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Master equation for correlators of normal-ordered field mode operators

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ABSTRACT We study the master equation for open quantum systems in the alternative form, preserving the normal form of the averaged normal-ordered operators. We give an example of using this equation for the correlators of normal-ordered field mode operators. We explore the properties of the system of linear equations for the higher-order field operators based on the example of a two-mode bosonic system.

KEYWORDS master equation, normal-ordered correlators, two-mode bosonic system

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1. Introduction

Quantum information processing is transfer, storage, and conversion of quantum information. It is essential for quantum protocols like quantum teleportation [1], quantum computing [2], quantum key distribution [3], dense coding, and quantum memory [4]. There are plenty of approaches to various problems for these protocols, but there is no general model that describes all kinds of them.

Models of open quantum systems are intended for solving problems of transport and storage of quantum information. This approach can be described in terms of completely positive trace-preserving (CPTP), or trace-non-increasing linear mappings known as quantum channels. The quantum master equation is the most effective way to determine the dynamics of open systems. More precisely, we use the Lindblad-type master equation [5, 6]. This equation was studied using physical methods [7–9] and mathematical techniques for the case of the single-mode Lindblad equation [6, 10–17]. In the case of multi-mode bosonic systems, the mathematical approach was used to construct the Fock-like eigenstates of the Lindblad superoperators using the Lie algebras [18].

This approach simplifies the determination of the dynamics of the averaged moments of the Stokes operator and the polarization of light [19]. These parameters are sufficient for a fairly accurate description of many physical processes. However, these averaged moments of the Stokes operator are inadequate in cases when other statistical parameters than the wave average intensity are recorded [20]. Hence, the investigation of the higher-order moment correlators is required for a more detailed description of these quantum processes. We offer a more convenient representation of the Lindblad equation for the calculation of higher-order moments. The equation saves the normal form of the operators: if the original averaged operator was represented in the normal form, then each term in the equation has the form of the average of the operator in the normal form.

2. Model

The starting point is the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) form of the master equation for polarized light in the two-mode bosonic system in optical fibers:

$$\frac{\partial \rho}{\partial t} = - i\sum_{n,m}\frac{1}{2}\Omega_{n,m}[a_n^\dagger a_m,\rho] - \sum_{n,m}\frac{1}{2}\Gamma_{n,m}\Big((n_T+1)(a_n^\dagger a_m\rho+\rho a_n^\dagger a_m - 2a_m\rho a_n^\dagger) + n_T(a_m a_n^\dagger\rho + \rho a_m a_n^\dagger - 2a_n^\dagger\rho a_m)\Big), \tag{1}$$

where indexes $n, m \in \{1, 2\}$, dagger denotes the Hermitian conjuration, ρ is the density matrix of a quantum state, $a_n^\dagger$ and a_n are the creation and the annihilation operators of the n-th mode, Ω and Γ are the frequency and the relaxation matrices [21], and n_T is the mean number of thermal photons:

$$n_T = \frac{1}{\exp\left(\frac{\hbar\Omega_0}{k_B T}\right) - 1},$$

where Ω_0 is the free-space frequency, $\hbar$ is the reduced Planck constant, T is the environment temperature. k_B is the Boltzmann constant. The frequency (Ω) and the relaxation (Γ) matrices are Hermitian: $\Omega = \Omega^\dagger, \Gamma = \Gamma^\dagger$. The positivity condition for the relaxation matrix has the following (conventional) form: $z^*\Gamma z \geqslant 0, z \in \mathbb{C}^2$.

3. Alternative form for the Lindblad equation

Let us apply some operator $\hat{A}$ to the both parts of equation (1) and calculate the trace of the obtained operators. Note that the average A of the operator $\hat{A}$ is given by the following formula $A = \langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\rho)$. Keeping in mind that the trace is invariant under cyclic permutations, we obtain the following relation:

$$\frac{\partial A}{\partial t} = \mathrm{Tr}\Big(-\frac{i}{2}\sum_{n,m}\Omega_{n,m}(\hat{A}a_n^\dagger a_m\rho - a_n^\dagger a_m\hat{A}\rho)- \\ -\frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}(n_T+1)(\hat{A}a_n^\dagger a_m\rho + a_n^\dagger a_m\hat{A}\rho - 2a_n^\dagger A a_m\rho)- \\ -\frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}n_T(\hat{A}a_m a_n^\dagger\rho + a_m a_n^\dagger\hat{A}\rho - 2a_m\hat{A}a_n^\dagger\rho)\Big). \quad (2)$$

Our goal is to bring the equation to such a form that if the operator A is in the normal form then the equation contains operators in the normal form only. Consider each of the sums in (2), starting with the first one:

$$\mathrm{Tr}\Big(-\frac{i}{2}\sum_{n,m}\Omega_{n,m}(\hat{A}a_n^\dagger a_m\rho - a_n^\dagger a_m\hat{A}\rho)\Big) = \\ \mathrm{Tr}\Big(-\frac{i}{2}\sum_{n,m}\Omega_{n,m}(\hat{A}a_n^\dagger a_m - a_n^\dagger\hat{A}a_m + a_n^\dagger\hat{A}a_m - a_n^\dagger a_m\hat{A})\rho\Big) = \\ \mathrm{Tr}\Big(-\frac{1}{2}\sum_{n,m}(i\Omega_{n,m}[\hat{A},a_n^\dagger]a_m - i\Omega_{n,m}a_n^\dagger[a_m,\hat{A}])\rho\Big). \quad (3)$$

As for the second sum, one subtracts and adds $a_n^\dagger a_m\hat{A}\rho$ to obtain commutators:

$$-\frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}(n_T+1)\Big(\hat{A}a_n a_m\rho + a_n^\dagger a_m\hat{A}\rho - 2a_n^\dagger\hat{A}a_m\rho\Big) = \\ -\frac{1}{2}\sum_{n,m}(n_T+1)\mathit{\Gamma}_{n,m}\Big(([\hat{A},a_n^\dagger]a_m + a_n^\dagger[a_m,\hat{A}])\rho\Big). \quad (4)$$

Let us consider the third sum in more details following the procedure used above:

$$\mathrm{Tr}\Big(-\frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}n_T(\hat{A}a_m a_n^\dagger\rho + a_m a_n^\dagger\hat{A}\rho - 2a_m\hat{A}a_n^\dagger\rho)\Big) = \\ \mathrm{Tr}\Big(-\frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}n_T(\hat{A}a_m a_n^\dagger - a_m a_n^\dagger\hat{A} + 2[a_m,[a_n^\dagger,\hat{A}]] + 2[a_n^\dagger,A]a_m)\rho\Big) = \\ \mathrm{Tr}\Big(-\sum_{n,m}\mathit{\Gamma}_{n,m}n_T([a_m,[a_n^\dagger,A]])\rho - \frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}n_T(Aa_n^\dagger a_m - a_n^\dagger a_m A + 2[a_n^\dagger,A]a_m)\rho\Big) = \\ \mathrm{Tr}\Big(-\sum_{n,m}\mathit{\Gamma}_{n,m}n_T[[\hat{A},a_n^\dagger],a_m]\rho\Big) + \mathrm{Tr}\Big(\frac{1}{2}\sum_{n,m}\mathit{\Gamma}_{n,m}n_T([\hat{A},a_n^\dagger]a_m + a_n^\dagger[a_m,\hat{A}])\rho\Big). \quad (5)$$

Substituting (3), (4) and (5) in (2), we obtain the following relation for averaging of any operator $\hat{A}$:

$$\frac{\partial A}{\partial t} = \\ \frac{1}{2}\sum_{n,m}(\Big\langle[\hat{A},a_n^\dagger]a_m\Big\rangle(-i\Omega_{n,m} - \Gamma_{n,m}) + \Big\langle a_n^\dagger[a_m,\hat{A}]\Big\rangle(i\Omega_{n,m} - \Gamma_{n,m})) - -n_t\sum_{n,m}\Gamma_{n,m}\Big\langle[[\hat{A},a_n^\dagger],a_m]\Big\rangle. \quad (6)$$

One can note that this form of the master equation saves the normal form of the operators. An example showing this property is presented below.

4. Second normal-ordered moments

Let us assume the operator $\hat{A} = a_p^\dagger a_q$ has the form of a field operator. Hence, the commutators in the expression (6) can be rewritten as follows:

$$[a_p^\dagger a_q, a_n^\dagger] = \delta_{qn} a_p^\dagger,$$
$$[[a_p^\dagger a_q, a_n^\dagger], a_m] = -\delta_{qn}\delta_{pm},$$
$$[a_m, a_p^\dagger a_q] = \delta_{pm} a_q$$

and equation (6) takes the form:

$$\frac{\partial \left\langle a_p^\dagger a_q \right\rangle}{\partial t} = \frac{1}{2}\sum_m \left\langle a_p^\dagger a_m \right\rangle (-i\Omega_{q,m} - \Gamma_{q,m}) + \frac{1}{2}\sum_n \left\langle a_n^\dagger a_q \right\rangle (i\Omega_{n,p} - \Gamma_{n,p}) + n_t \Gamma_{q,p}. \qquad (7)$$

Note that all operations in (6) are very simple. All transformations are reduced to counting of three simple commutators. The commutators in the first and the second sums preserve the order of the operators, and the double commutator in the third sum lowers the order by 2. This property takes place for any field operator. Furthermore, it is valid for any operator, if we expand it in respect to the basis of the field operators.

Let the set of operators $\hat{A}_i(\hat{B}_i)$ consist of the normal-ordered operators. Let any operator $\hat{A}_i$ be a product of α operators of annihilation and β operators of creations in any possible variants of modes; any operator $\hat{B}_i$ is a product of $\alpha - 1$ operators of annihilation and $\beta - 1$ operators of creations in any possible variants of modes. Then the system of differential equations for the set $\hat{A}_i$ derived from equation (6) gives one the expression containing two sets of operators $\hat{A}_i$ and $\hat{B}_i$. This means that the equations of higher orders depend on the lower ones recursively. Example (7) shows this property in the simplest case $(\alpha = \beta = 1)$.

We can mention the amount of non-recurring expressions in the system. Only an amount of possible variations of operators of the creation (annihilation) in the first mode affects the quantity of all variants, because of $[a_p^\dagger, a_q^\dagger] = [a_p, a_q] = 0$. Hence, the amount of equations is equal to $(\alpha + 1)(\beta + 1)$.

5. Conclusion

In this paper, we presented the equation for the normal-ordered moments and its property of saving the normal order. We started with the master equation in the GKSL form, converted it to the convenient form for the normal-ordered moments (6), and gave the example for the two-mode bosonic system. This is a logical continuation for the description of open quantum systems. We will try to continue the idea in terms of hidden polarization of light.

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