and nonlinear impulsive effect

$$x\left(t_i^+\right) - x\left(t_i^-\right) = F_i\left(x\left(t_i\right)\right),\quad i = 1,2,...,p, \tag{3}$$

where $0 < \bar{t} < T$, $\bar{t} \neq t_i$, $i = 1,2,...,p$, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $x, y \in X$, X is a closed bounded domain in the space $\mathbb{R}^n$, ∂X is its border, $f \in \mathbb{R}^n$, $x\left(t_i^+\right) = \lim\limits_{\nu\to 0^+} x\left(t_i + \nu\right)$, $x\left(t_i^-\right) = \lim\limits_{\nu\to 0^-} x\left(t_i - \nu\right)$ are right-hand side and left-hand side limits of function $x(t)$ at the point $t = t_i$, respectively. The function f is T-periodic, $F_i = F_{i+p}$, $t_{i+p} = t_i + T$.

$C([0,T],\mathbb{R}^n)$ is the notation of the Banach space, which consists of continuous vector functions $x(t)$, defined on the segment $[0,T]$, with values in $\mathbb{R}^n$ and with the norm

$$\| x(t) \| = \sqrt{\sum_{j=1}^{n} \max_{0\leq t\leq T} | x_j(t) |}.$$

$PC([0,T],\mathbb{R}^n)$ is the notation of the linear vector space

$$PC([0,T],\mathbb{R}^n) = \{x : [0,T] \to \mathbb{R}^n;\ x(t) \in C((t_i, t_{i+1}], \mathbb{R}^n),\ i = 1, ..., p\},$$

where $x(t_i^+)$ and $x(t_i^-)$ $(i = 0, 1, ..., p)$ exist and are bounded; $x(t_i^-) = x(t_i)$. Note, that the linear vector space $PC([0,T],\mathbb{R}^n)$ is Banach space with the following norm

$$\| x(t) \|_{PC} = \max\left\{ \| x \|_{C((t_i,t_{i+1}])},\ i = 1, 2, ..., p \right\}.$$

Formulation of problem. To find the T-periodic function $x(t) \in PC([0,\ T],\mathbb{R}^n)$, which for all $t \in [0,T]$, $t \neq t_i$, $i = 1, 2, ..., p$ satisfies the system of differential equation (1), periodic condition (2) and for $t = t_i$, $i = 1, 2, ..., p$, $0 < t_1 < t_2 < ... < t_p < T$ satisfies the nonlinear limit condition (3) and goes through x_0 at $t = 0$.

3. Reduction to an functional-integral equation

Let the function $x(t) \in PC([0,T],\mathbb{R}^n)$ is a solution of the periodic boundary value problem (1)–(3). Then by integration of the equation (1) on the intervals: $(0, t_1]$, $(t_1, t_2]$, ..., $(t_p, t_{p+1}]$, we obtain:

$$\int_0^{t_1} f(s,x,y)\,ds = \int_0^{t_1} x'(s)\,ds = x(t_1^-) - x(0^+),\ \ t \in (0, t_1],$$

$$\int_{t_1}^{t_2} f(s,x,y)\,ds = \int_{t_1}^{t_2} x'(s)\,ds = x(t_2^-) - x(t_1^+),\ \ t \in (t_1, t_2],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{t_p}^{t_{p+1}} f(s,x,y)\,ds = \int_{t_p}^{t_{p+1}} x'(s)\,ds = x(t_{p+1}^-) - x(t_p^+),\ \ t \in (t_p, t_{p+1}],$$

where $f(s,x,y) = f(t, x(t), \max\{x(\tau)\,|\tau \in [t-h,t]\})$.

Hence, taking $x(0^+) = x(0)$, $x(t_{k+1}^-) = x(t)$ into account, on the interval $(0,T]$ we have

$$\int_0^{t} f(s,x,y)\,ds = \left[x(t_1) - x(0^+)\right] + \left[x(t_2) - x(t_1^+)\right] + ... + \left[x(t) - x(t_p^+)\right] =$$

$$= -x(0) - \left[x(t_1^+) - x(t_1)\right] - \left[x(t_2^+) - x(t_2)\right] - ... - \left[x(t_p^+) - x(t_p)\right] + x(t).$$

Taking into account the condition (3), the last equality we rewrite as

$$x(t) = x(0) + \int_0^{t} f(s,x,y)\,ds + \sum_{0<t_i<t} F_i(x(t_i)). \tag{4}$$

We subordinate the function $x(t) \in PC([0,T],\mathbb{R}^n)$ in (4) to satisfy the periodic condition (2):

$$x(T) = x(0) + \int_0^{T} f(s,x,y)\,ds + \sum_{0<t_i<T} F_i(x(t_i)).$$

Hence, taking the condition (2) into account, we obtain:

$$\int_0^{T} f(s,x,y)\,ds + \sum_{0<t_i<T} F_i(x(t_i)) = 0.$$

Consequently, the differential equation (1) one can write as

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau)\,|\tau \in [t-h, t]\right\}\right) - \\ - \frac{1}{T}\int\limits_0^T f\left(t, x(t), \max\left\{x(\tau)\,|\tau \in [t-h, t]\right\}\right)\, dt - \frac{1}{T}\sum_{i=1}^{p} F_i\left(x\left(t_i\right)\right). \quad (5)$$

Then by integration of the equation (5) on the intervals: $(0, t_1]\,,\ (t_1, t_2]\,,\ \ldots,\ (t_p, t_{p+1}]$, instead (4) we obtain the following equation:

$$x(t) = x_0 + \int\limits_0^t \Big[f\left(s, x(s), \max\left\{x(\tau)\,|\tau \in [s-h, s]\right\}\right) - \\ - \frac{1}{T}\int\limits_0^T f\left(\theta, x(\theta), \max\left\{x(\tau)\,|\tau \in [\theta-h, \theta]\right\}\right)\, d\theta - \frac{1}{T}\sum_{i=1}^{p} F_i\left(x\left(t_i\right)\right)\Big]\, ds + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right). \quad (6)$$

Lemma 1. For solution of equation (6) one has the following estimate

$$\parallel x(t) - x_0 \parallel_{PC} \le M\left(\frac{T}{2} + 2p\right), \quad (7)$$

where $M = \max\left\{\parallel f\left(t, x(t), y(t)\right) \parallel;\ \max\limits_{1\le i\le p} \parallel F_i\left(t, x(t)\right) \parallel\right\}$.

Proof. We rewrite the equation (6) as

$$x(t) - x_0 = \int\limits_0^t \Big[f\left(s, x(s), y(s)\right) - \frac{1}{T}\int\limits_0^T f\left(\theta, x(\theta), y(\theta)\right)\, d\theta - \frac{1}{T}\sum_{i=1}^{p} F_i\left(x\left(t_i\right)\right)\Big]\, ds + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right) =$$

$$= \int\limits_0^t f\left(s, x(s), y(s)\right)\, ds - \frac{t}{T}\int\limits_0^t f\left(s, x(s), y(s)\right)\, ds - \frac{t}{T}\int\limits_t^T f\left(s, x(s), y(s)\right)\, ds - \frac{t}{T}\sum_{i=1}^{p} F_i\left(x\left(t_i\right)\right) + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right).$$

Hence, implies that there is true the following estimate

$$\parallel x(t) - x_0 \parallel_{PC} \le \alpha(t) \cdot \parallel f\left(t, x(t), y(t)\right) \parallel + 2p \cdot \max_{1\le i\le p} \parallel F_i\left(t, x(t)\right) \parallel, \quad (8)$$

where $\alpha(t) = 2t\left(1 - \dfrac{t}{T}\right)$. It is easy to check that from (8) follows (7). Lemma 1 is proved.

Remark. T-periodic solution $x_{\varphi(t)} = \psi(t)$ of the system (1) with initial value function $\varphi(t)$ on the initial set $[-h, 0]$ is defined by the initial value function $\varphi(t)$, which is periodical continuation of the solution $\psi(t)$ into initial set $[-h, 0]$.

Lemma 2. For the difference of two functions with maxima there holds the following estimate

$$\parallel \max\left\{x(\tau)\,|\tau \in [t-h, t]\right\} - \max\left\{y(\tau)\,|\tau \in [t-h, t]\right\} \parallel \le \parallel x(t) - y(t) \parallel + 2h\left\|\frac{\partial}{\partial t}\left[x(t) - y(t)\right]\right\|. \quad (9)$$

Proof. We use obvious true relations

$$\max\left\{x(\tau)\,|\tau \in [t-h, t]\right\} = \max\left\{\left[x(\tau) - y(\tau) + y(\tau)\right]|\tau \in [t-h, t]\right\} \le \\ \le \max\left\{\left[x(\tau) - y(\tau)\right]|\tau \in [t-h, t]\right\} + \max\left\{y(\tau)\,|\tau \in [t-h, t]\right\}.$$

Hence, we obtain

$$\max\left\{x(\tau)\,|\tau \in [t-h, t]\right\} - \max\left\{x(\tau)\,|\tau \in [t-h, t]\right\} \le \max\left\{\left[x(\tau) - y(\tau)\right]|\tau \in [t-h, t]\right\}. \quad (10)$$

We denote by t_1 and t_2 the points of $[t-h, t]$, on which the maximums of the functions $x(t)$ and $y(t)$ are reached:

$$\max\left\{x(\tau)\,|\tau \in [t-h, t]\right\} = x(t_1),\ \max\left\{y(\tau)\,|\tau \in [t-h, t]\right\} = y(t_1), \\ \max\left\{\left[x(\tau) - y(\tau)\right]|\tau \in [t-h, t]\right\} = x(t_2) - y(t_2).$$

Then, taking (10) and last equalities, we have

$$\parallel \max\left\{x(\tau)\,|\tau \in [t-h, t]\right\} - \max\left\{y(\tau)\,|\tau \in [t-h, t]\right\} - x(t) + y(t) \parallel \le \\ \le \parallel \left[x(t) - y(t)\right] - \left[x(t_1) - y(t_1)\right] \parallel + \parallel \left[x(t_2) - y(t_2)\right] - \left[x(t_1) - y(t_1)\right] \parallel. \quad (11)$$

From another side, it is obvious that, there holds the estimate

$$\parallel \left[x(\bar{t}) - y(\bar{t})\right] - \left[x(\bar{\bar{t}}) - y(\bar{\bar{t}})\right] \parallel \le h\left\|\frac{d}{dt}\left[x(t^*) - y(t^*)\right]\right\| \le h\left\|\frac{d}{dt}\left[x(t) - y(t)\right]\right\|, \quad (12)$$

where $\bar{t},\, \bar{\bar{t}} \in [t-h, t],\ t^* \in (\bar{t},\, \bar{\bar{t}})$. From the estimates (11) and (12) we come to the following estimate:

$$\| \max\{x(\tau)\,|\tau \in [t-h,t]\} - \max\{y(\tau)\,|\tau \in [t-h,t]\} - x(t) + y(t) \| \leq 2h \left\| \frac{d}{dt}[x(t) - y(t)] \right\|.$$

Therefore, it is easy to check that there holds the inequality (9) and we complete the proof of the Lemma 2.

By the BD we denote the Banach space of functions on the interval $[0,T]$ with the norm

$$\| x(t) \|_{BD} \leq \| x(t) \|_{PC} + h \, \| x'(t) \|_{PC}\,.$$

Theorem 1. Assume that for all $t \in [0,T],\ t \neq t_i,\ \ i = 1, 2, ..., p$ are fulfilled the following conditions:
1. $\max\left\{ \| f(t, x(t), y(t)) \|\,;\ \max\limits_{1 \leq i \leq p} \| F_i(t, x(t)) \| \right\} = M < \infty$;
2. $\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leq L_1 [\| x_1 - x_2 \| + \| y_1 - y_2 \|]$;
3. $\| F_i(x_1) - F_i(x_2) \| \leq L_2 \| x_1 - x_2 \|$;
4. The radius of the inscribed ball in X is greater than $M \left(\frac{T}{2} + 2p \right)$;
5. $2L_1 \left(\frac{T}{2} + h \right) + L_2 p \left(2 + \frac{h}{T} \right) < 1$.

If the system (1) has a periodic solution for all $t \in [0,T],\ t \neq t_i,\ \ i = 1, 2, ..., p$, then this solution can be founded by the system of nonlinear functional-integral equations

$$x(t, x_0) = x_0 + \int_0^t \Big[f(s, x(s, x_0), \max\{x(\tau, x_0)\,|\tau \in [s-h, s]\}) -$$

$$- \frac{1}{T} \int_0^T f(\theta, x(\theta, x_0), \max\{x(\tau, x_0)\,|\tau \in [\theta - h, \theta]\})\, d\theta - \frac{1}{T} \sum_{i=1}^{p} F_i(x(t_i, x_0)) \Big] ds + \sum_{0<t_i<t} F_i(x(t_i, x_0)). \quad (13)$$

Proof. We will show that the right-hand side of the system of equations (13) as an operator maps a ball with radius $M \cdot \left(\frac{T}{2} + 2p \right)$ into itself and is a contraction operator. So, according to the lemma 1, from (7) we have

$$\| x(t, x_0) - x_0 \|_{PC} \leq M \left(\frac{T}{2} + 2p \right). \quad (14)$$

From the equation (5) we obtain

$$\| x'(t, x_0) \|_{PC} \leq M \left(2 + \frac{p}{T} \right). \quad (15)$$

We consider a difference $x(t, x_0) - \vartheta(t, x_0)$, where functions $x(t, x_0)$ and $\vartheta(t, x_0)$ satisfy the system of equations (13). By the conditions of the theorem, from (13) we have

$$\| x(t, x_0) - \vartheta(t, x_0) \| \leq L_1 \int_0^t \Big\{ \| x(s, x_0) - \vartheta(s, x_0) \| +$$

$$+ \| \max\{x(\tau, x_0)\,|\tau \in [s-h, s]\} - \max\{\vartheta(\tau, x_0)\,|\tau \in [s-h, s]\} \| + \frac{1}{T} \int_0^T [\| x(\theta, x_0) - \vartheta(\theta, x_0) \| +$$

$$+ \| \max\{x(\tau, x_0)\,|\tau \in [\theta-h, \theta]\} - \max\{\vartheta(\tau, x_0)\,|\tau \in [\theta-h, \theta]\} \|]\, d\theta \Big\} ds +$$

$$+ \sum_{i=1}^{p} L_2 \| x(t_i, x_0) - \vartheta(t_i, x_0) \| + \sum_{0<t_i<t} L_2 \| x(t_i, x_0) - \vartheta(t_i, x_0) \| \leq$$

$$\leq 2\alpha(t) L_1 \Big[\| x(t, x_0) - \vartheta(t, x_0) \|_{PC} + h \, \| x'(t, x_0) - \vartheta'(t, x_0) \|_{PC} \Big] + 2pL_2 \| x(t, x_0) - \vartheta(t, x_0) \|_{PC} \leq$$

$$\leq 2 \left(L_1 \frac{T}{2} + pL_2 \right) \| x(t, x_0) - \vartheta(t, x_0) \|_{PC} + L_1 T\, h \, \| x'(t, x_0) - \vartheta'(t, x_0) \|_{PC}. \quad (16)$$

Similarly, by the assumptions of the theorem 1, from (5) we have

$$\| x'(t, x_0) - \vartheta'(t, x_0) \|_{PC} \leq \left(2L_1 + L_2 \frac{p}{T} \right) \| x(t, x_0) - \vartheta(t, x_0) \|_{PC} + 2L_1 h \, \| x'(t, x_0) - \vartheta'(t, x_0) \|_{PC}. \quad (17)$$

Multiplying both sides of (17) to h and the result adding to (16), we obtain

$$\| x(t, x_0) - \vartheta(t, x_0) \|_{BD} \leq \rho \cdot \| x(t, x_0) - \vartheta(t, x_0) \|_{BD}, \quad \rho = 2L_1 \left(\frac{T}{2} + h \right) + L_2 p \left(2 + \frac{h}{T} \right). \quad (18)$$

According to the last condition of the theorem 1, $\rho < 1$. So, from the estimate (18) we deduce that the operator on right-hand side of (13) is compressing. From the estimates (14), (15) and (18) implies that there exists unique fixed point $x(t, x_0) \in BD$. The theorem 1 is proved.

From the estimate (18) it is easy to obtain that for $x_0,\ \bar{x}_0 \in X$ holds

$$\| x(t, x_0) - x(t, \bar{x}_0) \|_{BD} \leq \frac{| x_0 - \bar{x}_0 |}{1 - \rho}.$$

We note that the theorem 1 one can proof by the method of successive approximations, defining iteration process as

$$x_0(t, x_0) = x_0, \ \ x_{k+1}(t, x_0) = x_0 + \int\limits_0^t \Big[f\left(s, x_k(s, x_0), \max\left\{ x_k(\tau, x_0) \,|\tau \in [s-h, s] \right\}\right) -$$

$$- \frac{1}{T} \int\limits_0^T f\left(\theta, x_k(\theta, x_0), \max\left\{ x_k(\tau, x_0) \,|\tau \in [\theta-h, \theta] \right\}\right) d\theta - \frac{1}{T} \sum_{i=1}^{p} F_i\left(x_k\left(t_i, x_0\right)\right) \Big] ds +$$

$$+ \sum_{0<t_i<t} F_i\left(x_k\left(t_i, x_0\right)\right). \quad (19)$$

Now we will show the existence of periodic solutions of the system of impulsive differential equations (1). We introduce designations:

$$\Delta(x_0) = \frac{1}{T} \int\limits_0^T f\left(t, x_\infty(t, x_0), \max\left\{ x_\infty(\tau, x_0) \,|\tau \in [t-h, t] \right\}\right) dt + \frac{1}{T} \sum_{i=1}^{p} F_i\left(x_\infty\left(t_i, x_0\right)\right), \quad (20)$$

$$\Delta_k(x_0) = \frac{1}{T} \int\limits_0^T f\left(t, x_k(t, x_0), \max\left\{ x_k(\tau, x_0) \,|\tau \in [t-h, t] \right\}\right) dt + \frac{1}{T} \sum_{i=1}^{p} F_i\left(x_k\left(t_i, x_0\right)\right), \quad (21)$$

where $x(t, x_0) = \lim\limits_{k\to\infty} x_k(t, x_0) = x_\infty(t, x_0)$ is solution of the nonlinear system (13). Therefore, $x_\infty(t, x_0)$ is the solution of the system of impulsive differential equations (1) for $\Delta(x_0) = 0$ going through x_0 at $t = 0$. Consequently, the questions of existence of solution of the system of impulsive differential equations (1) we reduce to the questions of existence of zeros of function $\Delta(x_0)$ and we solve this problem, finding zeros of the function $\Delta_k(x_0)$.

Theorem 2. Assume that

1. All conditions of the theorem 1 are fulfilled;
2. There is a natural number k such that the function $\Delta_k(x_0)$ has an isolated singular point $\Delta_k(x_0) = 0$, index of which is nonzero;
3. There is a closed convex region $X_0 \subset X$, containing a single singular point such that on the its border ∂X_0 is fulfilled estimate

$$\inf_{x \in \partial X_0} \| \Delta_k(x) \| \geq \frac{M \rho^{k+1}}{1 - \rho}. \quad (22)$$

Then the system of impulsive differential equations (1) has a periodic solution for all $t \in [0, T],\ t \neq t_i,\ i = 1, 2, ..., p$ that $x(0) \in X_0$.

Proof. Let us consider families of everywhere continuous on ∂X vector fields

$$V(\sigma, x_0) = \Delta_k(x_0) + \sigma(\Delta(x_0) - \Delta_k(x_0)),$$

which connect the fields

$$V(0, x_0) = \Delta_k(x_0), \ \ V(1, x_0) = \Delta(x_0).$$

We note that there is true the estimate

$$\| \Delta(x_0) - \Delta_k(x_0) \| \leq \frac{M \rho^{k+1}}{1 - \rho}. \quad (23)$$

Therefore, the vector field $V(\sigma, x_0)$ does not vanish anywhere on ∂X_0. Indeed, from (22) and (23) implies that

$$\| V(\sigma, x_0) \| \geq \| \Delta_k(x_0) \| - \| \Delta(x_0) - \Delta_k(x_0) \| > 0. \quad (24)$$

The fields $\Delta_k(x_0)$ and $\Delta(x_0)$ are homotopic on ∂X and the rotations of the fields homotopic on the compact are equal. Hence, taking into account (24), we conclude that the rotation of the field $\Delta(x_0)$ on the ∂X_0 is equal to the index of the singular point x_0 of the field $\Delta_k(x_0)$ and nonzero. Consequently, the vector field $\Delta(x_0)$ on the X_0 has a singular point x_0, for which $\Delta(x_0) = 0$. Therefore, the system of impulsive differential equations (1) has a periodic solution for all $t \in [0, T],\ \ t \neq t_i,\ \ i = 1, 2, ..., p$ that $x(0) \in X_0$. In addition, we note that for $x_0,\ \bar{x}_0 \in X$ from (20) and (21) we have

$$\| \Delta(x_0) \|_{PC} \leq M \left(1 + \frac{p}{T} \right),$$

$$\| \Delta(x_0) - \Delta(\bar{x}_0) \|_{BD} \leq \frac{| x_0 - \bar{x}_0 |}{1-\rho}.$$

Theorem is proved.

4. Conclusion

The theory of differential equations plays an important role in solving applied problems. Especially, nonlocal boundary value problems for differential equations with impulsive actions have many applications in mathematical physics, mechanics and technology, in particular in nanotechnology. In this paper, we investigated the system of first order differential equations (1) with periodical boundary value condition (2) and with nonlinear condition (3) of impulsive effects for $t = t_i,\ i = 1, 2, \ldots, p,\ 0 < t_1 < t_2 < \cdots < t_p < T$. The nonlinear right-hand side of this equation consists of the construction of maxima. The questions of existence and uniqueness of the T-periodic solution of the boundary value problem (1)–(3) are studied. If the system (1) has a solution for all $t \in [0, T],\ t \neq t_i,\ \ i = 1, 2, ..., p$, then it is proved that this solution can be founded by the system of nonlinear functional-integral equations (13). The questions of existence of solution of the system of impulsive differential equations (1) we reduce to the questions of existence of zeros of function $\Delta(x_0)$ and we solve this problem finding zeros of the function $\Delta_k(x_0)$ in (21).

The results obtained in this work will allow us in the future to investigate another kind periodical boundary value problems for the heat equation and the wave equation with impulsive actions.

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