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Original article

Conditions for the existence of bound states of a two-particle Hamiltonian on a three-dimensional lattice

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ABSTRACT The Hamiltonian h of the system of two quantum particles moving on a 3-dimensional lattice interacting via some attractive potential is considered. Conditions for the existence of eigenvalues of the two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^3$, $\mu \in \mathbb{R}$, associated to the Hamiltonian h, are studied depending on the energy of the particle interaction $\mu \in \mathbb{R}$ and total quasi-momentum $k \in \mathbb{T}^3$ ($\mathbb{T}^3$ – three-dimensional torus).

KEYWORDS two-particle Hamiltonian, invariant subspace, unitary equivalent operator, virtual level, multiplicity of virtual level, eigenvalue.

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1. Introduction

To manipulate ultracold atoms and a unique setting for quantum simulations of interacting many-body systems, the coherent optical fields provide a strong tool because of their high-degree controllable parameters such as optical lattice geometry, dimension, particle mass, tunneling, two-body potentials, temperature etc. (See [1–4]). However, in such manipulations, due to diffraction, there is a fundamental limit for the length scale given by the wavelength of light [5] and therefore, the corresponding models are naturally restricted to a short-range case. The recent experimental and theoretical results show that integrating plasmonic systems with cold atoms, using optical potential fields formed from the near field scattering of light by an array of plasmonic nanoparticles, allows one to considerably increase the energy scales in the implementation of Hubbard models and engineer effective long-range interaction in many body dynamics [5–7].

In [8], the spectral properties of the two-particle operator depending on total quasi-momentum were investigated. In [9], the existence conditions and positiveness of eigenvalues of the two particle Hamiltonian with short range attractive perturbation was studied with respect to the quasi-momentum k and the virtual level at the lower edge of essential spectrum.

In [10], several numerical results for the bound state energies of one and two-particle systems was presented in two adjacent 3D layers, connected through a window. The authors investigated the relation between the shape of a window and energy levels, as well as number of eigenfunction's nodal domains.

In the recent work [11], the condition was obtained for the discrete two-particle Schrödinger operator with zero-range attractive potential to have an embedded eigenvalue in the essential spectrum depending on the dimension of the lattice. In [12], the discrete spectrum of the one-dimensional discrete Laplacian with short range attractive perturbation was studied.

In general, the Schrödinger operator $h(k)$, $k \in \mathbb{T}^d$, associated to the Lattice Hamiltonian h of two arbitrary particles with some dispersion relation and short range potential interaction acts in $L_2(\mathbb{T}^d)$ as (see [13])

$$h(k) = h_0(k) - \mathbf{v}, \quad k \in \mathbb{T}^d,$$

where $h_0(k)$ is a multiplication operator by $\mathcal{E}_k(p) = \dfrac{1}{m_1}\varepsilon(p) + \dfrac{1}{m_2}\varepsilon(p-k)$ and $\mathbf{v}$ is integral operator with kernel $v(p,s) = v(p-s)$.

The existence and absence of eigenvalues of the family $h(k)$ depending on the energy of interaction and quasi-momentum k were investigated in [14] and [15] for the cases $\varepsilon(p) = \sum\limits_{i=1}^{3}(1-\cos 2p_i)$, $v(p-s) = \sum\limits_{\alpha=1}^{3}\mu_\alpha \cos(p_\alpha - s_\alpha)$

and $\varepsilon(p) = \sum_{i=1}^{3}(1-\cos 2np_i)$, $v(p-q) = \sum_{l=1}^{N}\sum_{i=1}^{3}\mu_{li}\cos l(p_i-q_i)$, respectively. The spectral properties of this operator $h(k)$ for the one dimensional case was studied in [16]. The general case when the function $\varepsilon(p)$ satisfies some conditions and $v(p-s) = \mu_0 + \sum_{\alpha=1}^{d}\mu_\alpha\cos(p_\alpha - q_\alpha)$ was investigated in [17].

In [18], the Hamiltonian $\hat{h}_{\mu\lambda}$, $\mu, \lambda \geqslant 0$, describing the motion of one quantum particle on a three-dimensional lattice in an external field was considered. The authors completely investigated the dependence of the number of eigenvalues of this operator on the interaction energy for $\mu \geqslant 0$ and $\lambda \geqslant 0$. They showed that all eigenvalues arise either from the threshold virtual level (resonance) or from the threshold eigenvalues under a variation of the interaction energy.

In [19], the authors considered the two-particle Schrödinger operator $H(k)$, ($k \in \mathbb{T}^3 \equiv (-\pi,\pi]^3$ is the total quasi-momentum of a system of two particles) corresponding to the Hamiltonian of the two-particle system on the three-dimensional lattice $\mathbb{Z}^3$. It was proved that the number $N(\mathbf{k}) \equiv N(k^{(1)}, k^{(2)}, k^{(3)})$ of eigenvalues below the essential spectrum of the operator $H(k)$ is a nondecreasing function in each $k^{(i)} \in [0,\pi]$, $i = 1,2,3$. Under some additional conditions on the potential $\hat{v}$, the monotonicity of each eigenvalue $z_n(\mathbf{k}) \equiv z_n(k^{(1)}, k^{(2)}, k^{(3)})$ of the operator $H(k)$ in $k^{(i)} \in [0,\pi]$ with other coordinates k being fixed was proved.

In this work we study the Hamiltonian h for a system of two particles on the lattice $\mathbb{Z}^3$ interacting through *attractive short-range potential* V. We investigate the existence conditions of eigenvalues and bound states of the Hamiltonians $h_\mu(k)$, $k \in \mathbb{T}^3$, associated to the Hamiltonian h. To study $h_\mu(k)$, we first construct the invariant subspaces $\mathcal{H}_l \subset L_2(\mathbb{T}^3)$, $l = \overline{1,27}$ for the operator $h_\mu(k)$. Moreover, the investigation of spectral properties for $h_\mu(k)$ is reduced to study the operator $h_{\mu,l}(k) := h_\mu(k) : \mathcal{H}_l \to \mathcal{H}_l$, $l = \overline{1,27}$. Further, eigenvalue problem for $h_{\mu,l}(k)$ is reduced to study of a compact equation of rank one, which allows one to analyze the spectrum of $h_{\mu,l}(k)$.

2. Statement of the main result

The two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^3$, $\mu \in \mathbb{R}$, associated to the Hamiltonian h for a system of two particles on the lattice $\mathbb{Z}^3$ interacting via attractive short-range potential, is a self-adjoint operator which acts in $L_2(\mathbb{T}^3)$ as

$$h_\mu(k) = h_0(k) - \mu\mathbf{v}, \quad k = (k_1,k_2,k_3) \in \mathbb{T}^3, \quad \mu \in \mathbb{R},$$

where $h_0(k)$ is a multiplication operator by

$$\mathcal{E}_k(p) = \frac{1}{m_1}\varepsilon(p) + \frac{1}{m_2}\varepsilon(p-k), \;\; \varepsilon(p) = \sum_{i=1}^{3}(1-\cos 2p_i),$$

with $\mathbf{v}$ being an integral operator with kernel

$$v(p-s) = 1 + \sum_{\alpha=1}^{3}\cos(p_\alpha - s_\alpha) + \sum_{\gamma=1}^{3}\cos(p_\alpha - s_\alpha)\cos(p_\beta - s_\beta) + \prod_{\alpha=1}^{3}\cos(p_\alpha - s_\alpha),$$

Note that by the Weyl theorem on the essential spectrum [20] the essential spectrum $\sigma_{ess}(h_\mu(k))$ of the operator $h_\mu(k)$ coincides with the spectrum of the unperturbed operator $h_0(k)$

$$\sigma_{ess}(h_\mu(k)) = \sigma(h_0(k)) = [m(k), M(k)],$$

where $m(k) = \min_{p\in\mathbb{T}^3}\mathcal{E}_k(p)$, $M(k) = \max_{p\in\mathbb{T}^3}\mathcal{E}_k(p)$.

Since $\mathbf{v} \geqslant 0$ for $\mu > 0$,

$$\sup_{\|f\|=1}(h_\mu(k)f,f) \leqslant \sup_{\|f\|=1}(h_0(k)f,f) = M(k)(f,f), \qquad f \in L_2(\mathbb{T}^3).$$

Hence, $h_\mu(k)$ does not have eigenvalues lying to the right of the essential spectrum, i.e.,

$$\sigma(h_\mu(k)) \cap (M(k),\infty) = \varnothing.$$

Similarly, for $\mu < 0$

$$\inf_{\|f\|=1}(h_\mu(k)f,f) \geqslant \inf_{\|f\|=1}(h_0(k)f,f) = m(k)(f,f), \qquad f \in L_2(\mathbb{T}^3).$$

Therefore, $h_\mu(k)$ does not have eigenvalues lying to the left of the essential spectrum, i.e.,

$$\sigma(h_\mu(k)) \cap (-\infty, m(k)) = \varnothing.$$

Let functions φ_l be defined as

$$\varphi_l(p) = \prod_{\alpha=1}^{3}\eta_l(p_\alpha), \quad \{\eta_l(p_\alpha)\} \in \{1, \cos p_\alpha, \sin p_\alpha\}, \quad \alpha\{1,2,3\}. \tag{1}$$

This system consists of a 27 orthogonal functions $\{\varphi_l\}$. The operator $\mathbf{v}$ can be expressed via the functions $\{\varphi_l(\cdot)\}$, defined in (1), in the form

$$(\mathbf{v}f)(p) = \sum_{l=1}^{27} (\varphi_l, f)\varphi_l(p). \tag{2}$$

Below, we describe the conditions for the existence of eigenvalues of $h_\mu(k)$.

Let us denote by $n(\mu)$ the number of eigenvalues (with the multiplicities) lying outside the essential spectrum of the operator $h_\mu(k)$.

Remark that for the number $n(\mu)$ of eigenvalues of $h_\mu(k)$, $\mu > 0$ (resp. $\mu < 0$) and $k \in \mathbb{T}^3$, lying to the left (resp. to the right) of the essential spectrum the following estimate is true

$$0 \leqslant n(\mu) \leqslant 27.$$

Assumption 2.1. Assume that $m = m_1 = m_2$ and $k \in \Pi$, where Π is a set of $k = (k_1, k_2, k_3) \in \mathbb{T}^3$ with $k_\alpha = -\dfrac{\pi}{2}$ or $k_\alpha = \dfrac{\pi}{2}$ for some $\alpha \in \{1, 2, 3\}$.

We divide the set Π into three subsets Π_n, $n = 1, 2, 3$, defined as follows: Π_n contains elements $k \in \Pi$ such that precisely n of their coordinates are equal to $\pm\pi/2$.

Theorem 2.1. Let the Assumption 2.1 be fulfilled. Then the following statements are true

1. For any $\mu > 0$ (resp. $\mu < 0$) and $k \in \Pi_1$, the operator $h_\mu(k)$ has at least 12 eigenvalues lying to the left (resp. to the right) of the essential spectrum.

2. For any $\mu > 0$ (resp. $\mu < 0$) and $k \in \Pi_2$, the operator $h_\mu(k)$ has at least 18 eigenvalues lying to the left (resp. to the right) of the essential spectrum.

3. For any $\mu > 0$ (resp. $\mu < 0$) and $k \in \Pi_3$, the operator $h_\mu(k)$ has 27 eigenvalues lying to the left (resp. to the right) of the essential spectrum.

We introduce the following subspaces $\mathcal{H}_l$, $l = \overline{1, 27}$, of $L_2(\mathbb{T}^3)$ as

$$\mathcal{H}_1 = \mathcal{H}^{eee}_{000},\ \mathcal{H}_2 = \mathcal{H}^{eee}_{\pi 00},\ \mathcal{H}_3 = \mathcal{H}^{eee}_{0\pi 0},\ \mathcal{H}_4 = \mathcal{H}^{eee}_{00\pi},\ \mathcal{H}_5 = \mathcal{H}^{eee}_{0\pi\pi},\ \mathcal{H}_6 = \mathcal{H}^{eee}_{\pi 0\pi},\ \mathcal{H}_7 = \mathcal{H}^{eee}_{\pi\pi 0},$$
$$\mathcal{H}_8 = \mathcal{H}^{eee}_{\pi\pi\pi},\ \mathcal{H}_9 = \mathcal{H}^{oee}_{\pi 00},\ \mathcal{H}_{10} = \mathcal{H}^{oee}_{\pi\pi 0},\ \mathcal{H}_{11} = \mathcal{H}^{oee}_{\pi 0\pi},\ \mathcal{H}_{12} = \mathcal{H}^{oee}_{\pi\pi\pi},\ \mathcal{H}_{13} = \mathcal{H}^{eoe}_{0\pi 0},\ \mathcal{H}_{14} = \mathcal{H}^{eoe}_{0\pi\pi},$$
$$\mathcal{H}_{15} = \mathcal{H}^{eoe}_{\pi\pi 0},\ \mathcal{H}_{16} = \mathcal{H}^{eoe}_{\pi\pi\pi},\ \mathcal{H}_{17} = \mathcal{H}^{ooe}_{\pi\pi 0},\ \mathcal{H}_{18} = \mathcal{H}^{ooe}_{\pi\pi\pi},\ \mathcal{H}_{19} = \mathcal{H}^{eeo}_{00\pi}, \mathcal{H}_{20} = \mathcal{H}^{eeo}_{0\pi\pi},\ \mathcal{H}_{21} = \mathcal{H}^{eeo}_{\pi 0\pi},$$
$$\mathcal{H}_{22} = \mathcal{H}^{eeo}_{\pi\pi\pi},\ \mathcal{H}_{23} = \mathcal{H}^{eoo}_{0\pi\pi},\ \mathcal{H}_{24} = \mathcal{H}^{oeo}_{\pi 0\pi},\ \mathcal{H}_{25} = \mathcal{H}^{eoo}_{\pi\pi\pi}, \mathcal{H}_{26} = \mathcal{H}^{oeo}_{\pi\pi\pi},\ \mathcal{H}_{27} = \mathcal{H}^{ooo}_{\pi\pi\pi},$$

where o, e, 0 and π denote even, odd, π-even and π-odd notions of variable, respectively. For example $\mathcal{H}^{eeo}_{0\pi\pi}$ denotes a space of functions $f(p)$ which are even with respect to each variables p_1, p_2 and odd with respect to p_3, and π-even with respect to p_1, and π-odd with respect to each variables p_2, p_3, i.e.,

$$\mathcal{H}^{eeo}_{0\pi\pi} = \{f \in L_2(\mathbb{T}^3):$$
$$f(-p_1, p_2, p_3) = f(p_1, p_2, p_3),\ f(p_1, -p_2, p_3) = f(p_1, p_2, p_3),\ f(p_1, p_2, -p_3) = -f(p_1, p_2, p_3),$$
$$f(p_1 + \pi, p_2, p_3) = f(p_1, p_2, p_3),\ f(p_1, p_2 + \pi, p_3) = -f(p_1, p_2, p_3),\ f(p_1, p_2, p_3 + \pi) = -f(p_1, p_2, p_3)\}.$$

Remark that the operator $h_\mu(k)$ is invariant with respect to $\mathcal{H}_l$, $l = \overline{1, 27}$ (See Lemma 3.1). We denote by $h_{\mu,l}(k)$ the restriction $h_\mu(k)\big|_{\mathcal{H}_l}$ of $h_\mu(k)$ to $\mathcal{H}_l$.

Note that $\varphi_l \in \mathcal{H}_l$, $l = \overline{1, 27}$. Therefore, the operator $h_{\mu,l}(k)$, $l = \overline{1, 27}$ acts in $\mathcal{H}_l$ as

$$h_{\mu,l}(k) = h_0(k) - \mu \mathbf{v}_l,$$

where

$$(\mathbf{v}_l f)(p) = (\varphi_l, f)\varphi_l(p), \quad \varphi_l \in \mathcal{H}_l, \quad l = \overline{1, 27}.$$

Then we have

$$\sigma(h_\mu(k)) = \bigcup_{l=1}^{27} \sigma(h_{\mu,l}(k)).$$

Next, we study the operator $h_{\mu,l}(k)$, $l = \overline{1, 27}$.

We set

$$\xi_l(k; z) = \int_{\mathbb{T}^3} \frac{\varphi_l^2(s)ds}{\tilde{\mathcal{E}}_k(s) - z}, \quad \varphi_l \in \mathcal{H}_l, \quad l = \overline{1, 27}, \quad z \in \mathbb{C}\backslash[m(k), M(k)], \tag{3}$$

where

$$\tilde{\mathcal{E}}_k(p) = \sum_{i=1}^{3} \Big(\frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2k_i + \frac{1}{m_2^2}\cos 2p_i}\Big).$$

If Assumption 2.1 is not fulfilled, then the integral (3) converges as $z = m(k)$ ($z = M(k)$) (see Lemma 3.2 below).

We set

$$\mu_l^0(m(k)) = \frac{1}{\xi_l(k; m(k))}, \qquad \mu_l^0(M(k)) = \frac{1}{\xi_l(k; M(k))}, \quad l = \overline{1,27}.$$

Let $C(\mathbb{T}^3)$ be the Banach space of continuous (periodic) functions on $\mathbb{T}^3$ and $G_l(z)$, $l \in \{1, 2, \dots, 27\}$ be the (Birman–Schwinger) integral operator with the kernel

$$G_l(p, q; z) = \frac{\varphi_l(p)\varphi_l(q)}{\tilde{\mathcal{E}}_{\mathbf{0}}(q) - z}, \quad z \in (-\infty, m(\mathbf{0})] \cup [M(\mathbf{0}), +\infty), \quad m(\mathbf{0}) = 0, \quad M(\mathbf{0}) = 6\frac{m_1 + m_2}{m_1 m_2}.$$

Definition 2.1. If number 1 is an eigenvalue of the operator $G_l(0)$, (resp. $G_l(M(\mathbf{0}))$) and the corresponding eigenfunction ψ_l satisfies the condition

$$\frac{\psi_l(\cdot)}{\tilde{\mathcal{E}}_{\mathbf{0}}(\cdot)} \in L_1(\mathbb{T}^3) \backslash L_2(\mathbb{T}^3), \quad \Big(\text{resp. } \frac{\psi_l(\cdot)}{\tilde{\mathcal{E}}_{\mathbf{0}}(\cdot) - M(\mathbf{0})} \in L_1(\mathbb{T}^3) \backslash L_2(\mathbb{T}^3)\Big),$$

then it means that the operator $h_{\mu,l}(\mathbf{0})$ has a virtual level at the left edge (resp. at the right edge) of the essential spectrum.

Theorem 2.2. Suppose that Assumption 2.1 are not fulfilled. Then the following statements are true

1. For any $0 < \mu < \mu_l^0(k)$ (resp. $\mu_l^0(k) < \mu < 0$) the operator $h_{\mu,l}(k)$ has no eigenvalues lying to the left (resp. to the right) of the essential spectrum.

2. Let $0 < \mu = \mu_l^0(m(\mathbf{0}))$ (resp. $\mu_l^0(M(\mathbf{0})) = \mu < 0$). If $\varphi_l(\mathbf{0}) \neq 0$, then $h_{\mu,l}(\mathbf{0})$ has a virtual level at $z = 0$ (resp. at $z = 6\frac{m_1 + m_2}{m_1 m_2}$), if $\varphi_l(\mathbf{0}) = 0$, then the number $z = 0$ (resp. $z = 6\frac{m_1 + m_2}{m_1 m_2}$) is an eigenvalue of $h_{\mu,l}(\mathbf{0})$.

3. For any $k \in \mathbb{T}^3$ and $\mu > \mu_l^0(k) > 0$ (resp. $\mu < \mu_l^0(k) < 0$), the operator $h_{\mu,l}(k)$ has unique eigenvalue lying to the left (resp. to the right) of the essential spectrum.

Theorem 2.3. Let Assumption 2.1 be fulfilled. Then the following statements are true

1. For any $\mu > 0$ (resp. $\mu < 0$) and $k \in \Pi_1$, there exist $l_1, l_2, \dots, l_{12} \in \{1, 2, \dots 27\}$ such that the operator $h_{\mu,l_i}(k)$, $i = \overline{1,12}$ has a unique eigenvalue lying to the left (resp. to the right) of the essential spectrum.

2. For any $\mu > 0$ (resp. $\mu < 0$) and $k \in \Pi_2$, there exist $l_1, l_2, \dots, l_{18} \in \{1, 2, \dots 27\}$ such that the operator $h_{\mu,l_i}(k)$, $i = \overline{1,18}$ has a unique eigenvalue lying to the left (resp. to the right) of the essential spectrum.

3. For any $\mu > 0$ (resp. $\mu < 0$), $k \in \Pi_3$ and $l \in \{1, 2, \dots 27\}$ the operator $h_{\mu,l}(k)$ has unique eigenvalue lying to the left (resp. to the right) of the essential spectrum.

Remark 2.1. Note that Theorem 2.2, 2) shows that the number $z = 0$ (respectively, $z = 6\frac{m_1 + m_2}{m_1 m_2}$) might be a virtual level or an eigenvalue or a virtual level and an eigenvalue for the operator $h_\mu(\mathbf{0})$. For the case $\mu = \mu_1^0(m(\mathbf{0}))$ or $\mu = \mu_8^0(m(\mathbf{0}))$, number $z = 0$ is a simple virtual level of $h_\mu(\mathbf{0})$ with

$$f_1(p) = \frac{1}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)} \in L_1(\mathbb{T}^3) \backslash L_2(\mathbb{T}^3) \quad \text{or} \quad f_8(p) = \frac{\cos p_1 \cos p_2 \cos p_3}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)} \in L_1(\mathbb{T}^3) \backslash L_2(\mathbb{T}^3).$$

For the case $\mu = \mu_2^0(m(\mathbf{0})) = \mu_3^0(m(\mathbf{0})) = \mu_4^0(m(\mathbf{0}))$ or $\mu = \mu_5^0(m(\mathbf{0})) = \mu_6^0(m(\mathbf{0})) = \mu_7^0(m(\mathbf{0}))$, number $z = 0$ is a virtual level of $h_\mu(\mathbf{0})$ with multiplicity 3 with

$$f_{1+i}(p) = \frac{\cos p_i}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)} \in L_1(\mathbb{T}^3) \backslash L_2(\mathbb{T}^3) \quad \text{or} \quad f_{4+i}(p) = \frac{\cos p_\alpha \cos p_\beta}{\tilde{\mathcal{E}}_{\mathbf{0}}(p)} \in L_1(\mathbb{T}^3) \backslash L_2(\mathbb{T}^3),$$

$\{i, \alpha, \beta\} = \{1, 2, 3\}$.

3. Proof of the main results

Consider the operator $\tilde{h}_\mu(k)$ acting in $L_2(\mathbb{T}^3)$ by the formula

$$\tilde{h}_\mu(k) = \tilde{h}_0(k) - \mu \mathbf{v},$$

where $\tilde{h}_0(k)$ is the operator of multiplication by the function $\tilde{\mathcal{E}}_k(\cdot)$.

The operator $h_\mu(k)$ is unitary equivalent to the operator $\tilde{h}_\mu(k)$ (See Lemma 2 in [15]). The equivalence is performed by the unitary operator $U : L_2(\mathbb{T}^3) \to L_2(\mathbb{T}^3)$ as $\tilde{h}_\mu(k) = U^{-1} h_\mu(k) U$, where

$$(Uf)(p) = f(p - \frac{1}{2}\theta(k)),$$

$$\theta(k) = (\theta_1(k_1), \theta_2(k_2), \theta_3(k_3)), \quad \theta_i(k_i) = \arccos \frac{\frac{1}{m_1} + \frac{1}{m_2}\cos 2k_i}{\sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2k_i + \frac{1}{m_2^2}}}, \quad i = 1, 2, 3.$$

Lemma 3.1. 1. The following equality holds

$$L_2(\mathbb{T}^3) = \bigoplus_{l=1}^{27} \mathcal{H}_l. \tag{4}$$

2. The operator $h_\mu(k)$ is invariant with respect to $\mathcal{H}_l$, $l = \overline{1,27}$, i.e., $h_\mu(k) : \mathcal{H}_l \to \mathcal{H}_l$.

Proof. 1. For brevity, let us introduce some notations: o, e, 0 and π denote even, odd, π-even and π-odd notions of variables, respectively. For example, $\mathcal{H}^e$ denotes a space of functions $f(p)$ which are even with respect to variable p_1, similarly, $\mathcal{H}_0^{ee}$ denotes a space of functions $f(p)$ which are even with respect to each variables p_1, p_2 and π-even with respect to p_1.

We represent $f \in L_2(\mathbb{T}^3)$ as

$$f(p_1,p_2,p_3) = f^e(p_1,p_2,p_3) + f^o(p_1,p_2,p_3),$$

where

$$f^e(p_1,p_2,p_3) = \frac{f(p_1,p_2,p_3) + f(-p_1,p_2,p_3)}{2} \in \mathcal{H}^e,$$
$$f^o(p_1,p_2,p_3) = \frac{f(p_1,p_2,p_3) - f(-p_1,p_2,p_3)}{2} \in \mathcal{H}^o.$$

It is clear that $L_2(\mathbb{T}^3) = \mathcal{H}^e \oplus \mathcal{H}^o$.

Similarly, we represent the functions f^e and f^o as

$$f^e(p_1,p_2,p_3) = f^{ee}(p_1,p_2,p_3) + f^{eo}(p_1,p_2,p_3)$$

and

$$f^o(p_1,p_2,p_3) = f^{oe}(p_1,p_2,p_3) + f^{oo}(p_1,p_2,p_3),$$

where

$$f^{ee}(p_1,p_2,p_3) = \frac{f^e(p_1,p_2,p_3) + f^e(p_1,-p_2,p_3)}{2} \in \mathcal{H}^{ee},$$
$$f^{eo}(p_1,p_2,p_3) = \frac{f^e(p_1,p_2,p_3) - f^e(p_1,-p_2,p_3)}{2} \in \mathcal{H}^{eo},$$
$$f^{oe}(p_1,p_2,p_3) = \frac{f^o(p_1,p_2,p_3) + f^o(p_1,-p_2,p_3)}{2} \in \mathcal{H}^{oe},$$
$$f^{oo}(p_1,p_2,p_3) = \frac{f^o(p_1,p_2,p_3) - f^o(p_1,-p_2,p_3)}{2} \in \mathcal{H}^{oo}.$$

Then $\mathcal{H}^e = \mathcal{H}^{ee} \oplus \mathcal{H}^{eo}$ and $\mathcal{H}^o = \mathcal{H}^{oe} \oplus \mathcal{H}^{oo}$.

Arguing similarly step by step we obtain the equality of the direct sum of subspaces

$$L_2(\mathbb{T}^3) = \mathcal{H}^{eee} \oplus \mathcal{H}^{eeo} \oplus \mathcal{H}^{eoe} \oplus \mathcal{H}^{oee} \oplus \mathcal{H}^{eoo} \oplus \mathcal{H}^{oeo} \oplus \mathcal{H}^{ooe} \oplus \mathcal{H}^{ooo}. \tag{5}$$

Each subspace in (5) is represented via the direct sum of subspaces defined as combination of π-even and π-odd functions

$$\mathcal{H}^{eee} = \mathcal{H}^{eee}_{000} \oplus \mathcal{H}^{eee}_{\pi 00} \oplus \mathcal{H}^{eee}_{0\pi 0} \oplus \mathcal{H}^{eee}_{00\pi} \oplus \mathcal{H}^{eee}_{0\pi\pi} \oplus \mathcal{H}^{eee}_{\pi 0\pi} \oplus \mathcal{H}^{eee}_{\pi\pi 0} \oplus \mathcal{H}^{eee}_{\pi\pi\pi},$$
$$\mathcal{H}^{eeo} = \mathcal{H}^{eeo}_{00\pi} \oplus \mathcal{H}^{eeo}_{0\pi\pi} \oplus \mathcal{H}^{eeo}_{\pi 0\pi} \oplus \mathcal{H}^{eeo}_{\pi\pi\pi}, \quad \mathcal{H}^{eoe} = \mathcal{H}^{eoe}_{0\pi 0} \oplus \mathcal{H}^{eoe}_{0\pi\pi} \oplus \mathcal{H}^{eoe}_{\pi\pi 0} \oplus \mathcal{H}^{eoe}_{\pi\pi\pi},$$
$$\mathcal{H}^{oee} = \mathcal{H}^{oee}_{\pi 00} \oplus \mathcal{H}^{oee}_{\pi\pi 0} \oplus \mathcal{H}^{oee}_{\pi 0\pi} \oplus \mathcal{H}^{oee}_{\pi\pi\pi}, \quad \mathcal{H}^{eoo} = \mathcal{H}^{eoo}_{0\pi\pi} \oplus \mathcal{H}^{eoo}_{\pi\pi\pi},$$
$$\mathcal{H}^{oeo} = \mathcal{H}^{oeo}_{\pi 0\pi} \oplus \mathcal{H}^{oeo}_{\pi\pi\pi}, \qquad \mathcal{H}^{ooe} = \mathcal{H}^{ooe}_{\pi\pi 0} \oplus \mathcal{H}^{ooe}_{\pi\pi\pi}, \qquad \mathcal{H}^{ooo} = \mathcal{H}^{ooo}_{\pi\pi\pi}.$$

Substituting these equalities into (5), we obtain (4).

2. By (2), the operator $\mathbf{v}$ can be expressed via $\mathbf{v}_l$ as

$$(\mathbf{v}f)(p) = \sum_{l=1}^{27} (\mathbf{v}_l f)(p), \quad (\mathbf{v}_l f)(p) = (\varphi_l, f)\varphi_l(p), \quad \varphi_l \in \mathcal{H}_l, \quad l = \overline{1,27}.$$

Since $\{\varphi_l(\cdot)\}$, $l = \overline{1,27}$ is an orthogonal system in $L_2(\mathbb{T}^3)$, $\mathbf{v} : \mathcal{H}_l \to \mathcal{H}_l$. One can see that $\tilde{\mathcal{E}}_k(p)\varphi_l(p) \in \mathcal{H}_l$. Hence, $h_\mu(k) : \mathcal{H}_l \to \mathcal{H}_l$. □

The following lemma is proven in [14]

Lemma 3.2. Suppose Assumption 2.1 does not hold. Then the integral

$$\int_{\mathbb{T}^3} \frac{\varphi(s)ds}{\tilde{\mathcal{E}}_k(s) - m(k)}$$

converges for any $\varphi \in C(\mathbb{T}^3)$.

Lemma 3.3. A number z, $z \in \mathbb{C} \setminus [m(k), M(k)]$, is an eigenvalue of $h_{\mu,l}(k)$ iff $\Delta_l(\mu,k;z) = 0$, where

$$\Delta_l(\mu,k;z) = 1 - \mu\xi_l(k;z). \tag{6}$$

Proof of lemma 3.3. Let $z \in \mathbb{C} \setminus [m(k), M(k)]$ be an eigenvalue of $h_{\mu,l}(k)$ and f_l, $l = \overline{1,27}$ be the corresponding eigenfunction, i.e., the equation

$$h_{\mu,l}(k) f_l = z f_l$$

has a nontrivial solution f_l. Then

$$f_l = \mu\big(r_0(k,z)\mathbf{v}_l\big) f_l, \quad l = \overline{1,27}, \tag{7}$$

where $r_0(k,z)$ is a multiplication operator by the function $\dfrac{1}{\tilde{\mathcal{E}}_k(p) - z}$. Denote

$$\psi_l = (\varphi_l, f_l). \tag{8}$$

Then equation (7) can be represented as

$$f_l(p) = \frac{\mu \varphi_l(p)}{\tilde{\mathcal{E}}_k(p) - z} \psi_l. \tag{9}$$

By substituting (9) in (8), we obtain the following equation

$$\psi_l = \mu \int\limits_{\mathbb{T}^3} \frac{\varphi_l^2(s) ds}{\tilde{\mathcal{E}}_k(s) - z} \psi_l, \quad l = \overline{1,27}.$$

If $z \in \mathbb{C} \setminus [m(k), M(k)]$ is an eigenvalue of the operator $h_{\mu,l}(k)$ then $\Delta_l(\mu, k; z) = 0$.

Conversely, let $\Delta_l(\mu, k; z) = 0$ with $z \in \mathbb{C} \setminus [m(k), M(k)]$, i.e.,

$$1 - \mu \xi_l(k; z) = 0.$$

Then the function

$$\psi_l(p) = \frac{\varphi_l(p)}{\tilde{\mathcal{E}}_k(p) - z}$$

is an eigenfunction of the operator $h_{\mu,l}(k)$ corresponding to the eigenvalue $z \in \mathbb{C} \setminus [m(k), M(k)]$.

□

Lemma 3.3 gives the following result.

Corollary 3.1. A number z, $z \in \mathbb{C} \setminus [m(k), M(k)]$, is an eigenvalue of $h_\mu(k)$ iff $\Delta(\mu, k; z) = 0$, where

$$\Delta(\mu, k; z) = \prod_{l=1}^{27} \Delta_l(\mu, k; z).$$

Further we prove the main results for $\mu > 0$. The case $\mu < 0$ will be proven in a similar way.

Proof of Theorem 2.2. By Lemma 3.2, the integral

$$\int\limits_{\mathbb{T}^3} \frac{\varphi_l^2(s) ds}{\tilde{\mathcal{E}}_k(s) - m(k)}$$

converges for any $\varphi_l \in \mathcal{H}_l$, $l = \overline{1,27}$.

1. The function $\Delta_l(\cdot, k; \cdot)$ is monotonically decreasing for $z \in (-\infty, m(k))$ $\big(\mu \in (0, \infty)\big)$ for any fixed $\mu > 0$ $\big(z < m(k)\big)$. Then we have

$$\Delta_l(\mu, k; z) > \Delta_l(\mu, k; m(k)) > \Delta_l(\mu_l^0(k), k; m(k)) = 0 \quad \text{for all} \quad \mu \in (0, \mu_l^0(k)).$$

According to Lemma 3.3, the operator $h_{\mu,l}(k)$, $l = \overline{1,27}$ has no eigenvalues lying to the left of the essential spectrum.

2. Let $z = 0$ and $\mu = \mu_l^0(m(\mathbf{0}))$, $\mathbf{0} = (0,0,0) \in \mathbb{T}^3$. Then

$$\Delta_l\big(\mu_l^0(m(\mathbf{0})), \mathbf{0}; 0\big) = 1 - \mu_l^0\big(m(\mathbf{0})\big) \int\limits_{\mathbb{T}^3} \frac{\varphi_l^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s) - m(\mathbf{0})} = 0.$$

Then the function

$$f_l(p) = \frac{\varphi_l(p)}{\tilde{\mathcal{E}}_{\mathbf{0}}(p) - m(\mathbf{0})}, \quad l = \overline{1,27}.$$

is a solution of the equation $h_{\mu,l}(\mathbf{0}) f_l = 0$. Indeed,

$$h_{\mu,l}(\mathbf{0}) f_l = \varphi_l(p)\Big(1 - \mu_l^0\big(m(\mathbf{0})\big) \int\limits_{\mathbb{T}^3} \frac{\varphi_l^2(s) ds}{\tilde{\mathcal{E}}_{\mathbf{0}}(s) - m(\mathbf{0})}\Big) = 0.$$

Note that from the equation (1), we have

$$\varphi_l(\mathbf{0}) = \prod_{\alpha=1}^{3} \eta_l(\mathbf{0}) \neq 0, \;\; l = \overline{1,8}, \qquad \varphi_l(\mathbf{0}) = \prod_{\alpha=1}^{3} \eta_l(\mathbf{0}) = 0, \;\; l = \overline{9,27}.$$

Therefore,

$$f_l \in L_1(\mathbb{T}^3)\backslash L_2(\mathbb{T}^3),\ \ l=\overline{1,8} \qquad f_l \in L_2(\mathbb{T}^3),\ \ l=\overline{9,27}.$$

This yields that $h_{\mu,l}(\mathbf{0})$ has a virtual level at $z=0$ for any $l=\overline{1,8}$ and $z=0$ is an eigenvalue of $h_{\mu,l}(\mathbf{0})$ for any $l=\overline{9,27}$.

3. Let $\mu > \mu_l^0(k)$. Then

$$\lim_{z\to m(k)} \Delta_l(\mu,k;z) = \Delta_l(\mu,k;m(k)) = 1 - \frac{\mu}{\mu_l^0(k)} < 0.$$

Note that

$$\lim_{z\to -\infty} \Delta_l(\mu,k;z) = 1.$$

Then from the continuity and monotonicity of $\Delta_l(\mu,k;\cdot)$ in $(-\infty,m(k))$, we have that there exists unique $z_l \in (-\infty,m(k))$ such that

$$\Delta_l(\mu,k;z_l)=0.$$

According to Lemma 3.3, the operator $h_{\mu,l}(k)$ has unique eigenvalue lying to the left of the essential spectrum.

□

Proof of Theorem 2.3. 1. We prove theorem for the case $k\in\Pi_1$, $k_1=\pm\frac{\pi}{2}$. Then the function $\tilde{\mathcal{E}}_k(\cdot)$ does not depend of p_1 is expressed as

$$\tilde{\mathcal{E}}_k(p) = \frac{6}{m} - \sum_{i=2}^{3}\frac{1}{m}\sqrt{2+2\cos 2k_i}\cos 2p_i.$$

We separate the functions $\xi_l(k;\cdot)$ with $\varphi_l(p_1,0,0)\neq 0$. There are 12 such functions and after integrating them with respect to s_1 they can be represented as

$$\xi_1(k;z) = 2\pi\int\limits_{\mathbb{T}^2}\frac{ds}{\tilde{\mathcal{E}}_k(s)-z},\quad \xi_2(k;z)=\xi_9(k;z)=\pi\int\limits_{\mathbb{T}^2}\frac{ds}{\tilde{\mathcal{E}}_k(s)-z},$$

$$\xi_{i+1}(k;z) = 2\pi\int\limits_{\mathbb{T}^2}\frac{\cos^2 s_i ds}{\tilde{\mathcal{E}}_k(s)-z},\quad \xi_{i+3}(k;z)=\xi_{i+8}(k;z)=\pi\int\limits_{\mathbb{T}^2}\frac{\cos^2 s_i ds}{\tilde{\mathcal{E}}_k(s)-z},\quad i=2,3,$$

$$\xi_7(k;z) = 2\pi\int\limits_{\mathbb{T}^2}\frac{\cos^2 s_2\cos^2 s_3 ds}{\tilde{\mathcal{E}}_k(s)-z},\quad \xi_8(k;z)=\xi_{12}(k;z)=\pi\int\limits_{\mathbb{T}^2}\frac{\cos^2 s_2\cos^2 s_3 ds}{\tilde{\mathcal{E}}_k(s)-z}.$$

Since $(\tilde{\mathcal{E}}_k(p)-m(k)) = O(p^2)$ as $|p|\to 0$, the last equations give

$$\lim_{z\to m(k)}\Delta_l(\mu,k;z) = -\infty.$$

According to the continuity and monotonicity of $\Delta_l(\mu,k;\cdot)$ in $(-\infty,m(k))$ and

$$\lim_{z\to -\infty}\Delta_l(\mu,k;z)=1,$$

there exists unique $z_l\in(-\infty,m(k))$ such that

$$\Delta_l(\mu,k;z_l)=0,\quad l=\overline{1,12}.$$

The cases $k_i=\pm\frac{\pi}{2}$, $i=2,3$ can be considered in a similar way.

2. We prove theorem for the case $k\in\Pi_2$ with $k_1=k_2=\pm\frac{\pi}{2}$. The function $\tilde{\mathcal{E}}_k(\cdot)$ does not depend of p_1, p_2 and is expressed as

$$\tilde{\mathcal{E}}_k(p) = \frac{6}{m} - \frac{1}{m}\sqrt{2+2\cos 2k_3}\cos 2p_3.$$

Then there exist 18 functions $\xi_l(k;\cdot)$, $l=\overline{1,18}$ with $\varphi_l(p_1,p_2,0)\neq 0$. These functions are represented via integrals with respect to s_3 and contain a numerator function $\tilde{\varphi}_l(s_3)$ with $\tilde{\varphi}_l(0)\neq 0$. Since $(\tilde{\mathcal{E}}_k(p_3)-m(k)) = O(p_3^2)$ as $p_3\to 0$, the last equations give

$$\lim_{z\to m(k)}\xi_l(k;z)=+\infty,\qquad \lim_{z\to m(k)}\Delta_l(\mu,k;z)=-\infty,\quad l=\overline{1,18}.$$

Hence there exists unique $z_l\in(-\infty,m(k))$ such that

$$\Delta_l(\mu,k;z_l)=0,\quad l=\overline{1,18}.$$

The remaining cases with $k\in\Pi_2$ are proved in a similar way.

3. The case $k\in\Pi_3$ can also be considered by similar discussions as in parts 1) and 2). □

Theorem 2.3 leads to Theorem 2.1.

4. Conclusion

We investigate the existence conditions for eigenvalues of the two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^3$, $\mu \in \mathbb{R}$ corresponding to the Hamiltonian of the two-particle system on the three-dimensional lattice, where $h_\mu(k)$ is considered as a perturbation of free Hamiltonian $h_0(k)$ by the potential operator $\mu\mathbf{v}$ with rank 27.

To study spectral properties of $h_\mu(k)$, we first constructed the invariant subspaces $\mathcal{H}_l \subset L_2(\mathbb{T}^3)$, $l = \overline{1,27}$ for the operator $h_\mu(k)$. Moreover, investigation of the spectral properties of the operator $h_\mu(k)$ is reduced to the study of the operator $h_{\mu,l}(k) := h_\mu(k) : \mathcal{H}_l \to \mathcal{H}_l$, $l = \overline{1,27}$. Further, eigenvalue problem for $h_{\mu,l}(k)f = zf, z \notin \sigma_{ess}(h_\mu(k))$ is reduced to the study of an integral operator $\mu G_l(z)$ of rank one. This allowed us to analyze the eigenvalue problem of $h_{\mu,l}(k)$ for any $\mu \in \mathbb{R}$.

Particularly, if $k = \mathbf{0}$, then there exist the numbers $\mu_l^0(m(\mathbf{0})) > 0$ and $\mu_l^0(M(\mathbf{0})) < 0$, $l = \overline{1,27}$ such that

(i) for any μ with $0 < \mu < \mu_l^0(\mathbf{0})$ (resp. $\mu_l^0(\mathbf{0}) < \mu < 0$) the operator $h_{\mu,l}(\mathbf{0})$ has no eigenvalues lying to the left (resp. to the right) of the essential spectrum;

(ii) for $\mu = \mu_l^0(m(\mathbf{0}))$ (resp. $\mu_l^0(M(\mathbf{0})) = \mu$), if $\varphi_l(\mathbf{0}) \neq 0$, then $h_{\mu,l}(\mathbf{0})$ has a virtual level at $z = 0$ (resp. at $z = 6\dfrac{m_1+m_2}{m_1 m_2}$), if $\varphi_l(\mathbf{0}) = 0$, then the number $z = 0$ (resp. $z = 6\dfrac{m_1+m_2}{m_1 m_2}$) is an eigenvalue of $h_{\mu,l}(\mathbf{0})$;

(iii) for any μ, $\mu > \mu_l^0(\mathbf{0}) > 0$ (resp. $\mu < \mu_l^0(\mathbf{0}) < 0$), the operator $h_{\mu,l}(\mathbf{0})$ has unique eigenvalue lying to the left (resp. to the right) of the essential spectrum.

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