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Original article

Inverse problem for Fredholm integro-differential equation with final redefinition conditions at the end of the interval

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ABSTRACT The questions of solvability and construction of solutions of an inverse problem for second-order Fredholm integro-differential equation with degenerate kernel, final conditions at the end of the interval, two parameters, and two redefinition data are considered. The sets of regular parameter values are determined and the corresponding solutions are constructed. The specific features of the inverse problem are studied. Criteria for the unique solvability of the posed inverse problem are established.

KEYWORDS integro-differential equation, degenerate kernel, final conditions at the end of a segment, parameters, redefinition data, solvability.

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1. Problem statement

The differential and integro-differential equations have applications in biological, chemical and physical sciences, ecology, biotechnology, industrial robotics, pharmacokinetics, biophysics at micro- and nano-scales [1–13]. Today, for ordinary integro-differential equations, new problems are posed and a large number of papers, devoting to study of integro-differential equations, are published. Problems with nonlocal conditions for differential and integro-differential equations were considered in [14–35]. Integro-differential equations with degenerate kernel were considered in [36–40].

In this paper, we study the solvability of the inverse problem for second-order ordinary Fredholm integro-differential equation with degenerate kernel, two parameters, and final conditions at the end of the interval. This paper differs from papers mentioned above in requirement of finding two unknown redefinition data. This inverse problem has features related with the corresponding direct problem. Let us describe the latter one. We consider on the segment $[0;T]$ integro-differential equation of the form

$$u''(t) + \left(\lambda^2 - \alpha(t)\right) u(t) = \nu \int\limits_0^T K(t,s)\, u(s)\, ds, \tag{1}$$

where $T, T>0$, is given real number, $\lambda, \lambda>0$, is real parameter, ν is real nonzero parameter, $\alpha(t) \in C\,[0;T]$ is positive function, $K(t,s) = \sum\limits_{i=1}^{k} a_i(t)\, b_i(s)$, $a_i(t),\, b_i(s) \in C\,[0;T]$. It is assumed that the systems of functions $\{a_i(t)\}$ and $\{b_i(s)\}$, $i=\overline{1,k}$ are linear independent.

We consider equation (1) with the following conditions

$$u\,(T) = \varphi_1, \quad u'(T) = \varphi_2, \tag{2}$$

$$u(t_1) = \psi_1, \quad u'(t_1) = \psi_2, \tag{3}$$

where $0 < t_1 < T < \infty$, $\varphi_j = \mathrm{const}$, φ_j are constant quantities of redefinition, $\psi_j = \mathrm{const}$, $j = 1,2$. The choice of conditions (2) with the final data is related to the fact that in many practical applications, it is not possible to determine the initial conditions. For example, when studying the technological process of aluminum production, before the start of the production cycle, the raw material passes through firing and the state of the raw material at the beginning of the production cycle is not known. However, the final expected state of the output will be known or we can find it from known intermediate state.

Formulation of the problem. It is required to find a triple of unknowns

$$\left\{u(t) \in C^2[0;T],\ \varphi_i \in \mathbb{R},\ i=1,2\right\},$$

where the first one is a function satisfying equation (1), the second and the third are values from conditions (2) and (3).

Note that the problem is formulated in such a way that the direct problem (1), (2) has a unique solution for all values of the parameter λ, and the inverse problem (1)–(3) has a unique solution only for certain values of this parameter λ. In addition, the second parameter ν also plays an important role in the issue of solvability.

2. Solution of the direct problem (1), (2)

Taking into account the degeneracy of the kernel, we rewrite equation (1) in the following form

$$u''(t) + \lambda^2 u(t) = \nu \sum_{i=1}^{k} a_i(t)\, \tau_i + \alpha(t)\, u(t), \tag{4}$$

where

$$\tau_i = \int_0^T b_i(s)\, u(s)\, ds. \tag{5}$$

Solving the inhomogeneous differential equation (4) by the method of variation of arbitrary constants, we obtain the representation

$$u(t) = A_1 \cos \lambda t + A_2 \sin \lambda t + \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \int_0^t \sin \lambda (t-s)\, a_i(s)\, ds + \frac{1}{\lambda} \int_0^t \sin \lambda (t-s)\, \alpha(s)\, u(s)\, ds, \tag{6}$$

where A_1, A_2 are yet arbitrary constants. By differentiating (6) one time, we obtain

$$u'(t) = -\lambda A_1 \sin \lambda t + \lambda A_2 \cos \lambda t + \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \int_0^t \lambda \cos \lambda (t-s)\, a_i(s)\, ds + \frac{1}{\lambda} \int_0^t \lambda \cos \lambda (t-s)\, \alpha(s)\, u(s)\, ds. \tag{7}$$

To find the unknown coefficients, we use the final conditions (2). Then, from representations (6) and (7) we arrive at a system of algebraic equations (SAE)

$$\begin{cases} A_1 \cos \lambda T + A_2 \sin \lambda T = \gamma_1, \\ -A_1 \sin \lambda T + A_2 \cos \lambda T = \gamma_2, \end{cases} \tag{8}$$

where

$$\gamma_1 = \varphi_1 - \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \beta_{1i} - \frac{1}{\lambda} \int_0^T \sin \lambda (T-s)\, \alpha(s)\, u(s)\, ds, \tag{9}$$

$$\gamma_2 = \varphi_2 - \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \beta_{2i} - \frac{1}{\lambda} \int_0^T \cos \lambda (T-s)\, \alpha(s)\, u(s)\, ds, \tag{10}$$

$$\beta_{1i} = \int_0^T \sin \lambda (T-s)\, a_i(s)\, ds, \quad \beta_{2i} = \int_0^T \cos \lambda (T-s)\, a_i(s)\, ds.$$

For the unique solvability of SAE (8), the condition

$$\delta_0 = \begin{vmatrix} \cos \lambda T & \sin \lambda T \\ -\sin \lambda T & \cos \lambda T \end{vmatrix} \neq 0$$

should be fulfilled. Since $\delta_0 = 1$, this condition is fulfilled for all values of the parameter λ. Consequently, SAE (8) has the unique solution

$$A_1 = \delta_1 = \begin{vmatrix} \gamma_1 & \sin \lambda T \\ \gamma_2 & \cos \lambda T \end{vmatrix} = \varphi_1 \cos \lambda T - \varphi_2 \sin \lambda T +$$

$$+ \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \int_0^T \sin \lambda s\, a_i(s)\, ds + \frac{1}{\lambda} \int_0^T \sin \lambda s\, \alpha(s)\, u(s)\, ds, \tag{11}$$

$$A_2 = \delta_2 = \begin{vmatrix} \cos\lambda T & \gamma_1 \\ -\sin\lambda T & \gamma_2 \end{vmatrix} = \varphi_1 \sin\lambda T + \varphi_2 \cos\lambda T +$$

$$+ \frac{\nu}{\lambda}\sum_{i=1}^{k}\tau_i \int_0^T \cos\lambda s\, a_i(s)\, d s + \frac{1}{\lambda}\int_0^T \cos\lambda s\, \alpha(s)\, u(s)\, ds. \quad (12)$$

Substituting (11) and (12) into representation (6), we obtain

$$u\,(t) = \varphi_1\,\chi_1(t) + \varphi_2\,\chi_2(t) + \frac{\nu}{\lambda}\sum_{i=1}^{k}\tau_i\,\chi_{3i}(t) + \frac{1}{\lambda}\int_0^t H(t,s,\lambda)\,\alpha(s)\,u(s)\,ds, \quad (13)$$

where

$$\chi_1(t) = \cos\lambda\,(T-t) - \sin\lambda\,(T-t), \quad \chi_2(t) = \cos\lambda\,(T+t) - \sin\lambda\,(T-t),$$

$$\chi_{3i}(t) = \int_0^T H(t,s,\lambda)\,a_i(s)\,ds,$$

$$H(t,s,\lambda) = \begin{cases} \sin\lambda\,(t+s), & t < s \le T, \\ \sin\lambda\,(t-s) + \cos\lambda t\,\sin\lambda s + \lambda\,\sin\lambda t\,\sin\lambda s, & 0 \le s < t. \end{cases}$$

Although function (13) is a solution to the direct problem (1), (2), it contains quantities that are still unknown. To find these quantities τ_i, we substitute representation (13) into (5) and arrive at a new SAE:

$$\tau_i - \frac{\nu}{\lambda}\sum_{j=1}^{k}\tau_j\,\sigma_{3ij}(t) = \varphi_1\,\sigma_{1i} + \varphi_2\,\sigma_{2i} + \sigma_{4i}, \quad (14)$$

where

$$\sigma_{1i} = \int_0^T b_i(s)\cos\lambda\,(T-s)\,ds, \quad \sigma_{2i} = -\int_0^T b_i(s)\sin\lambda\,(T-s)\,ds,$$

$$\sigma_{3ij} = \int_0^T b_i(s)\int_0^T H(s,\theta,\lambda)\,a_j(\theta)\,d\theta\,ds, \quad \sigma_{4i} = \frac{1}{\lambda}\int_0^T b_i(s)\int_0^T H(s,\theta,\lambda)\,\alpha(\theta)\,u(\theta)\,d\theta\,ds.$$

To establish the unique solvability of SAE (14), we introduce the following matrix

$$\Theta_0\,(\nu,\,\lambda) = \begin{pmatrix} 1 - \frac{\nu}{\lambda}\sigma_{31\,1} & \frac{\nu}{\lambda}\sigma_{31\,2} & \dots & \frac{\nu}{\lambda}\sigma_{31\,k} \\ \frac{\nu}{\lambda}\sigma_{32\,1} & 1 - \frac{\nu}{\lambda}\sigma_{32\,2} & \dots & \frac{\nu}{\lambda}\sigma_{32\,k} \\ \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda}\sigma_{3k\,1} & \frac{\nu}{\lambda}\sigma_{3k\,2} & \dots & 1 - \frac{\nu}{\lambda}\sigma_{3\,k\,k} \end{pmatrix}$$

and consider the values of the parameter ν, for which the Fredholm determinant differs from zero:

$$\Delta_0(\nu,\lambda) = \det\,\Theta_0\,(\nu,\lambda) \ne 0. \quad (15)$$

Determinant $\Delta_0(\nu,\,\lambda)$ in (15) is a polynomial with respect to $\frac{\nu}{\lambda}$ of the degree not higher than k. The algebraic equation $\Delta_0(\nu,\,\lambda) = 0$ has no more than k different real roots. We denote them by μ_l $(l = \overline{1,\,p},\ 1 \le p \le k)$. Then $\nu = \nu_l = \lambda\,\mu_l$ are called the characteristic (irregular) values of the kernel of the integro-differential equation (1). So, we introduce the following two designations

$$\Omega_1 = \{(\nu,\,\lambda) : \,\nu = \lambda\,\mu_l,\ \lambda \in (0,\infty)\}\,, \quad \Omega_2 = \{(\nu,\,\lambda) : \,\nu \ne \lambda\,\mu_l,\ \lambda \in (0,\infty)\}\,.$$

The set Ω_1 is the set of irregular values of the kernel of the integro-differential equation (1). While the set Ω_2 is the set of regular values of the kernel.

On the number set Ω_2 we consider a matrix

$$\Theta_{im}(\nu,\,\lambda) = \begin{pmatrix} 1 - \frac{\nu}{\lambda}\sigma_{31\,1} & \dots & \frac{\nu}{\lambda}\sigma_{31\,(i-1)} & \sigma_{m\,1} & \frac{\nu}{\lambda}\sigma_{31\,(i+1)} & \dots & \frac{\nu}{\lambda}\sigma_{31\,k} \\ \frac{\nu}{\lambda}\sigma_{32\,1} & \dots & \frac{\nu}{\lambda}\sigma_{32\,(i-1)} & \sigma_{m\,2} & \frac{\nu}{\lambda}\sigma_{32\,(i+1)} & \dots & \frac{\nu}{\lambda}\sigma_{332\,k} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\nu}{\lambda}\sigma_{3k\,1} & \dots & \frac{\nu}{\lambda}\sigma_{3k\,(i-1)} & \sigma_{m\,k} & \frac{\nu}{\lambda}\sigma_{3k\,(i+1)} & \dots & 1 - \frac{\nu}{\lambda}\sigma_{3k\,k} \end{pmatrix},$$

$m = 1, 2, 4$. Taking into account the known properties of the matrix $\Theta_{im}(\nu, \lambda)$, we apply the Cramer method on the spectral set Ω_2 and obtain solutions of SAE (14) in the form

$$\tau_i = \varphi_1 \frac{\Delta_{1i}(\nu, \lambda)}{\Delta_0(\nu, \lambda)} + \varphi_2 \frac{\Delta_{2i}(\nu, \lambda)}{\Delta_0(\nu, \lambda)} + \frac{\Delta_{4i}(\nu, \lambda, u)}{\Delta_0(\nu, \lambda)}, \quad i = \overline{1, k}, \quad (\nu, \lambda) \in \Omega_2, \tag{16}$$

where $\Delta_{im}(\nu, \lambda) = \det \Theta_{im}(\nu, \lambda)$, $m = 1, 2, 4$. Substituting solutions (16) into function (13), we obtain

$$u(t, \nu, \lambda) = \varphi_1 h_1(t, \nu, \lambda) + \varphi_2 h_2(t, \nu, \lambda) + \\ + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_{4i}(\nu, \lambda, u)}{\Delta_0(\nu, \lambda)} \chi_{3i}(t) + \int_0^T \bar{H}(t, s, \lambda)\, u(s, \nu, \lambda)\, ds, \quad (\nu, \lambda) \in \Omega_2, \tag{17}$$

where

$$h_j(t, \nu, \lambda) = \chi_j(t) + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_j(\nu, \lambda)}{\Delta_0(\nu, \lambda)} \chi_{3i}(t), \quad j = 1, 2,$$

$$\chi_{3i}(t) = \int_0^T H(t, s, \lambda)\, a_i(s)\, ds, \quad \bar{H}(t, s, \lambda) = \frac{1}{\lambda} H(t, s, \lambda)\, \alpha(s).$$

Note that representation (17) is equivalent to the direct problem (1), (2) for regular values of the parameter ν. However, φ_1 and φ_2 have not been determined yet.

3. Solution of the inverse problem (1)–(3)

For convenience, representation (17) can be written in the following form

$$u(t, \nu, \lambda) = \varphi_1 \left[\cos \lambda (T - t) - \sin \lambda (T - t) + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_1(\nu, \lambda)}{\Delta_0(\nu, \lambda)} \chi_{3i}(t) \right] + \\ + \varphi_2 \left[\cos \lambda (T + t) - \sin \lambda (T - t) + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_2(\nu, \lambda)}{\Delta_0(\nu, \lambda)} \chi_{3i}(t) \right] + \\ + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_{4i}(\nu, \lambda, u)}{\Delta_0(\nu, \lambda)} \chi_{3i}(t) + \int_0^T \bar{H}(t, s, \lambda)\, u(s, \nu, \lambda)\, ds, \quad (\nu, \lambda) \in \Omega_2. \tag{18}$$

We differentiate (18) one time:

$$u'(t, \nu, \lambda) = \varphi_1 \left[\lambda \sin \lambda(T - t) + \lambda \cos \lambda(T - t) + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_1(\nu, \lambda)}{\Delta_0(\nu, \lambda)} \chi'_{3i}(t) \right] + \\ + \varphi_2 \left[-\lambda \sin \lambda(T + t) + \lambda \cos \lambda(T - t) + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_2(\nu, \lambda)}{\Delta_0(\nu, \lambda)} \chi'_{3i}(t) \right] + \\ + \frac{\nu}{\lambda} \sum_{i=1}^{k} \frac{\Delta_{4i}(\nu, \lambda, u)}{\Delta_0(\nu, \lambda)} \chi'_{3i}(t) + \int_0^T \bar{H}'(t, s, \lambda)\, u(s, \nu, \lambda)\, ds, \quad (\nu, \lambda) \in \Omega_2, \tag{19}$$

where

$$\chi'_{3i}(t) = \int_0^T H'(t, s, \lambda)\, a_i(s)\, d\,s,$$

$$H'(t, s, \omega) = \begin{cases} \lambda \cos \lambda (t + s), \quad t < s \le T, \\ \lambda \cos \lambda (t - s) - \lambda \sin \lambda t \sin \lambda s + \lambda^2 \cos \lambda t \sin \lambda s, \ \ 0 \le s < t. \end{cases}$$

$$\bar{H}'(t, s, \lambda) = \frac{1}{\lambda} H'(t, s, \lambda)\, \alpha(s).$$

Then, applying intermediate conditions (3) to functions (18) and (19), we arrive at the solution of the following SAE:

$$\begin{cases} \varphi_1 \left[\chi_1(t_1, \lambda) + \varepsilon_{11} \right] + \varphi_2 \left[\chi_2(t_1, \lambda) + \varepsilon_{12} \right] = \bar{\psi}_1, \\ \varphi_1 \left[\chi'_1(t_1, \lambda) + \varepsilon_{21} \right] + \varphi_2 \left[\chi'_{2n}(t_1, \lambda) + \varepsilon_{22} \right] = \bar{\psi}_2, \end{cases} \tag{20}$$

where

$$\varepsilon_{1j} = \frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_j(\nu,\lambda)}{\Delta_0(\nu,\lambda)}\,\chi_{3\,i}(t_1), \quad \varepsilon_{2j} = \frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_j(\nu,\lambda)}{\Delta_0(\nu,\lambda)}\,\chi'_{3\,i}(t_1), \quad j=1,2,$$

$$\bar{\psi}_1 = \psi_1 - \frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_{4\,i}(\nu,\lambda,u)}{\Delta_0(\nu,\lambda)}\,\chi_{3\,i}(t_1) + \int_0^T \bar{H}(t_1,s,\lambda)\,u(s,\nu,\lambda)\,ds, \tag{21}$$

$$\bar{\psi}_2 = \psi_2 - \frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_{4\,i}(\nu,\lambda,u)}{\Delta_0(\nu,\lambda)}\,\chi'_{3\,i}(t_1) + \int_0^T \bar{H}'(t_1,s,\lambda)\,u(s,\nu,\lambda)\,ds. \tag{22}$$

The fulfillment of the following condition ensures the unique solvability of SAE (20):

$$V_0(\lambda) = \begin{vmatrix} \chi_1(t_1,\lambda)+\varepsilon_{11} & \chi_2(t_1,\lambda)+\varepsilon_{12} \\ \chi'_1(t_1,\lambda)+\varepsilon_{21} & \chi'_2(t_1,\lambda)+\varepsilon_{22} \end{vmatrix} =$$
$$= -\lambda\,\sin 2\lambda T - \lambda\,\cos 2\lambda T + 2\lambda\,\sin\lambda(T-t_1)\,\cos\lambda(T-t_1) - \lambda\,\cos 2\lambda(T-t_1) -$$
$$-\lambda\,\varepsilon_{11}[\sin\lambda(T+t_1)+\cos\lambda(T-t_1)] - \lambda\varepsilon_{12}[\sin\lambda(T-t_1)+\cos\lambda(T-t_1)] -$$
$$-\varepsilon_{21}[\cos\lambda(T+t_1) - \lambda\,\sin\lambda(T-t_1)] - \varepsilon_{22}[\sin\lambda(T-t_1) - \lambda\,\cos\lambda(T-t_1)] +$$
$$+\varepsilon_{11}\varepsilon_{22} - \varepsilon_{21}\varepsilon_{12} \neq 0. \tag{23}$$

Before proceeding to the solution of SAE (20), we consider condition (23) for the general case. To do this, suppose the opposite:

$$-\lambda\,\sin 2\lambda T - \lambda\,\cos 2\lambda T + 2\lambda\,\sin\lambda(T-t_1)\,\cos\lambda(T-t_1) - \lambda\,\cos 2\lambda(T-t_1) -$$
$$-\lambda\,\varepsilon_{11}[\sin\lambda(T+t_1)+\cos\lambda(T-t_1)] - \lambda\varepsilon_{12}[\sin\lambda(T-t_1)+\cos\lambda(T-t_1)] -$$
$$-\varepsilon_{21}[\cos\lambda(T+t_1) - \lambda\,\sin\lambda(T-t_1)] - \varepsilon_{22}[\sin\lambda(T-t_1) - \lambda\,\cos\lambda(T-t_1)] +$$
$$+\varepsilon_{11}\varepsilon_{22} - \varepsilon_{21}\varepsilon_{12} = 0. \tag{24}$$

Condition (24) is a transcendental equation and the set of its solutions with respect to λ denote by $\Im$. So, on the set

$$\Omega_3 = \{(\nu_n,\lambda):\, |\,\Delta_0(\nu,\lambda)\,| > 0,\;\; \nu_n \neq \lambda\,\mu_l,\;\; \lambda\in\Im\}$$

SAE (20) is not one valued solvable. But, on the other set

$$\Omega_4 = \{(\nu_n,\lambda)\,:\,|\,\Delta_0(\nu,\lambda)\,| > 0,\, |\,V_0(\lambda)\,| > 0,\, \nu_n \neq \lambda\,\mu_l,\, \lambda\in(0;\infty)\setminus\Im\}$$

SAE (20) is one valued solvable. Taking into account notations (21) and (22), we obtain

$$\varphi_j = \psi_1 w_{j1} + \psi_2 w_{j2} + \frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_{4\,i}(\nu,\lambda,u)}{\Delta_0(\nu,\lambda)}\,w_{j3\,i} + \int_0^T W_j(s,\lambda)\,u(s,\nu,\lambda)\,ds, \;\; j=1,2, \tag{25}$$

where

$$w_{11} = V_0^{-1}\left(\chi'_2(t_1)+\varepsilon_{22}\right), \quad w_{12} = V_0^{-1}\left(-\chi_2(t_1)+\varepsilon_{12}\right),$$
$$w_{21} = V_0^{-1}\left(\chi'_1(t_1)+\varepsilon_{21}\right), \quad w_{22} = V_0^{-1}\left(\chi_1(t_1)+\varepsilon_{11}\right),$$
$$w_{13}(\lambda) = -\left[\chi_{3i}(t_1,\lambda)w_{11}(\lambda) + \chi'_{3i}(t_1,\lambda)\,w_{12}(\lambda)\right],$$
$$w_{23}(\lambda) = -\left[\chi_{3i}(t_1,\lambda)\,w_{21}(\lambda) + \chi'_{3i}(t_1,\lambda)\,w_{22}(\lambda)\right],$$
$$W_1(s,\lambda) = H(t_1,s)\,w_{11}(\lambda) + H'(t_1,s)\,w_{12}(\lambda),$$
$$W_2(s,\lambda) = H(t_1,s)\,w_{21}(\lambda) + H'(t_1,s)\,w_{22}(\lambda).$$

Representations in (25) are expressions of unknown quantities φ_1 and φ_2 in terms of an unknown function $u(t,\nu,\lambda)$. Therefore, we need to uniquely define the function $u(t,\nu,\lambda)$. Substituting representations (25) into equation (17), we obtain in the final form the following functional-integral equation

$$u(t,\nu,\lambda) = W(t,\nu,\lambda,u) \equiv \psi_1 g_1(t,\nu,\lambda) + \psi_2 g_2(t,\nu,\lambda) +$$
$$+\frac{\nu}{\lambda}\sum_{i=1}^{k}\frac{\Delta_{4\,i}(\nu,\lambda,u)}{\Delta_0(\nu,\lambda)}\,g_{3\,i}(t) + \int_0^T G(t,s,\nu,\lambda)\,u(s,\nu,\lambda)\,ds, \quad (\nu,\lambda)\in\Omega_5, \tag{26}$$

where

$$g_1\,(t,\nu,\lambda) = w_{11}(\lambda)h_1\,(t,\nu,\lambda) + w_{21}(\lambda)h_2\,(t,\nu,\lambda),$$
$$g_2\,(t,\nu,\lambda) = w_{12}(\lambda)\,h_1\,(t,\nu,\lambda) + w_{22}(\lambda)\,h_2\,(t,\nu,\lambda),$$
$$g_{3\,i}\,(t) = g_1\,(t,\nu,\lambda)\,\chi_{3i}(t_1) + g_2\,(t,\nu,\lambda)\,\chi'_{3i}(t_1) + \chi_{3i}(t),$$

$$G\,(t,s,\nu,\lambda)=g_1\,(t,\nu,\lambda)\,\bar H(t_1,s)+g_2\,(t,\nu,\lambda)\,\bar H'(t_1,s)+\bar H(t,s).$$

Note that this functional-integral equation makes sense only for values of parameters $\nu,\ \lambda$ from the set Ω_4. In addition, in the functional-integral equation (26), the unknown function $u(t,\nu,\lambda)$ is under the sign of the determinant and under the sign of the integral. Let us investigate this equation in the question of unique solvability. To this end, consider the space of functions continuous on the interval $[0;T]$. In this space, we introduce the norm as follows: $\|\,u(t)\,\|=\max\limits_{0\le t\le T}\,|\,u(t)\,|$.

Theorem. Let be fulfilled the following condition

$$\rho=\frac{|\,\nu\,|}{\lambda}\sum_{i=1}^{k}\left|\frac{\bar\Delta_{4\,i}(\nu,\lambda)}{\Delta_0(\nu,\lambda)}\right|\,\|\,g_{3\,i}(t)\,\|+\int\limits_0^T\|\,G(t,s,\nu,\lambda)\,\|\,ds<1,$$

where $\left|\,\bar\Delta_{4\,i}(\nu,\lambda)\,\right|$ is defined from (30). Then the functional-integral equation (26) has the unique solution for values of parameters ν,λ from the set Ω_4 and for all $t\in[0;T]$. This solution can be found from the following iterative Picard process

$$\begin{cases} u_0(t,\nu,\lambda)=\psi_1\,g_1(t,\nu,\lambda)+\psi_2\,g_2(t,\nu,\lambda),\\ u_k(t,\nu,\lambda)=W(t,\nu,\lambda,u_{k-1}),\quad k=1,2,3,...\end{cases}$$

Proof. Since ψ_j and $g_j(t,\nu,\lambda),\ j=1,2$, are bounded, then for zero approximation we obtain the estimate

$$|\,u_0(t,\nu,\lambda)\,|=|\,\psi_1\,|\,|\,g_1(t,\nu,\lambda)\,|+|\,\psi_2\,|\,|\,g_2(t,\nu,\lambda)\,|\le M\,(|\,\psi_1\,|+|\,\psi_2\,|)<\infty, \tag{27}$$

where $M=\max\,\{\|\,g_1(t,\nu,\lambda)\,\|\,,\,\|\,g_2(t,\nu,\lambda)\,\|\}$.

Similarly to (27), taking into account the property of the determinant, we obtain an estimate for the first difference of the approximation

$$|\,u_1(t,\nu,\lambda)-u_0(t,\nu,\lambda)\,|\le$$

$$+\frac{|\,\nu\,|}{\lambda}\sum_{i=1}^{k}\left|\frac{\Delta_{4\,i}(\nu,\lambda,u_0)}{\Delta_0(\nu,\lambda)}\right|\,\|\,g_{3\,i}(t)\,\|+M\,(|\,\psi_1\,|+|\,\psi_2\,|)\int\limits_0^T\|\,G(t,s,\nu,\lambda)\,\|\;ds. \tag{28}$$

where $|\,\Delta_{i4}(\nu,\lambda,u_0)\,|=|\,\det\,\Theta_{i4}\,(\nu,\lambda,\,u_0)\,|$,

$$\Theta_{i4}(\nu,\lambda,u_0)=\begin{pmatrix} 1-\frac{\nu}{\lambda}\sigma_{31\,1} & \cdots & \frac{\nu}{\lambda}\sigma_{31\,(i-1)} & \sigma_{4\,1}(u_0) & \frac{\nu}{\lambda}\sigma_{31\,(i+1)} & \cdots & \frac{\nu}{\lambda}\sigma_{31\,k}\\ \frac{\nu}{\lambda}\sigma_{32\,1} & \cdots & \frac{\nu}{\lambda}\sigma_{32\,(i-1)} & \sigma_{4\,2}(u_0) & \frac{\nu}{\lambda}\sigma_{32\,(i+1)} & \cdots & \frac{\nu}{\lambda}\sigma_{332\,k}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \frac{\nu}{\lambda}\sigma_{3k\,1} & \cdots & \frac{\nu}{\lambda}\sigma_{3k\,(i-1)} & \sigma_{4\,k}(u_0) & \frac{\nu}{\lambda}\sigma_{3k\,(i+1)} & \cdots & 1-\frac{\nu}{\lambda}\sigma_{3k\,k}\end{pmatrix},$$

$$\sigma_{4i}(u_0)=M\,(|\,\psi_1\,|+|\,\psi_2\,|)\int\limits_0^T|\,b_i(s)\,|\int\limits_0^T\left|\,\bar H(s,\theta,\lambda)\,\right|\,d\theta\,ds.$$

Continuing this process, we obtain by induction that

$$|\,u_k(t,\nu,\lambda)-u_{k-1}(t,\nu,\lambda)\,|\le$$

$$+\,\frac{|\,\nu\,|}{\lambda}\sum_{i=1}^{k}\left|\frac{\Delta_{4\,i}(\nu,\lambda,u_{k-1})-\Delta_{4\,i}(\nu,\lambda,u_{k-2})}{\Delta_0(\nu,\lambda)}\right|\,\|\,g_{3\,i}(t)\,\|+$$

$$+\int\limits_0^T\|\,G(t,s,\nu,\lambda)\,\|\;|\,u_{k-1}(s,\nu,\lambda)-u_{k-2}(s,\nu,\lambda)\,|\,ds.$$

This implies the estimate

$$\|\,u_k(t,\nu,\lambda)-u_{k-1}(t,\nu,\lambda)\,\|\le\rho\;\|\,u_{k-1}(t,\nu,\lambda)-u_{k-2}(t,\nu,\lambda)\,\|\,, \tag{29}$$

where

$$\rho=\frac{|\,\nu\,|}{\lambda}\sum_{i=1}^{k}\left|\frac{\bar\Delta_{4\,i}(\nu,\lambda)}{\Delta_0(\nu,\lambda)}\right|\,\|\,g_{3\,i}(t)\,\|+\int\limits_0^T\|\,G(t,s,\nu,\lambda)\,\|\,ds,$$

$$\left|\,\bar\Delta_{4\,i}(\nu,\lambda)\,\right|=\left|\,\det\,\bar\Theta_{4\,i}(\nu,\lambda)\,\right|, \tag{30}$$

$$\bar{\Theta}_{4\,i}(\nu,\lambda) = \begin{pmatrix} 1-\dfrac{\nu}{\lambda}\sigma_{31\,1} & \cdots & \dfrac{\nu}{\lambda}\sigma_{31\,(i-1)} & \bar{\sigma}_{4\,1} & \dfrac{\nu}{\lambda}\sigma_{31\,(i+1)} & \cdots & \dfrac{\nu}{\lambda}\sigma_{31\,k} \\ \dfrac{\nu}{\lambda}\sigma_{32\,1} & \cdots & \dfrac{\nu}{\lambda}\sigma_{32\,(i-1)} & \bar{\sigma}_{4\,2} & \dfrac{\nu}{\lambda}\sigma_{32\,(i+1)} & \cdots & \dfrac{\nu}{\lambda}\sigma_{332\,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\nu}{\lambda}\sigma_{3k\,1} & \cdots & \dfrac{\nu}{\lambda}\sigma_{3k\,(i-1)} & \bar{\sigma}_{4\,k} & \dfrac{\nu}{\lambda}\sigma_{3k\,(i+1)} & \cdots & 1-\dfrac{\nu}{\lambda}\sigma_{3k\,k} \end{pmatrix},$$

$$\bar{\sigma}_{4i} = \int_0^T |\, b_i(s) \,| \int_0^T \left|\, \bar{H}(s,\theta,\lambda) \,\right| \, d\theta \, ds.$$

According to the theorem, $\rho < 1$. Estimates (27), (28) and (29) imply that the operator on the right side of (26) is contractive. This implies the existence of a single fixed point. Consequently, the functional-integral equation (26) has the unique solution on the set Ω_4 of values of parameters ν, λ for all $t \in [0;T]$. The theorem is proved.

Now, the function $u(t,\nu,\lambda) \in C^2[0;T]$ is already known. In order to determine the unknown quantities φ_1 and φ_2, we will substitute the solution $u(t,\nu,\lambda) \in C^2[0;T]$ of equation (26) into representations (25). Then, the quantities φ_1 and φ_2, are uniquely determined.

4. Conclusion

We have considered questions of the unique solvability of the inverse problem for the second-order Fredholm integro-differential equation (1) with a degenerate kernel, final conditions (2) at the end of the interval, two parameters, and two redefinition data. Sets of regular parameter values are defined. The features that arise when solving the inverse problem (1)–(3) are studied. Criteria for the unique solvability of the posed inverse problem are established.

Remark 1. For values of parameters (ν, λ) from the set Ω_3, the uniqueness of the set of solutions to the inverse problem (1)–(3) is violated. Because in this case condition (23) is not satisfied.

Remark 2. For values of parameters (ν, λ) from the set Ω_1, the inverse problem (1)–(3) does not make sense. Because in this case condition (15) is not satisfied. But, the direct problem (1), (2) has an infinite set of solutions, if $\varphi_1 = \varphi_2 = 0$ and $\alpha(t) \equiv 0$ for all $t \in [0;T]$.

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