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Original article

Inverse problem for a second order impulsive system of integro-differential equations with two redefinition vectors and mixed maxima

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ABSTRACT An inverse problem for a second order system of ordinary integro-differential equations with impulsive effects, mixed maxima and two redefinition vectors is investigated. A system of nonlinear functional integral equations is obtained by applying some transformations. The existence and uniqueness of the solution of the nonlinear inverse problem is reduced to the unique solvability of the system of nonlinear functional integral equations in Banach space $PC\left([0,T],\mathbb{R}^n\right)$. The method of successive approximations in combination with the method of compressing mapping is used in the proof of unique solvability of the nonlinear functional integral equations. Then values of redefinition vectors are founded.

KEYWORDS inverse problem, second order system, impulsive integro-differential equations, two-point nonlinear boundary value conditions, two redefinition vectors, mixed maxima, existence and uniqueness of solution.

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1. Introduction

It is known that the dynamics of evolving processes undergoes sometimes abrupt changes, for example, upheavals, natural disasters and shocks. Such short-term, but very painful, perturbations are often interpreted as impulses. That is, we actually have a dynamic system with impulsive actions. Dynamic systems with mixed maxima naturally describe processes with impulsive actions. It is presented by differential equations having solutions with first kind "discontinuities" at fixed or non-fixed time moments. This type of differential and integro-differential equations have applications in biological, chemical and physical sciences, ecology, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. [1–5]. In particular, such kind of problems appear in biophysics at micro- and nano-scales [6–10]. Such differential equations with "discontinuities" at fixed or non-fixed time moments are called differential equations with impulsive effects. There are a lot of publications of devoted to differential equations with impulsive effects, which describe many natural and technical processes [11–25].

Two-point and multi-point boundary value problems for the differential and integro-differential equations are studied by many authors (see, for example [26–29]). However, second-order differential equations with nonlocal boundary value conditions and impulsive effects are almost not studied. It is related to the fact that the reduction of such problem to equivalent functional integral equation faces difficulties. In this paper, we investigate an inverse problem for a system of second order integro-differential equations with impulsive effects, two-point nonlinear boundary value conditions and mixed maxima. The questions of existence and uniqueness of the solution to the nonlinear inverse problem are investigated. We note that when studying the solvability problem for the differential and integro-differential equations with mixed maxima one should deal with singularity. Moreover, the jumps of solutions are a natural things for differential equations with mixed maxima [30].

We consider the existence problem and constructive method for calculating the unique solutions of the second order system of nonlinear ordinary integro-differential equations on the interval $[0,T]$ for $t\neq t_i, \quad i=1,2,...,p$

$$x''(t)=f\left(t,x(t),\int\limits_0^T\Theta\left(t,s,\max\left\{x(\tau)\middle|\tau\in[\lambda_1(s):|:\lambda_2(s)]\right\}\right)ds\right), \tag{1}$$

where $t\neq t_i,\ i=1,2,...,p,\ 0=t_0<t_1<...<t_p<t_{p+1}=T,\ x\in X,\ X$ is the closed bounded domain in the space $\mathbb{R}^n,\ f(t,x,y)\in C\big([0,T]\times X\times Y,\mathbb{R}^n\big),\ Y$ is the closed bounded domain in $\mathbb{R}^n,\ 0<\lambda_j(t)<T,\ j=1,2,$ $[\lambda_1(t):|:\lambda_2(t)]=[\min\left\{\lambda_1(t),\lambda_2(t)\right\};\max\left\{\lambda_1(t),\lambda_2(t)\right\}],\ \lambda_j(t)=\lambda_j(t,x(t))\in C\big([0,T]\times X,\mathbb{R}\big),\ j=1,2,$

$$\max_{0\leq t\leq T}\int\limits_0^T|\,\Theta(t,s,x)\,|ds<\infty.$$

We study equation (1) with two nonlinear two-point conditions

$$A_1(t)x(0^+) + B_1(t)x(T^-) = C_1 + D_1(t, x(t)), \tag{2}$$

$$A_2(t)x'(0^+) + B_2(t)x'(T^-) = C_2 + D_2(t, x(t)) \tag{3}$$

and two nonlinear impulsive conditions

$$x\left(t_i^+\right) - x\left(t_i^-\right) = F_i\left(x\left(t_i\right)\right), \quad i = 1, 2, ..., p, \tag{4}$$

$$x'\left(t_i^+\right) - x'\left(t_i^-\right) = G_i\left(x\left(t_i\right)\right), \quad i = 1, 2, ..., p, \tag{5}$$

where $A_i(t),\ B_i(t)$ are $n\times n$-dimensional matrix-functions, $C_1 \in \mathbb{R}^n$ and $C_2 \in \mathbb{R}^n$ are redefinition vectors, $D_i(t, x(t)) \in C\big([0, T] \times X, \mathbb{R}^n\big)$ is nonlinear vector-function, $i = 1, 2,\ F_i, G_i \in C\big(X, \mathbb{R}^n\big),\ x\left(t_i^+\right) = \lim\limits_{\nu\to 0^+} x\left(t_i + \nu\right),\ x\left(t_i^-\right) = \lim\limits_{\nu\to 0^-} x\left(t_i - \nu\right)$ are right-hand side and left-hand side limits of function $x(t)$ at the point $t = t_i$, respectively.

In order to find redefinition vectors, we use the following two intermediate conditions

$$x(\bar t) = E_1, \quad E_1 \in \mathbb{R}^n, \quad 0 < \bar t < T, \ \ \bar t \neq t_i, \ \ i = 1, 2, ..., p, \tag{6}$$

$$x'(\bar t) = E_2, \quad E_2 \in \mathbb{R}^n, \quad 0 < \bar t < T, \ \ \bar t \neq t_i, \ \ i = 1, 2, ..., p. \tag{7}$$

We use the Banach space $C\left([0, T], \mathbb{R}^n\right)$, which consists of continuous vector-functions $x(t)$ on the segment $[0, T]$ with the norm

$$\|\, x \,\| = \sqrt{\sum_{j=1}^{n} \max_{0\leq t\leq T} |\, x_j(t)\,|}.$$

$PC\left([0, T], \mathbb{R}^n\right)$ is the linear vector space:

$$PC\left([0, T], \mathbb{R}^n\right) = \left\{x : [0, T] \to \mathbb{R}^n;\ x(t) \in C\left(\left(t_i, t_{i+1}\right], \mathbb{R}^n\right),\ i = 1, ..., p\right\},$$

where $x\left(t_i^+\right)$ and $x\left(t_i^-\right)$ $(i = 0, 1, ..., p)$ exist and they are bounded; $x\left(t_i^-\right) = x\left(t_i\right)$. Note, that the linear vector space $PC\left([0, T], \mathbb{R}^n\right)$ is the Banach space with the following norm

$$\|\, x \,\|_{PC} = \max\left\{\|\, x \,\|_{C((t_i, t_{i+1}])},\ i = 1, 2, ..., p\right\}.$$

Formulation of the problem. Find a triple of unknown quantities

$$\Big\{x(t) \in PC\left([0, T], \mathbb{R}^n\right),\ C_j \in \mathbb{R}^n,\ j = 1, 2\Big\},$$

that the function $x(t)$ satisfies the second-order integro-differential equation (1) for all $t \in [0,\ T],\ \ t \neq t_i,\ \ i = 1, 2, ..., p$, nonlinear two-point conditions (2), (3) and for $t = t_i,\ \ i = 1, 2, ..., p,\ \ 0 < t_1 < t_2 < ... < t_p < T$ satisfies the nonlinear limit conditions (4), (5) and intermediate conditions (6), (7).

2. Reduction of the direct problem (1)–(5) to nonlinear system of functional integral equations

Let function $x(t) \in PC\left([0, T], \mathbb{R}^n\right)$ be a solution of the second order two-point boundary value problem (1)–(5). Then, integrating the integro-differential equation (1) one time over intervals: $(0, t_1],\ (t_1, t_2],\ \ldots,\ (t_p, t_{p+1}] \in [0, T],\ \ t_{p+1} = t$, we obtain:

$$\int_0^{t_1} f(x)ds = \int_0^{t_1} x''(s)ds = x'(t_1^-) - x'(0^+),$$

$$\int_{t_1}^{t_2} f(s)ds = \int_{t_1}^{t_2} x''(s)ds = x'\left(t_2^-\right) - x'\left(t_1^+\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int_{t_p}^{t_{p+1}} f(s)ds = \int_{t_p}^{t_{p+1}} x''(s)ds = x'\left(t_{p+1}^-\right) - x'\left(t_p^+\right),$$

where, for convenience, we put

$$f(t) = f\left(t, x(t), \int_0^T \Theta\left(t, s, \max\left\{x(\tau)\middle|\tau \in [\lambda_1(s) : | : \lambda_2(s)]\right\}\right) ds\right).$$

Hence, taking $x'(0^+) = x'(0),\ x'(t_{p+1}^-) = x'(t)$ into account, we have on the interval $(0, T]$

$$\int_0^t f(s)ds = \left[x'\left(t_1\right) - x'\left(0^+\right)\right] + \left[x'\left(t_2\right) - x'\left(t_1^+\right)\right] + \cdots + \left[x'(t) - x'\left(t_p^+\right)\right] =$$

$$= -x'(0) - \left[x'\left(t_1^+\right) - x'\left(t_1\right)\right] - \left[x'\left(t_2^+\right) - x'\left(t_2\right)\right] - \cdots - \left[x'\left(t_p^+\right) - x'\left(t_p\right)\right] + x'(t).$$

Taking into account the impulsive condition (5), we rewrite the last equality as follows

$$x'(t) = x'(0) + \int_0^t f(s)\,ds + \sum_{0<t_i<t} G_i\left(x(t_i)\right). \tag{8}$$

Subordinate the function $x'(t) \in PC\left([0,T],\mathbb{R}^n\right)$ in presentation (8) to satisfy the nonlinear two-point boundary condition (3):

$$x'(T) = x'(0) + \int_0^T f(s)ds + \sum_{0<t_i<T} G_i\left(x\left(t_i\right)\right). \tag{9}$$

Substituting (9) into condition (3), we find $x'(0)$ as follows:

$$x'(0) = Q_2^{-1}(t)\left[C_2 + D_2(t,x(t)) - B_2(t)\int_0^T f(s)ds - B_2(t)\sum_{0<t_i<T} G_i\left(x\left(t_i\right)\right)\right], \tag{10}$$

where $\det\, Q_2(t) \neq 0,\;\; Q_2(t) = A_2(t) + B_2(t)$.

Substituting (10) into presentation (8), we obtain:

$$x'(t) = Q_2^{-1}(t)\left[C_2 + D_2(t,x(t)) - B_2(t)\int_0^T f(s)ds - B_2(t)\sum_{0<t_i<T} G_i\left(x\left(t_i\right)\right)\right] + \int_0^t f(s)ds + \sum_{0<t_i<t} G_i\left(x\left(t_i\right)\right). \tag{11}$$

Then, integrating integro-differential equation (11) one time over the intervals $(0,t_1]\,,\,(t_1,t_2]\,,\,\ldots\,,\,(t_p,t_{p+1}]$ and taking $x'(0^+) = x'(0),\; x'(t_{p+1}^-) = x'(t)$ into account, we have on the interval $(0,T]$:

$$\int_0^t Q_2^{-1}(s)\left[C_2 + D_2(s,x(s)) - B_2(s)\int_0^T f(\theta)d\theta - B_2(s)\sum_{0<t_i<T} G_i\left(x\left(t_i\right)\right)\right]ds +$$

$$+\int_0^t\left[\int_0^s f(\theta)\,d\theta + \sum_{0<t_i<s} G_i\left(x\left(t_i\right)\right)\right]ds =$$

$$= \left[x\left(t_1\right) - x\left(0^+\right)\right] + \left[x\left(t_2\right) - x\left(t_1^+\right)\right] + ... + \left[x(t) - x\left(t_p^+\right)\right] =$$

$$= -x(0) - \left[x\left(t_1^+\right) - x\left(t_1\right)\right] - \left[x\left(t_2^+\right) - x\left(t_2\right)\right] - ... - \left[x\left(t_p^+\right) - x\left(t_p\right)\right] + x(t). \tag{12}$$

Taking into account the nonlinear impulsive condition (4), we derive the following formula from the equality (12)

$$x(t) = x(0) + \int_0^t Q_2^{-1}(s)\left[C_2 + D_2(s,x(s)) - B_2(s)\int_0^T f(\theta)d\theta - B_2(s)\sum_{0<t_i<T} G_i\left(x\left(t_i\right)\right)\right]ds +$$

$$+\int_0^t\left[\int_0^s f(\theta)d\theta + \sum_{0<t_i<s} G_i\left(x\left(t_i\right)\right)\right]ds + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right). \tag{13}$$

Applying the two-point nonlinear condition (2) to equation (13), we find the value of $x(0)$ as follows:

$$x(0) = Q_1^{-1}(t)\left[C_1 + D_1(t,x(t))\right] - \int_0^T Q_1^{-1}(t)B_1(t)Q_2^{-1}(s)\left[C_2 + D_2(s,x(s))\right]ds +$$

$$+\int_0^T Q_1^{-1}(t)B_1(t)Q_2^{-1}(s)B_2(s)\int_0^T f(\theta)d\theta ds +$$

$$+\int_0^T Q_1^{-1}(t)B_1(t)Q_2^{-1}(s)B_2(s)\sum_{0<t_i<t} G_i\left(x\left(t_i\right)\right)ds - Q_1^{-1}(t)B_1(t)\int_0^T\int_0^s f(\theta)d\theta ds -$$

$$-Q_1^{-1}(t)B_1(t)\int_0^T \sum_{0<t_i<t} G_i\left(x\left(t_i\right)\right)ds - Q_1^{-1}(t)B_1(t)\sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right). \tag{14}$$

When obtaining (14), we used the well known formulas suggested by Dirichlet:

$$\int_0^T g(t,s)\int_0^s f(\theta)d\theta ds=\int_0^T f(s)\int_s^T g(t,\theta)d\theta\, ds,$$

$$\int_0^T g(t,s)\sum_{0<t_i<t} I_i\left(x\left(t_i\right)\right)\, ds=\sum_{0<t_i<T}\int_{t_i}^T g(t,s)\, ds\, I_i\left(x\left(t_i\right)\right).$$

Then, we rewrite (14) as follows

$$x(0)=Q_1^{-1}(t)\left[C_1+D_1(t,x(t))\right]-\int_0^T V_0(t,s)\left[C_2+D_2(s,x(s))\right]ds+$$

$$+\int_0^T V_1(t,s)f(s)ds+\sum_{0<t_i<T} V_1(t,t_i)G_i\left(x\left(t_i\right)\right)-Q_1^{-1}(t)B_1(t)\sum_{0<t_i<T} F_i\left(x\left(t_i\right)\right), \tag{15}$$

where $V_0(t,s)=Q_1^{-1}(t)B_1(t)Q_2^{-1}(s)$, $\det Q_1(t)\neq 0$, $Q_1(t)=A_1(t)+B_1(t)$,

$$V_1(t,s)=Q_1^{-1}(t)B_1(t)\int_s^T Q_2^{-1}(\theta)\left[A_2(\theta)+2B_2(\theta)\right]d\theta.$$

Substituting (15) into presentation (13), we obtain nonlinear system of functional integral equations:

$$x(t)=Q_1^{-1}(t)\left[C_1+D_1(t,x(t))\right]+\int_0^T W_0(t,s)\left[C_2+D_2(s,x(s))\right]ds+$$

$$+\int_0^T W_1(t,s)f\bigg(s,x(s),\int_0^T\Theta\left(s,\theta,\max\left\{x(\tau)\middle|\tau\in\left[\lambda_1(\theta):|:\lambda_2(\theta)\right]\right\}\right)d\theta\bigg)ds+$$

$$+\sum_{0<t_i<T} W_1(t,t_i)G_i\left(x\left(t_i\right)\right)+\sum_{0<t_i<T} W_2(t_i)F_i\left(x\left(t_i\right)\right), \tag{16}$$

where

$$W_0(t,s)=\begin{cases} -V_0(t,s), & t<s\leq T,\\ -V_0(t,s)+Q_2^{-1}(s), & 0\leq s<t,\end{cases}$$

$$W_1(t,s)=\begin{cases} V_1(t,s), \quad t<s\leq T,\\ V_1(t,s)-\int_0^t Q_2^{-1}(\theta)\,B_2(\theta)\,d\theta+\int_s^t Q_2^{-1}(\theta)\,\left[A_2(\theta)+B_2(\theta)\right]\,d\theta, \quad 0\leq s<t,\end{cases}$$

$$W_2(s)=\begin{cases} -Q_1^{-1}(s)\,B_1(s), & t<s\leq T,\\ Q_1^{-1}(s)\,A_1(s), & 0\leq s<t.\end{cases}$$

In the nonlinear system of functional integral equations (16), the vectors C_1 and C_2 are redefinition vectors. We will redefine these constant vectors C_1 and C_2.

3. Inverse problem (1)–(7)

By virtue of intermediate condition (6), we obtain from presentation (16)

$$C_1=Q_1(t)E_1-D_1(t,x(t))-P(t)C_2-\int_0^T W_0(t,s)D_2(s,x(s))ds-$$

$$-\int_0^T W_1(t,s)f\bigg(s,x(s),\int_0^T\Theta\left(s,\theta,\max\left\{x(\tau)\middle|\tau\in\left[\lambda_1(\theta):|:\lambda_2(\theta)\right]\right\}\right)d\theta\bigg)ds-$$

$$-\sum_{0<t_i<T} W_1(t,t_i)G_i\left(x\left(t_i\right)\right)-\sum_{0<t_i<T} W_2(t_i)F_i\left(x\left(t_i\right)\right), \tag{17}$$

where $P(t) = \int\limits_0^T W_0(t,s)ds$.

As we see in (17), there is unknown constant vector C_2. To find C_2, we use intermediate condition (7). Then, from presentation (11), we have

$$C_2 = Q_2(t)E_2 - D_2(t,x(t)) + \sum_{0<t_i<T} K_0(t) G_i\left(x\left(t_i\right)\right) +$$

$$+ \int\limits_0^T K_0(t) f\Big(s, x(s), \int\limits_0^T \Theta\left(s, \theta, \max\left\{x(\tau)\middle|\tau \in [\lambda_1(\theta) : | : \lambda_2(\theta)]\right\}\right) d\theta\Big) ds, \tag{18}$$

where

$$K_0(t) = \begin{cases} B_2(t), \ \ t < s \le T, \\ B_2(t) - Q_2(t), \quad 0 \le s < t. \end{cases}$$

Substituting (18) into presentation (17), we obtain

$$C_1 = Q_1(t)E_1 - P(t)Q_2(t)E_2 - D_1(t,x(t)) + P(t)D_2(t,x(t)) - \int\limits_0^T W_0(t,s)D_2(s,x(s))ds +$$

$$+ \int\limits_0^T K_1(t,s) f\Big(s, x(s), \int\limits_0^T \Theta\left(s, \theta, \max\left\{x(\tau)\middle|\tau \in [\lambda_1(\theta) : | : \lambda_2(\theta)]\right\}\right) d\theta\Big) ds +$$

$$+ \sum_{0<t_i<T} K_1(t,t_i) G_i\left(x\left(t_i\right)\right) - \sum_{0<t_i<T} W_2(t_i) F_i\left(x\left(t_i\right)\right), \tag{19}$$

where

$$K_1(t,s) = \begin{cases} -W_1(t,s) - P(t)B_2(t), \ \ t < s \le T, \\ -W_1(t,s) - P(t)\left[B_2(t) - Q_2(t)\right], \quad 0 \le s < t. \end{cases}$$

Formulas (18) and (19) allow one to determine constant vectors C_1 and C_2. However, there is unknown function $x(t)$ in these expressions. We substitute expressions (18) and (19) into equation (16) and obtain the following nonlinear system of functional integral equations

$$x(t) = J(t;x) \equiv \Phi_0(t) + Q_1^{-1}(t)[D_1(t,x(t)) - D_1(\bar{t}, x(\bar{t}))] +$$

$$+ \Phi_1(t) D_2(\bar{t}, x(\bar{t})) + \int\limits_0^T \Phi_2(t,s) D_2(s,x(s)) ds +$$

$$+ \int\limits_0^T \Phi_3(t,s) f\Big(s, x(s), \int\limits_0^T \Theta\left(s, \theta, \max\left\{x(\tau)\middle|\tau \in [\lambda_1(\theta, x(\theta)) : | : \lambda_2(\theta, x(\theta))]\right\}\right) d\theta\Big) ds +$$

$$+ \sum_{0<t_i<T} \Phi_3(t,t_i) G_i\left(x\left(t_i\right)\right) + \sum_{0<t_i<T} \Phi_4(t,t_i) F_i\left(x\left(t_i\right)\right), \tag{20}$$

where $\Phi_0(t) = Q_1^{-1}(t)Q_1^{-1}(\bar{t})E_1 + Q_2(\bar{t})[Q_1^{-1}(t)P(\bar{t}) + P(t)]E_2, \ \ \Phi_1(t) = Q_1^{-1}(t)P(\bar{t}) - P(t),$

$$\Phi_2(t,s) = Q_1^{-1}(t)W_0(\bar{t},s) + W_0(t,s), \quad \Phi_3(t,s) = Q_1^{-1}(t)K_1(\bar{t},s) + P(t)K_0(\bar{t}) + W_1(t,s),$$

$$\Phi_4(t,s) = \left[1 - Q_1^{-1}(t)\right] W_2(s).$$

4. Unique solvability

Theorem. Suppose that the following conditions are fulfilled:

1). $M_f = \max\limits_{0\leq t\leq T}\left| f\Big(t, 0, \int\limits_0^T \Theta\left(t, s, 0\right)\, ds\Big)\right| < \infty;\ M_{D_i} = \max\limits_{0\leq t\leq T} \left| D_j(t, 0) \right| < \infty,\ j = 1, 2;$

2). $m_F = \max\limits_{i\in\{1,2,\ldots,p\}} \left| F_i(0) \right| < \infty, \quad m_G = \max\limits_{i\in\{1,2,\ldots,p\}} \left| G_i(0) \right| < \infty;$

3). For all $t \in [0, T],\ x, y \in \mathbb{R}^n$, the following inequality holds

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq M_1(t)\ |\, x_1 - x_2 \,| + M_2(t)\ |\, y_1 - y_2 \,|\,;$$

4). For all $t, s \in [0, T]^2,\ x \in \mathbb{R}^n$, the following inequality holds

$$|\Theta(t, s, x_1) - \Theta(t, s, x_2)| \leq M_3(t, s)\ |\, x_1 - x_2 \,|\,;$$

5). For all $t \in [0, T],\ x \in \mathbb{R}^n$, the following inequality holds

$$|\lambda_j(t, x_1) - \lambda_j(t, x_2)| \leq M_{4j}(t)\ |\, x_1 - x_2 \,|\,,\quad j = 1, 2;$$

6). For all $t \in [0, T],\ x \in \mathbb{R}^n$, the following inequality holds

$$|D_j(t, x_1) - D_j(t, x_2)| \leq M_{5j}(t)\ |\, x_1 - x_2 \,|\,,\quad j = 1, 2;$$

7). For all $x \in \mathbb{R}^n,\quad i = 0, 1, ..., p$, the following inequality hold

$$|F_i(x_1) - F_i(x_2)| \leq m_{1i}\ \ |\, x_1 - x_2 \,|\,,\quad |G_i(x_1) - G_i(x_2)| \leq m_{2i}\ \ |\, x_1 - x_2 \,|\,;$$

8). $\rho = \chi_1 + ... + \chi_5 < 1$, where $\chi_1, \ldots, \chi_5$ are defined by formulas (25)–(27) below.

Then equation (20) has unique solution $x(t) \in PC\left([0, T], \mathbb{R}^n\right)$. This solution can be found by the following iterative process:

$$\begin{cases} x^k\left(t\right) = J\big(t; x^{k-1}\big),\ \ k = 1, 2, 3, ... \\ x^0\left(t\right) = \Phi_0(t),\ \ t \in (t_i, t_{i+1}),\ \ i = 0, 1, 2, ..., p. \end{cases} \tag{21}$$

Proof. We consider the following operator

$$J : PC\left([0, T]; \mathbb{R}^n\right) \to PC\left([0, T] \times \mathbb{R}^n; \mathbb{R}^n\right),$$

defined by the right-hand side of equation (20). Applying the principle of contracting operators to (20), we show that the operator J has unique fixed point.

Taking the first and the second conditions of the theorem, we obtain the following estimates for zero approximations and the first difference of the approximations (21):

$$\left\|\, x^0(t) \,\right\| \leq \max_{0\leq t\leq T} \left| \Phi_0(t) \right| = \delta_1 < \infty, \tag{22}$$

$$\begin{aligned} \left\|\, x^1(t) - x^0(t) \,\right\| &\leq 2 \max_{0\leq t\leq T} \left| Q_1^{-1}(t) \right| \cdot \left| D_1(t, 0) \right| + \max_{0\leq t\leq T} \left| \Phi_1(t) \right| \cdot \left| D_2(\bar{t}, 0) \right| + \\ &+ \max_{0\leq t\leq T} \int\limits_0^T \left| \Phi_2(t, s) \right| \cdot \left| D_2(s, 0) \right| ds + \max_{0\leq t\leq T} \int\limits_0^T \left| \Phi_3(t, s) \right| \left| f\Big(s, 0, \int\limits_0^T \Theta(s, \theta, 0) d\theta\Big) \right| ds + \\ &+ \max_{0\leq t\leq T} \sum_{i=1}^p \left| \Phi_3(t, t_i) \right| \cdot \left| G_i(0) \right| + \sum_{i=1}^p \left| \Phi_4(t_i) \right| \cdot \left| F_i(0) \right| \leq \\ &\leq 2 \left\|\, Q_1^{-1}(t) \,\right\| M_{D_1} + \sigma_0 M_{D_2} + \sigma_{11} M_f + \sigma_{12} m_G + \sigma_2 m_F = \delta_2 < \infty, \end{aligned} \tag{23}$$

where

$$\sigma_0 = \max_{0\leq t\leq T} \left| \Phi_2(t) \right| + \max_{0\leq t\leq T} \int\limits_0^T \left| \Phi_2(t, s) \right| ds,\quad \sigma_{11} = \max_{0\leq t\leq T} \int\limits_0^T \left| \Phi_3(t, s) \right| ds,$$

$$\sigma_{12} = \max_{0\leq t\leq T} \sum_{i=1}^p \left| \Phi_3(t, t_i) \right|, \quad \sigma_2 = \sum_{i=1}^p \left| \Phi_4(t_i) \right|.$$

Then, by the third–seventh conditions of the theorem, for difference of arbitrary consecutive approximations and arbitrary $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} \left\|\, x^{k+1}(t) - x^k(t) \,\right\| &\leq 2 \max_{0\leq t\leq T} \left| Q_1^{-1}(t) \right| M_{51}(t)\ \left| x^k(t) - x^{k-1}(t) \right| + \\ &+ \max_{0\leq t\leq T} \left| \Phi_1(t) \right| \cdot M_{52}(\bar{t})\ \left| x^k(\bar{t}) - x^{k-1}(\bar{t}) \right| + \end{aligned}$$

$$+\max_{0\le t\le T}\int\limits_0^T |\,\Phi_2(t,s)\,|\,M_{52}(s)\,\left|\,x^k(s)-x^{k-1}(s)\,\right|ds+$$

$$+\max_{0\le t\le T}\int\limits_0^T |\,\Phi_3(t,s)\,|\,\big[M_1(s)\,\left|\,x^k(s)-x^{k-1}(s)\,\right|+$$

$$+M_2(s)\int\limits_0^T M_3(s,\theta)\left|\max\left\{x^k(\tau)\left|\tau\in[\lambda_1(\theta,x^k(\theta)):|:\lambda_2(\theta,x^k(\theta))]\right.\right\}-\right.$$

$$\left.-\max\left\{x^{k-1}(\tau)\left|\tau\in[\lambda_1(\theta,x^{k-1}(\theta)):|:\lambda_2(\theta,x^{k-1}(\theta))]\right.\right\}\right|d\theta\big]\,ds+$$

$$+\max_{0\le t\le T}\sum_{i=1}^p|\,\Phi_3(t,t_i)\,|\;m_{2i}\,\left|\,x^k(t_i)-x^{k-1}(t_i)\,\right|+\sum_{i=1}^p|\,\Phi_4(t_i)|\;m_{1i}\,\left|\,x^k(t_i)-x^{k-1}(t_i)\,\right|.$$

Hence, by virtue of the following estimate

$$M_2(s)\int\limits_0^T M_3(s,\theta)\left|\max\left\{x^k(\tau)\left|\tau\in[\lambda_1(\theta,x^k(\theta)):|:\lambda_2(\theta,x^k(\theta))]\right.\right\}-\right.$$

$$\left.-\max\left\{x^{k-1}(\tau)\left|\tau\in[\lambda_1(\theta,x^{k-1}(\theta)):|:\lambda_2(\theta,x^{k-1}(\theta))]\right.\right\}\right|d\theta\le$$

$$\le M_2(s)\int\limits_0^T M_3(s,\theta)\Big[\,\Big\|\max\left\{x^k(\tau)\left|\tau\in[\lambda_1(\theta,x^k(\theta)):|:\lambda_2(\theta,x^k(\theta))]\right.\right\}-$$

$$-\max\left\{x^{k-1}(\tau)\left|\tau\in[\lambda_1(\theta,x^k(\theta)):|:\lambda_2(\theta,x^k(\theta))]\right.\right\}\Big\|+$$

$$+\Big\|\max\left\{x^{k-1}(\tau)\left|\tau\in[\lambda_1(\theta,x^k(\theta)):|:\lambda_2(\theta,x^k(\theta))]\right.\right\}-$$

$$-\max\left\{x^{k-1}(\tau)\left|\tau\in[\lambda_1(\theta,x^{k-1}(\theta)):|:\lambda_2(\theta,x^{k-1}(\theta))]\right.\right\}\Big\|\,\Big]d\theta\le$$

$$\le M_2(s)\int\limits_0^T M_3(s,\theta)\Big\{\,\left\|\,x^k(\theta)-x^{k-1}(\theta)\,\right\|+$$

$$+(\delta_1+\delta_2)\Big[\,\left|\lambda_1(\theta,x^k(\theta))-\lambda_1(\theta,x^{k-1}(\theta))\right|+\left|\lambda_2(\theta,x^k(\theta))-\lambda_2(\theta,x^{k-1}(\theta))\right|\,\Big]\Big\}d\theta\le$$

$$\le\max_{0\le t\le T}M_2(t)\int\limits_0^T M_3(t,s)\Big\{\,\left\|\,x^k(s)-x^{k-1}(s)\,\right\|+$$

$$+(\delta_1+\delta_2)(M_{41}(s)+M_{42}(s))\left\|\,x^k(s)-x^{k-1}(s)\,\right\|\,\Big\}ds,$$

and by the introduced norm in the space $PC\,([0,T],\mathbb{R}^n)$, we obtain

$$\left\|\,x^k(t)-\,x^{k-1}(t)\,\right\|_{PC}\le\rho\cdot\left\|\,x^{k-1}(t)-\,x^{k-2}(t)\,\right\|_{PC},\tag{24}$$

where $\rho=\chi_1+...+\chi_5$,

$$\chi_1=2\max_{0\le t\le T}\left|\,Q_1^{-1}(t)\,\right|M_{51}(t),\;\;\chi_2=\max_{0\le t\le T}\left[\,|\,\Phi_1(t)\,|\,M_{52}(\bar t)+\int\limits_0^T|\,\Phi_2(t,s)\,|\,M_{52}(s)ds\right],\tag{25}$$

$$\chi_3=\int\limits_0^T\|\Phi_3(t,s)\|\left[M_1(s)+M_2(s)\int\limits_0^T M_3(s,\theta)\,[1+(\delta_1+\delta_2)(M_{41}(\theta)+M_{42}(\theta))]\,d\theta\right]ds,\tag{26}$$

$$\chi_4=\max_{0\le t\le T}\sum_{i=1}^p|\,\Phi_3(t,t_i)\,|\;m_{2i},\quad\chi_5=\sum_{i=1}^p|\,\Phi_4(t_i)\,|\;m_{1i}.\tag{27}$$

According to the last condition of the theorem, we have $\rho<1$. Therefore, from the estimate (24), it follows that

$$\left\|\,x^k(t)-\,x^{k-1}(t)\,\right\|_{PC}<\left\|\,x^{k-1}(t)-\,x^{k-2}(t)\,\right\|_{PC}.\tag{28}$$

It implies from (28), that the operator J on the right-hand side of equation (20) is contracting. According to fixed point principle in the Banach space $PC\,([0,T],\mathbb{R}^n)$ and taking into account estimates (22), (23) and (28), we conclude that the operator J has unique fixed point. Consequently, equation (20) has unique solution $x(t)\in PC\,([0,T],\mathbb{R}^n)$. $\square$

Substituting this solution $x(t) \in PC\left([0,T], \mathbb{R}^n\right)$ into presentations (18) and (19), we obtain the redefinition vectors C_1 and C_2.

5. Conclusion

The theory of differential equations plays an important role in solving applied problems of sciences and technology. Especially, nonlocal boundary value problems for differential equations with impulsive actions have many applications in mathematical physics, mechanics and technology, in particular in nanotechnology.

In this paper, we investigated the questions of unique solvability of the system of second order integro-differential equations (1) with nonlinear two-point boundary value conditions (2) and (3), with nonlinear conditions (4) and (5) of impulsive effects for $t = t_i,\ i = 1, 2, \ldots, p,\ 0 < t_1 < t_2 < \cdots < t_p < T$ and intermediate conditions (6) and (7). The nonlinear right-hand side of this equation consists of the construction of mixed maxima. The problems of existence and uniqueness of the solution of the inverse problem (1)–(7) are studied. If the system (1) has a solution for all $t \in [0, T],\ t \neq t_i,\ \ i = 1, 2, ..., p$, then it is proved that this solution can be found by the system of nonlinear functional integral equations (20).

The results obtained in this work will allow us to investigate another kind of inverse problems for the equations of mathematical physics with impulsive actions. We hope that our work will stimulate the study of various kind of inverse boundary value problems for impulsive partial differential and integro-differential equations with many redefinition functions and results of investigations find the applications in mechanics, technology and nanotechnology.

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