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Original article

An inversion formula for the weighted Radon transform along family of cones

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ABSTRACT In this paper, an inversion problem for the weighted Radon transform along family of cones in three-dimensional space is considered. An inversion formula for the weighted Radon transform is obtained for the case when the range is a space of infinitely smooth functions.

KEYWORDS integral geometry problem, weighted Radon transform, inversion formula

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1. Introduction

The standard X-ray transform consists of recovering a function supported in a bounded domain from its integrals along straight lines through this domain. In dimension two ($n = 2$), it coincides with the Radon transform [1], which provides the theoretical underpinning for several medical imaging techniques such as Computed Tomography (CT) and Positron Emission Tomography (PET). The X-ray tomography method for three-dimensional object reconstruction is offered in [2] which is based on the $3D$ Radon transform and is compatible with anisotropic beam conditions.

Weighted and limited data Radon transforms ($d = n - 1$) have been studied in [3]. The Coherent X-ray Diffraction Imaging (CXDI) with its application to nanostructures are given in [4]. In [5] examples are constructed for non injectivity weighted ray and Radon transforms along hyperplanes in R^d, $d \geq 3$ with a non-trivial kernel in the space of finite infinitely smooth functions. The weighted Radon transforms in multidimensions are studied in [6]. Authors introduced an analog of Chang approximate inversion formula for such transforms and describe all weights for which this formula is exact. They indicated possible tomographic applications of inversion methods for weighted Radon transforms in $3D$. In paper [7], the reconstruction approach proposed in [6] for weighted ray transforms (weighted Radon transforms along oriented straight lines) in $3D$ is investigated numerically.

Some problems of integral geometry as restoring a function on some linear space from the set of values of this function on a given family of manifolds embedded in this space are studied in [8, 9]. In [10–12] a problem of reconstruction of a function in a strip from their given integrals with known weight function along polygonal and parabolic lines is considered. The uniqueness theorems are proved and the stability estimates for solutions in Sobolev spaces are obtained.

In this paper, we consider an inversion problem for the weighted Radon transform along family of cones in three-dimensional space. The problem is reduced to study of Volterra integral equation. Under some natural conditions, an inversion formula for the weighted Radon transform is obtained for the case when the range is a space of infinitely smooth functions.

2. Problem statement and the Main results

We introduce the following notations

$$(x, y, z), (\xi, \eta, \zeta) \in \mathbb{R}^3, \lambda, \mu \in R^1,$$

$$\Omega = \{(x, y, z) : \ x,\, y \in \mathbb{R},\ z \in [0, h] \ \ h < \infty\}.$$

A family of cones $S(x, y, z)$ is considered on Ω, which are uniquely parameterized using the coordinates of their vertices $(x, y, z) \in \Omega$:

$$S(x, y, z) = \left\{(\xi, \eta, \zeta) : (x - \xi)^2 + (y - \eta)^2 = (z - \zeta)^2, \quad \xi \in \mathbb{R}, \quad 0 \leq \zeta \leq z\right\}.$$

We denote by $\mathcal{J}(\mathbb{R}^2, C[0, h])$ the set of infinitely differentiable functions $\varphi(x, y, z)$ in $x, y \in R^2$ and continuous in $z \in [0, h]$ for which

$$\sup \left| x^{\alpha_1} y^{\alpha_2} \frac{\partial^{\beta+\gamma} \varphi(x, y, z)}{\partial^\beta x \partial^\gamma y} \right| < \infty$$

for all nonnegative integer numbers α_1, α_2, β and γ.

Problem A. Determine a function of three variables $u(x, y, z)$, if the integrals of the function $u(x, y, z)$ over a family of cones $S(x, y, z)$ are known for all (x, y, z) of the layer Ω as

$$\iint_{S(x,y,z)} g(x-\xi, y-\eta)\, u(\xi, \eta, \zeta)\, ds = f(x, y, z), \tag{1}$$

where g is the weight function on $\mathbb{R}^2$ and $f \in \mathcal{J}(\mathbb{R}^2, C[0, h])$.

Let us mark that the left hand side of equation (1) can be considered as the generalized Radon transform $\mathbf{R}$ over a family of cones S (See [13] and [14]), i.e. $\mathbf{R}u = f$.

Denote by $\hat{\hat{f}}(\cdot, \cdot, z)$ the Fourier transform of $f(x, y, z) \in \mathcal{J}(\mathbb{R}^2, C[0, h])$ with respect to $(x, y) \in \mathbb{R}^2$, i.e.

$$\hat{\hat{f}}(\lambda, \mu, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\lambda x + \mu y)} f(x, y, z) dx dy.$$

The following theorem describes the uniqueness conditions and the exact solution for the Problem A. Moreover, the solution expresses the inversion formula for the generalized Radon transform i.e. $u = \mathbf{R}^{-1} f$.

Theorem 2.1. Let $g(x, y) = ax + by$, $|a| + |b| \neq 0$ and $f \in \mathcal{J}(\mathbb{R}^2, C[0, h])$ have continuous partial derivatives up to the third order with respect to variable z. We suppose that $\frac{\partial^i}{\partial z^i} f \in \mathcal{J}(\mathbb{R}^2, C[0, h])$, $i = 1, 2, 3$, $f(x, y, 0) = \frac{\partial}{\partial z} f(x, y, 0) = \frac{\partial^2}{\partial z^2} f(x, y, 0) = \frac{\partial^3}{\partial z^3} f(x, y, 0) = 0$ for all $x, y \in \mathbb{R}$ and $\hat{\hat{f}}(\lambda, \mu, z) = 0$ for all $(\lambda, \mu) \in \{\lambda, \mu \in \mathbb{R} : a\lambda + b\mu = 0\}$ and $z \in [0, h]$. Then the Problem A has unique solution $u(x, y, z)$ which is continuous on Ω.

Additionally, if $f \in \mathcal{J}(\mathbb{R}^2, C[0, h])$ has continuous partial derivatives up to the ninth order with respect to variable z and $\frac{\partial^i}{\partial z^i} f \in \mathcal{J}(\mathbb{R}^2, C[0, h])$, $i = 1, \ldots, 9$, then the function $u(x, y, z)$ has the following form

$$u(x, y, z) = \frac{1}{18\sqrt{2}\pi i} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(\lambda x + \mu y)} \frac{(\frac{d^2}{dt^2} + \lambda^2 + \mu^2)^5}{(a\lambda + b\mu)(\lambda^2 + \mu^2)} \int_0^z \int_0^t (t-\tau)^2 J_2\big((t-\tau)\sqrt{\lambda^2 + \mu^2}\big) \hat{\hat{f}}(\lambda, \mu, \tau) d\tau dt d\mu d\lambda, \tag{2}$$

where $J_n(\cdot)$ is the Bessel function of the first kind of order n i.e.

$$J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+n}}{2^{2k+n} k! \Gamma(n + k + 1)}.$$

3. Proof of the Main result

In the integral on the left hand side of (1), we provide the following changing of variable

$$\zeta = z - \sqrt{(x-\xi)^2 + (y-\eta)^2}.$$

Then

$$\zeta'_\xi = -\frac{x-\xi}{\sqrt{(x-\xi)^2 + (y-\eta)^2}},\ \zeta'_\eta = -\frac{y-\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}},\ ds = \sqrt{1 + \left(\zeta'_\xi\right)^2 + \left(\zeta'_\eta\right)^2} d\xi d\eta.$$

Hence, equation (1) is represented as follows

$$\sqrt{2} \iint_{D(x,y,z)} g(x-\xi, y-\eta) u(\xi, \eta, z - \sqrt{(x-\xi)^2 + (y-\eta)^2}) d\xi d\eta = f(x, y, z). \tag{3}$$

Making the following substitutions in (3)

$$\xi = x - \rho\cos\phi, \quad \eta = y - \rho\sin\phi,$$

we get

$$\sqrt{2} \int_0^{2\pi} \int_0^z g(\rho\cos\phi, \rho\sin\phi)\, u(x - \rho\cos\phi, y - \rho\sin\phi, z - \rho)\, \rho d\rho d\phi = f(x, y, z). \tag{4}$$

We apply the Fourier transform to the both sides of equation (4) with respect to variables x and y

$$\hat{\hat{f}}(\lambda, \mu, z) = \sqrt{2} \int_0^{2\pi} \int_0^z e^{i\rho(\lambda cos\phi + \mu\sin\phi)} g(\rho\cos\phi, \rho\sin\phi)\, \hat{u}(\lambda, \mu, z - \rho)\, \rho d\rho d\phi. \tag{5}$$

where

$$\hat{u}\left(\lambda,\mu,z\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}e^{i(\lambda x+\mu y)}u\left(x,y,z\right)dxdy.$$

Remark that $\hat{\hat{f}}\in\mathcal{J}(\mathbb{R}^2,C[0,h])$, since $f\in\mathcal{J}(\mathbb{R}^2,C[0,h])$.

Making the change of variable $t=z-\rho$ in equation (5), we obtain

$$\hat{\hat{f}}(\lambda,\mu,z)=\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}e^{i(z-t)(\lambda cos\phi+\mu\sin\phi)}g((z-t)\cos\phi,(z-t)\sin\phi)\hat{\hat{u}}(\lambda,\mu,t)(z-t)dtd\phi. \tag{6}$$

Thus, equation (1) is equivalent to equation (6). Since $g(x,y)=ax+by$, equation (6) yields

$$\hat{\hat{f}}(\lambda,\mu,z)=\int\limits_0^{z}G(\lambda,\mu,z-t)\hat{\hat{u}}(\lambda,\mu,t)dt, \tag{7}$$

where

$$G(\lambda,\mu,z-t)=\sqrt{2}(z-t)^2\int\limits_0^{2\pi}e^{i(z-t)(\lambda\cos\phi+\mu\sin\phi)}(a\cos\phi+b\sin\phi)d\phi.$$

According to the equality

$$\lambda\cos\phi+\mu\sin\phi=\sqrt{\lambda^2+\mu^2}\cos(\phi-\psi),\quad \cos\psi=\frac{\lambda}{\sqrt{\lambda^2+\mu^2}},$$

changing variable $\phi=\theta+\psi$ gives us

$$G(\lambda,\mu,z-t)=\sqrt{2}(z-t)^2\int\limits_0^{2\pi}e^{i\beta\cos\theta}(a\cos(\theta+\psi)+b\sin(\theta+\psi))d\theta,$$

where

$$\beta=(z-t)\sqrt{\lambda^2+\mu^2}.$$

Since

$$e^{i\beta\cos\theta}=\cos(\beta\cos\theta)+i\sin(\beta\cos\theta),$$

the function $G(\lambda,\mu,z-t)$ is represented as

$$G(\lambda,\mu,z-t)=\sqrt{2}a(z-t)^2\int\limits_0^{2\pi}\cos(\beta\cos\theta)\cos(\theta+\psi)d\theta+i\sqrt{2}a(z-t)^2\int\limits_0^{2\pi}\sin(\beta\cos\theta)\cos(\theta+\psi)d\theta+$$

$$+\sqrt{2}b(z-t)^2\int\limits_0^{2\pi}\cos(\beta\cos\theta)\sin(\theta+\psi)d\theta+i\sqrt{2}b(z-t)^2\int\limits_0^{2\pi}\sin(\beta\cos\theta)\sin(\theta+\psi)d\theta.$$

Using the identities

$$\cos(\theta+\psi)=\cos\theta\cos\psi-\sin\theta\sin\psi,$$
$$\sin(\theta+\psi)=\sin\theta\cos\psi+\cos\theta\sin\psi,$$

we have

$$G(\lambda,\mu,z-t)=\sqrt{2}a(z-t)^2\cos\psi\int\limits_0^{2\pi}\cos(\beta\cos\theta)\cos\theta d\theta-\sqrt{2}a(z-t)^2\sin\psi\int\limits_0^{2\pi}\cos(\beta\cos\theta)\sin\theta d\theta+$$

$$+i\sqrt{2}a(z-t)^2\cos\psi\int\limits_0^{2\pi}\sin(\beta\cos\theta)\cos\theta d\theta-i\sqrt{2}a(z-t)^2\sin\psi\int\limits_0^{2\pi}\sin(\beta\cos\theta)\sin\theta d\theta+$$

$$+\sqrt{2}b(z-t)^2\cos\psi\int\limits_0^{2\pi}\cos(\beta\cos\theta)\sin\theta d\theta+\sqrt{2}b(z-t)^2\sin\psi\int\limits_0^{2\pi}\cos(\beta\cos\theta)\cos\theta d\theta+$$

$$+i\sqrt{2}b(z-t)^2\cos\psi\int\limits_0^{2\pi}\sin(\beta\cos\theta)\sin\theta d\theta+i\sqrt{2}b(z-t)^2\sin\psi\int\limits_0^{2\pi}\sin(\beta\cos\theta)\cos\theta d\theta.$$

Now, according to the integral equalities

$$\int_0^{2\pi} \cos(\beta\cos\theta)\cos\theta d\theta = 2\int_0^{\pi} \cos(\beta\cos\theta)\cos\theta d\theta,$$

$$\int_0^{2\pi} \sin(\beta\cos\theta)\cos\theta d\theta = 2\int_0^{\pi} \sin(\beta\cos\theta)\cos\theta d\theta$$

and

$$\int_0^{2\pi} \cos(\beta\cos\theta)\sin\theta d\theta = 0, \quad \int_0^{2\pi} \sin(\beta\cos\theta)\sin\theta d\theta = 0,$$

we obtain

$$G(\lambda,\mu,z-t) = 2\sqrt{2}(z-t)^2(a\cos\psi + b\sin\psi)\{\int_0^{\pi} \cos(\beta\cos\theta)\cos\theta d\theta + i\int_0^{\pi} \sin(\beta\cos\theta)\cos\theta d\theta\}.$$

The equalities (see, pages 419-420 in [16])

$$\int_0^{\pi} \cos(\beta\cos\theta)\cos(n\theta)d\theta = \pi\cos\frac{n\pi}{2}J_n(\beta),$$

$$\int_0^{\pi} \sin(\beta\cos\theta)\cos(n\theta)d\theta = \pi\sin\frac{n\pi}{2}J_n(\beta),$$

give one

$$\int_0^{\pi} \cos(\beta\cos\theta)\cos\theta d\theta = 0, \quad \int_0^{\pi} \sin(\beta\cos\theta)\cos\theta d\theta = \pi J_1(\beta).$$

Then equation (7) is written as

$$\hat{\hat{f}}(\lambda,\mu,z) = 2\sqrt{2}i\pi\frac{a\lambda+b\mu}{\sqrt{\lambda^2+\mu^2}}\int_0^z (z-t)^2 J_1\{(z-t)\sqrt{\lambda^2+\mu^2}\}\hat{u}(\lambda,\mu,t)dt, \tag{8}$$

Remark that if equation (8) has a solution then $\hat{\hat{f}}$ should be $\hat{\hat{f}}(\lambda,\mu,z) = 0$ for all $(\lambda,\mu) \in \{\lambda,\mu\in\mathbb{R} : a\lambda+b\mu = 0\}$ and $z\in[0,h]$. This is provided by the condition of the theorem. To prove the existence of a solution of (8), we write

$$\hat{\hat{f}}(z) = \int_0^z G(z,t)\hat{u}(t)dt, \tag{9}$$

where

$$G(z,t) := 2\sqrt{2}i\pi(z-t)^2\frac{a\lambda+b\mu}{\sqrt{\lambda^2+\mu^2}}J_1\{(z-t)\sqrt{\lambda^2+\mu^2}\}, \quad \hat{u}(t) := \hat{u}(\lambda,\mu,t), \hat{\hat{f}}(t) := \hat{\hat{f}}(\lambda,\mu,t).$$

Note that the smoothness of the first kind Bessel function $J_1(\cdot)$ and $J_1(0) = 0$ yield the smoothness of the function $G(\cdot)$ for any fixed $\lambda,\mu\in\mathbb{R}$. The conditions $f(x,y,0) = \frac{\partial}{\partial z}f(x,y,0) = \frac{\partial^2}{\partial z^2}f(x,y,0) = \frac{\partial^3}{\partial z^3}f(x,y,0) = 0$ yield $\hat{\hat{f}}(\lambda,\mu,0) = \frac{\partial}{\partial z}\hat{\hat{f}}(\lambda,\mu,0) = \frac{\partial^2}{\partial z^2}\hat{\hat{f}}(\lambda,\mu,0) = \frac{\partial^3}{\partial z^3}\hat{\hat{f}}(\lambda,\mu,0) = 0$. Since $G(z,z) = G_z^{'}(z,z) = G_z^{''}(z,z) = 0, G_z^{'''}(z,z) \neq 0$, equation (9) has unique solution $\hat{u}(\lambda,\mu,t)$ for any fixed $\lambda,\mu\in\mathbb{R}$. The function $u(t) := \hat{u}(\lambda,\mu,t)$ is a solution of the equation

$$K_z''(z,z)u(z) + \int_0^z K_z'''(z,t)u(t)dt = \hat{\hat{f}}_z'''(z),$$

where $\hat{\hat{f}}(t) := \hat{\hat{f}}(\lambda,\mu,t)$. Note that $\frac{\partial^3}{\partial z^3}\hat{\hat{f}} \in \mathcal{J}(\mathbb{R}^2, C[0,h])$ since $\frac{\partial^3}{\partial z^3}f \in \mathcal{J}(\mathbb{R}^2, C[0,h])$. Therefore, the function $\hat{u}(\lambda,\mu,t)$ is continuous on Ω and, hence, $u(x,y,t)$ is continuous on Ω.

Next, we present the explicit formula for $u(x,y,t)$. The equality $\hat{\hat{f}}(\lambda,\mu,z) = 0$ for all $(\lambda,\mu) \in \{\lambda,\mu\in\mathbb{R} : a\lambda+b\mu = 0\}$ and $z\in[0,h]$ yields continuity of the ratio function $\frac{\sqrt{\lambda^2+\mu^2}}{2\sqrt{2}\pi i(a\lambda+b\mu)}\hat{\hat{f}}(\lambda,\mu,z)$ on Ω.

We rewrite equation (7) as

$$F(z)=\int\limits_0^z K(z,t)U(t)dt. \tag{10}$$

where

$$F(z)=\frac{\sqrt{\lambda^2+\mu^2}}{2\sqrt{2}\pi i(a\lambda+b\mu)}\hat{\hat{f}}(\lambda,\mu,z),$$

$$K(z,t)=(z-t)^2J_1\{(z-t)\sqrt{\lambda^2+\mu^2}\}\hat{u}(\lambda,\mu,t),\quad U(t)=\hat{u}(\lambda,\mu,t).$$

Since the function $f\in\mathcal{J}(\mathbb{R}^2,C[0,h])$ has continuous partial derivatives up to the ninth order with respect to variable $z\in(0,h)$, the same is valid for $\hat{\hat{f}}(\lambda,\mu,z)$. The solution of equation (10) has the form (see, section 1.8-1 of [15])

$$U(z)=\int\limits_0^z V(t)dt, \tag{11}$$

where

$$V(t)=\frac{1}{9\sqrt{\lambda^2+\mu^2}^3}\left(\frac{d^2}{dt^2}+\lambda^2+\mu^2\right)^5\int\limits_0^t(t-\tau)^2J_2\{(t-\tau)\sqrt{\lambda^2+\mu^2}\}F(\tau)d\tau.$$

Thus, from (11), we obtain

$$\hat{u}(\lambda,\mu,z)=\frac{(\frac{d^2}{dt^2}+\lambda^2+\mu^2)^5}{18\sqrt{2}\pi i(a\lambda+b\mu)(\lambda^2+\mu^2)}\int\limits_0^z\int\limits_0^t(t-\tau)^2J_2\{(t-\tau)\sqrt{\lambda^2+\mu^2}\}\hat{\hat{f}}(\lambda,\mu,\tau)d\tau dt. \tag{12}$$

Hence, applying the inverse Fourier transform with respect to variables μ and λ to equation (12), we obtain equation (2).

The Theorem 2.1 is proved.

4. Conclusion

Consider equation (1) as the generalized Radon transform $\mathbf{R}$ over a family of cones S, i.e. $\mathbf{R}u=f$. Let $\mathbf{R}$ consist of functions $f\in\mathcal{J}(\mathbb{R}^2,C[0,h])$ satisfying the following conditions $\frac{\partial^i}{\partial z^i}f\in\mathcal{J}(\mathbb{R}^2,C[0,h]), i=1,2,3,$ $f(x,y,0)=\frac{\partial}{\partial z}f(x,y,0)=\frac{\partial^2}{\partial z^2}f(x,y,0)=\frac{\partial^3}{\partial z^3}f(x,y,0)=0$ for all $x,y\in\mathbb{R}$ and $\hat{\hat{f}}(\lambda,\mu,z)=0$ for all $(\lambda,\mu)\in\{\lambda,\mu\in\mathbb{R}:a\lambda+b\mu=0\}$ and $z\in[0,h]$. Then $\mathbf{R}$ is invertible for the case when the rang of $\mathbf{R}$ is $\mathcal{R}$, where $u=\mathbf{R}^{-1}f$ is continuous on Ω. Additionally, if $f\in\mathcal{R}$ has continuous partial derivatives up to the ninth order with respect to variable z and $\frac{\partial^i}{\partial z^i}f\in\mathcal{J}(\mathbb{R}^2,C[0,h]), i=1,\ldots,9$, then $\mathbf{R}^{-1}f$ is expressed by (2).

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