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Existence of the eigenvalues of a tensor sum of the Friedrichs models with rank 2 perturbation

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ABSTRACT In the paper we consider a tensor sum $H_{\mu,\lambda}$, $\mu,\lambda>0$ of two Friedrichs models $h_{\mu,\lambda}$ with rank two perturbation. The Hamiltonian $H_{\mu,\lambda}$ is associated with a system of three quantum particles on one-dimensional lattice. We investigate the number and location of the eigenvalues of $H_{\mu,\lambda}$. The existence of eigenvalues located respectively inside, in the gap, and below the bottom of the essential spectrum of $H_{\mu,\lambda}$ is proved.

KEYWORDS tensor sum, Hamiltonian, lattice, quantum particles, non-local interaction, Friedrichs model, eigenvalue, perturbation.

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1. Introduction

The problem of the existence of bound states of quantum particles (from the mathematical point of view the existence of the eigenvalues of the corresponding Hamiltonian) is very important in modern mathematical physics. Its usefulness is demonstrated by the particle storage problem. As an example of such an application the storage of hydrogen in nanolayered structures can be given. These systems can be used to produce an effective and safe fuel containers. It is well known that a particle can be stored in disturbed (curved, deformed, etc.) nanolayers. Experiments show that after intercalation into a nanolayered structure, hydrogen ions are concentrated near defects of the structure. In terms of mathematics, this means that the discrete spectrum of the corresponding Hamiltonian is not empty, and the eigenfunctions are localized near the structure defects. The problem of quantum particles storage in a nanolayered structures is considered in [1] and the system of a few electrons in 2D quantum waveguides coupled through a window is studied. The bound states induced by the disturbance are investigated. In [2] a number of numerical results for the bound state energies of one and two-particle systems in two adjacent 3D layers, connected through a window is presented and the relation between the shape of the window and the energy levels, as well as the number of eigenfunction's nodal domains are investigated. For mathematical models of quantum wave guides, it was shown in [3] that in some situations two interacting particles can be trapped more easily than a single particle. In this paper, we consider the existence of an eigenvalue for the Hamiltonian of a three-particle system on a lattice (three-particle discrete Schrödinger operator), the case where the discrete spectrum of the considered Hamiltonian is not empty. Here, the role of the two-particle discrete Schrödinger operator is played by the Friedrichs model.

Spectral properties of the operators known as the Friedrichs model [4] are important in the problems of analysis, mathematical physics, and probability theory. The latter operators act in the Hilbert space $L_2(\mathcal{M},d\mu)$, where $(\mathcal{M},d\mu)$ is a manifold with measure, according to the rule

$$(Af)(p)=u(p)f(p)-\alpha\int\limits_{\mathcal{M}}D(p,q)f(q)d\mu(q),\quad f\in L_2(\mathcal{M},d\mu),\tag{1.1}$$

where the function $u(\cdot)$ is a function on the manifold $\mathcal{M}\subset\mathbb{R}^{\mathrm{d}}$, the function $D(\cdot,\cdot)$ is a function of two variables on the manifold $\mathcal{M}^2$, and α is a coupling constant. In [4] it was established that in the case where $\mathcal{M}=[-1;1]\subset\mathbb{R}$, $u(q)=q$ and $\alpha>0$ is small, the operator A up to finitely many eigenvalues has an absolutely continuous spectrum and that this operator in its absolutely continuous subspace is unitarily equivalent to the operator A_0 of the form

$$(A_0f)(p)=u(p)f(p),\quad f\in L_2(\mathcal{M},d\mu).\tag{1.2}$$

Later, a more general model (in which $\mathcal{M}$ is an arbitrary interval in $\mathbb{R}$, the function f takes values in a Hilbert space $\mathcal{H}$, and the kernel D is replaced with a bounded operator in $\mathcal{H}$) was suggested in [5], and the results in [4] were transferred to it. This generalization essentially extended the range of application of the theory. It was shown in [6], developing the results of [5], and already entirely completely in [7] that the requirement that the parameter α be small can be dropped under certain restrictions on the kernel D (namely, under the assumption that it is compact and that it belongs to the Hölder class with an exponent $\mu > 1/2$). In this case, the spectrum of A_0 consists of an absolutely continuous part filling an interval $\mathcal{M} \subset \mathbb{R}$ and possibly finitely many eigenvalues with finite multiplicity. The existence of wave operators related to A_0 was also proved in [5] and [7] (also see [8] and [9] for the Friedrichs model).

In the present paper, we consider a tensor sum $H_{\mu,\lambda}$, $\mu, \lambda > 0$ of two Friedrichs models $h_{\mu,\lambda}$ with rank two perturbation. The Hamiltonian $H_{\mu,\lambda}$ is associated with a system of three quantum particles on one-dimensional lattice. We investigate the number and location of the eigenvalues of $H_{\mu,\lambda}$. The existence of eigenvalues located respectively inside, in the gap, and below the bottom of the essential spectrum of $H_{\mu,\lambda}$ is proved.

2. Friedrichs model

We denote by $\mathbb{T}$ the one-dimensional torus. The operations addition and multiplication by real numbers elements of $\mathbb{T} \subset \mathbb{R}$ should be regarded as operations on $\mathbb{R}$ modulo $2\pi\mathbb{Z}$. For example, if $x = 3\pi/5$, $y = 2\pi/3 \in \mathbb{T}$, then $x + y = -11\pi/15$, $10x = 0 \in \mathbb{T}$.

Let $L_2(\mathbb{T})$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}$.

We consider the bounded and self-adjoint Friedrichs model $h_{\mu,\lambda}$ acting on the Hilbert space $L_2(\mathbb{T})$ as

$$h_{\mu,\lambda} := h_{0,0} - \mu k_1 + \lambda k_2,$$

where $h_{0,0}$ is the multiplication operator by the function $u(\cdot)$:

$$(h_{0,0} g)(x) = u(x) g(x),$$

and k_α, $\alpha = 1, 2$ are non-local interaction (integral) operators:

$$(k_\alpha g)(x) = v_\alpha(x) \int_{\mathbb{T}} v_\alpha(t) g(t) dt, \quad \alpha = 1, 2.$$

Here $g \in L_2(\mathbb{T})$; $\mu, \lambda > 0$ are positive reals, $u(\cdot)$ and $v_\alpha(\cdot)$, $\alpha = 1, 2$ are real-valued continuous functions on $\mathbb{T}$.

It is remarkable that the choice of operators k_1 and k_2 with different signs allows to study the eigenvalues of $h_{\mu,\lambda}$ located to the left and to the right of its essential spectrum (see Theorem 2.4).

Throughout this paper, we assume the following additional assumptions.

Assumption 2.1. *The function $u(\cdot)$ is a twice continuously differentiable function on $\mathbb{T}$, has minima at the points $x_1, \cdots, x_n \in \mathbb{T}$ and has maxima at the points $y_1, \cdots, y_m \in \mathbb{T}$.*

Assumption 2.2. *We suppose that*

$$\mathrm{mes}(\mathrm{supp}\{v_1(\cdot)\} \cap \mathrm{supp}\{v_2(\cdot)\}) = 0, \tag{2.1}$$

where $\mathrm{mes}(\cdot)$ is the Lebesgue measure on $\mathbb{R}$ and $\mathrm{supp}\{v_\alpha(\cdot)\})$ is the support of the function $v_\alpha(\cdot)$.

The following example shows that the class of functions $u(\cdot)$ and $v_\alpha(\cdot)$, $\alpha = 1, 2$ satisfying above mentioned Assumptions 2.1 and 2.2 is nonempty. To prove this fact, we introduce the functions of the form:

$$\begin{aligned} u(x) &= 1 - \cos(3x); \\ v_1(x) &= \begin{cases} \sin x, & x \in [0; \pi]; \\ 0, & \text{othervice}; \end{cases} \\ v_2(x) &= \sin x - v_1(x), \quad x \in \mathbb{T}. \end{aligned} \tag{2.2}$$

Then, it is easy to check that points $x_1 = 0$, $x_2 = 2\pi/3$, $x_3 = -2\pi/3$ and $y_1 = \pi$, $y_2 = \pi/3$, $y_3 = -\pi/3$ are the extremal points for the function $u(\cdot)$. Validness of the condition (2.1) follows from the constructions of $v_\alpha(\cdot)$, $\alpha = 1, 2$.

The spectrum, the essential spectrum and the discrete spectrum of a bounded self-adjoint operator will be denoted by $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$, respectively.

By the definition, the perturbation $-\mu k_1 + \lambda k_2$ of the operator $h_{0,0}$ is a self-adjoint operator of rank two. Therefore, in accordance with the Weyl theorem [10] about the invariance of the essential spectrum under the finite rank perturbations, the essential spectrum of the operator $h_{\mu,\lambda}$ coincides with the spectrum of $h_{0,0}$:

$$\sigma_{\mathrm{ess}}(h_{\mu,\lambda}) = \sigma(h_{0,0}) = [m; M],$$

where the numbers m and M are defined by

$$m := \min_{x \in \mathbb{T}} u(x), \quad M := \max_{x \in \mathbb{T}} u(x).$$

In order to study the spectral properties of the operator $h_{\mu,\lambda}$, we introduce the following bounded self-adjoint operators (Friedrichs model with rank one perturbation) $h_\mu^{(1)}$, $h_\lambda^{(2)}$ acting on $L_2(\mathbb{T})$ in accordance with the rule

$$h_\mu^{(1)} := h_{0,0} - \mu k_1, \quad h_\lambda^{(2)} := h_{0,0} + \lambda k_2. \tag{2.3}$$

Let $\mathbb{C}$ be the field of complex numbers. We define the analytic functions $\Delta_\mu^{(1)}(\cdot)$ and $\Delta_\lambda^{(2)}(\cdot)$ (the Fredholm determinant associated with the operator $h_\mu^{(1)}$ and $h_\lambda^{(2)}$, respectively) in $\mathbb{C}\setminus[m;M]$ by

$$\Delta_\mu^{(1)}(z):=1-\mu I_1(z),\quad \Delta_\lambda^{(2)}(z):=1+\lambda I_2(z),$$

$$I_i(z):=\int\limits_{\mathbb{T}}\frac{v_i^2(t)dt}{u(t)-z},\quad i=1,2.$$

Then the Birman-Schwinger principle and the Fredholm theorem imply that [11, 12] the operator $h_\mu^{(1)}$ ($h_\lambda^{(2)}$) has an eigenvalue $z\in\mathbb{C}\setminus[m;M]$ if and only if $\Delta_\mu^{(1)}(z)=0$ ($\Delta_\lambda^{(2)}(z)=0$). From here, it follows that for the discrete spectrum of $h_\mu^{(1)}$ and $h_\lambda^{(2)}$ the equalities

$$\sigma_{\rm disc}(h_\mu^{(1)})=\{z\in\mathbb{C}\setminus[m;M]:\ \Delta_\mu^{(1)}(z)=0\};\tag{2.4}$$

$$\sigma_{\rm disc}(h_\lambda^{(2)})=\{z\in\mathbb{C}\setminus[m;M]:\ \Delta_\lambda^{(2)}(z)=0\}\tag{2.5}$$

hold.

The following lemma establishes a relation between eigenvalues of $h_{\mu,\lambda}$ and $h_\mu^{(1)}$, $h_\lambda^{(2)}$.

Lemma 2.3. *Let Assumption* 2.2 *be fulfilled. The number $z\in\mathbb{C}\setminus[m;M]$ is an eigenvalue of $h_{\mu,\lambda}$ if and only if z is an eigenvalue one of the operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$.*

Proof. Let the number $z\in\mathbb{C}\setminus[m;M]$ be an eigenvalue of $h_{\mu,\lambda}$ and $g\in L_2(\mathbb{T})$ be the corresponding eigenfunction. Then g satisfies the equation

$$(u(x)-z)g(x)-\mu v_1(x)\int\limits_{\mathbb{T}}v_1(t)g(t)dt+\lambda v_2(x)\int\limits_{\mathbb{T}}v_2(t)g(t)dt=0.\tag{2.6}$$

It is easy to see that for any $z\in\mathbb{C}\setminus[m;M]$ the relation $u(x)-z\neq 0$ holds for all $x\in\mathbb{T}$. Then equation (2.6) implies

$$g(x)=\frac{\mu v_1(x)d_1-\lambda v_2(x)d_2}{u(x)-z},\tag{2.7}$$

where

$$d_\alpha:=\int\limits_{\mathbb{T}}v_\alpha(t)g(t)dt,\quad \alpha=1,2.\tag{2.8}$$

Substituting the expression (2.7) for g into (2.8) and using Assumption 2.2, that is, the condition (2.1), we conclude that equation (2.6) has a nonzero solution if and only if the system of equations

$$\Delta_\mu^{(1)}(z)d_1=0,$$
$$\Delta_\lambda^{(2)}(z)d_2=0$$

has a nonzero solution, i.e., if the condition $\Delta_\mu^{(1)}(z)\Delta_\lambda^{(2)}(z)=0$ holds. If we set $v_1(x)\equiv 0$ ($v_2(x)\equiv 0$), then by the definitions of $h_{\mu,\lambda}$ and $h_\mu^{(1)}$ ($h_\lambda^{(2)}$), we obtain that $h_{\mu,\lambda}=h_\mu^{(1)}$ ($h_{\mu,\lambda}=h_\lambda^{(2)}$). The last two facts complete the proof. □

By Lemma 2.3, if the Assumption 2.2 is valid, then the discrete spectrum of $h_{\mu,\lambda}$ and $h_\mu^{(1)}$, $h_\lambda^{(2)}$ are related by the equality

$$\sigma_{\rm disc}(h_{\mu,\lambda})=\sigma_{\rm disc}(h_\mu^{(1)})\cup\sigma_{\rm disc}(h_\lambda^{(2)}).$$

We note that the operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$ have a structure simpler than that of $h_{\mu,\lambda}$, and therefore, the latter equality plays an important role in further investigating the spectrum of $h_{\mu,\lambda}$.

In what follows, C_1, C_2 and C_3 denote various positive constants, the values of which are not specified.

Theorem 2.4. *Let Assumptions* 2.1 *and* 2.2 *be fulfilled. If $v_1(x_\alpha)\neq 0$ for some $\alpha\in\{1,2,...,n\}$ and $v_2(y_\beta)\neq 0$ for some $\beta\in\{1,2,...,m\}$, then for all values of $\mu,\lambda>0$ the operator $h_{\mu,\lambda}$ has two simple eigenvalues $E_\mu^{(1)}\in(-\infty;m)$ and $E_\lambda^{(2)}\in(M;\infty)$.*

Proof. By Assumption 2.1, the function $u(\cdot)$ is a twice continuously differentiable function on $\mathbb{T}$, has minima at the points $x_1,\cdots,x_n\in\mathbb{T}$. Therefore, there exist numbers $C_1,C_2>0$ and $\delta>0$ such that

$$C_1|x-x_j|^2\leq u(x)-m\leq C_2|x-x_j|^2,\quad x\in U_\delta(x_j),\quad j=\overline{1,n}.\tag{2.9}$$

If $v_1(x_\alpha)\neq 0$ for some $\alpha\in\{1,2,...,n\}$, then there exist numbers $C_1>0$ and $\delta>0$ such that

$$v_1(x)\geq C_1,\quad x\in U_\delta(x_\alpha).\tag{2.10}$$

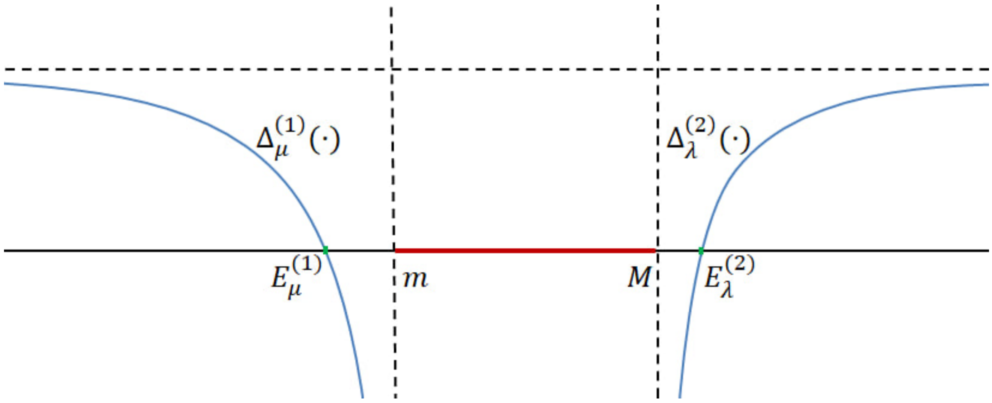

FIG. 1. Spectrum of $h_{\mu,\lambda}$

Using estimates (2.9) and (2.10), we have

$$\int_{\mathbb{T}} \frac{v_1^2(t)dt}{u(t)-m} \geq \int_{U_\delta(x_\alpha)} \frac{v_1^2(t)dt}{u(t)-m} \geq C_1 \int_{U_\delta(x_\alpha)} \frac{dt}{|t-x_\alpha|^2} = \infty.$$

From here, we obtain that $\Delta_\mu^{(1)}(m) = -\infty$. One can see that $\Delta_\mu^{(1)}(z) \to 1$ as $z \to -\infty$. Since the function $\Delta_\mu^{(1)}(\cdot)$ is decreasing on $(-\infty; m]$ for the case $v_1(x_\alpha) \neq 0$ we obtain that the operator $h_\mu^{(1)}$ has unique eigenvalue $E_\mu^{(1)}$ smaller than m. By Lemma 2.3, the number $E_\mu^{(1)}$ is an eigenvalue of $h_{\mu,\lambda}$. It is clear that $\Delta_\lambda^{(2)}(z) > 1$ for all $z < m$. Therefore, the operator $h_\lambda^{(2)}$ has no eigenvalues smaller than m. Consequently, the number $E_\mu^{(1)}$ is the unique eigenvalue of $h_{\mu,\lambda}$.

We note that $u(\cdot)$ is a twice continuously differentiable function on $\mathbb{T}$, has maxima at the points $y_1, \cdots, y_n \in \mathbb{T}$. Therefore, there exist $C_1, C_2 > 0$ and $\delta > 0$ such that

$$C_1|x-y_j|^2 \leq M - u(x) \leq C_2|x-y_j|^2, \quad y \in U_\delta(y_j), \quad j = \overline{1,n}. \tag{2.11}$$

If $v_2(y_\beta) \neq 0$ for some $\beta \in \{1,2,...,n\}$ then there exist $C_1 > 0$ and $\delta > 0$ such that

$$v_2(y) \geq C_1, \quad y \in U_\delta(y_\beta). \tag{2.12}$$

Using estimates (2.11) and (2.12), we have

$$\int_{\mathbb{T}} \frac{v_2^2(t)dt}{u(t)-M} \leq -\int_{U_\delta(y_\beta)} \frac{v_2^2(t)dt}{M-u(t)} \leq -C_1 \int_{U_\delta(y_\beta)} \frac{dt}{|t-y_\beta|^2} = -\infty.$$

The latter assertion yields that $\Delta_\lambda^{(2)}(M) = -\infty$. It is easy to check that $\Delta_\lambda^{(2)}(z) \to 1$ as $z \to \infty$. Since the function $\Delta_\lambda^{(2)}(\cdot)$ is decreasing on $[M;\infty)$ from the fact $v_2(y_\beta) \neq 0$ we obtain that the operator $h_\lambda^{(2)}$ has an eigenvalue $E_\lambda^{(2)}$ bigger than M. By Lemma 2.3, the number $E_\lambda^{(2)}$ is an eigenvalue of $h_{\mu,\lambda}$. By the construction of $\Delta_\mu^{(1)}(\cdot)$, we have $\Delta_\mu^{(1)}(z) > 1$ as $z > M$. So, the operator $h_\mu^{(1)}$ has no eigenvalues bigger than M. Therefore, the number $E_\lambda^{(2)}$ is the unique eigenvalue of $h_{\mu,\lambda}$. □

3. Three-particle model operator

In this section, at first, we provide an overview of the model Schrödinger operator.

We consider the discrete Schrödinger operator $\widehat{A} := \widehat{A}_0 - \widehat{K}$ acting in the space $l_2(\mathbb{Z}^2)$. The kinetic energy $\widehat{A}_0$ is given by a convolution with a function of the general form:

$$(\widehat{A}_0\widehat{\psi})(s_1,s_2) = \sum_{n_1,n_2\in\mathbb{Z}} u_0(s_1-n_1, s_2-n_2)\widehat{\psi}(n_1,n_2),$$

and the potential energy $\widehat{K}$ is defined as follows

$$(\widehat{K}\widehat{\psi})(s_1,s_2) = (u_1(s_1)+u_2(s_2))\widehat{\psi}(s_1,s_2).$$

We assume that the functions $u_0(\cdot,\cdot)$ and $u_\alpha(\cdot)$, $\alpha = 1,2$ satisfy the conditions

$$|u_0(s_1,s_2)| \leq C_0 \exp(-a(|s_1|+|s_2|)), \quad a > 0;$$
$$|u_\alpha(s_1)| \leq C_\alpha \exp(-b_\alpha|s_1|), \quad b_\alpha > 0, \quad \alpha = 1,2,$$

where C_α, $\alpha = 0,1,2$ are positive constants.

The operator $\widehat{A}$ is a particular case of the lattice model Hamiltonian studied in [13, 14].

It is remarkable that in [15] the diffraction approach applications to the quantum scattering problems of three three-dimensional charged quantum particles are considered for the construction of the Schrödinger operator eigenfunction asymptotics in different domains of configuration space.

Let $L_2(\mathbb{T}^n)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^n$, $n = 1, 2$ and $\mathcal{F} : l_2(\mathbb{Z}^2) \to L_2(\mathbb{T}^2)$ be the standard Fourier transformation:

$$(\mathcal{F}\widehat{\psi})(x, y) = \frac{1}{2\pi} \sum_{n_1, n_2 \in \mathbb{Z}} \widehat{\psi}(n_1, n_2) \exp(i[(x, n_1) + (y, n_2)]).$$

Then (see [14]), the operator

$$A := \mathcal{F}\widehat{A}\mathcal{F}^{-1} : L_2(\mathbb{T}^2) \to L_2(\mathbb{T}^2)$$

can be represented as $A := A_0 - K_1 - K_2$, where the operators A_0 and K_α, $\alpha = 1, 2$ are defined by

$$(A_0 f)(x, y) = k_0(x, y) f(x, y), \quad f \in L_2(\mathbb{T}^2);$$

$$(K_1 f)(x, y) = \int_{\mathbb{T}} k_1(x - s) f(s, y) ds, \quad (K_2 f)(x, y) = \int_{\mathbb{T}} k_2(y - s) f(x, s) ds, \quad f \in L_2(\mathbb{T}^2).$$

Here $k_0(\cdot, \cdot)$ and $k_\alpha(\cdot)$ are the Fourier transforms of the functions $u_0(\cdot, \cdot)$ and $u_\alpha(\cdot)$, $\alpha = 1, 2$, respectively. Usually, the operator A is called the momentum representation of the discrete operator $\widehat{A}$.

Let $L_2^{\mathrm{s}}(\mathbb{T}^2)$ be the Hilbert space of square integrable symmetric (complex) functions defined on $\mathbb{T}^2$.

Let us consider the Hamiltonian of the form

$$H_{\mu,\lambda} : L_2^{\mathrm{s}}(\mathbb{T}^2) \to L_2^{\mathrm{s}}(\mathbb{T}^2), \quad H_{\mu,\lambda} := H_{0,0} - \mu(V_{11} + V_{12}) + \lambda(V_{21} + V_{22}), \tag{3.1}$$

where $H_{0,0}$ is the multiplication operator by the function $u(x) + u(y)$:

$$(H_{0,0} f)(x, y) = (u(x) + u(y)) f(x, y),$$

and $V_{\alpha\beta}$, $\alpha, \beta = 1, 2$ are non-local interaction operators:

$$(V_{\alpha 1} f)(x, y) = v_\alpha(x) \int_{\mathbb{T}} v_\alpha(t) f(t, y) dt, \quad (V_{\alpha 2} f)(x, y) = v_\alpha(y) \int_{\mathbb{T}} v_\alpha(t) f(x, t) dt, \quad \alpha = 1, 2.$$

Here $f \in L_2^{\mathrm{s}}(\mathbb{T}^2)$. By the definition, the operators V_{ij}, $i, j = 1, 2$ are partial integral operators with degenerate kernel of rank 1.

It is clear that the operator $H_{\mu\,\lambda}$ is bounded and self-adjoint in $L_2^{\mathrm{s}}(\mathbb{T}^2)$.

We note that the operator $H_{\mu,\lambda}$ is related with the Schrödinger operators of a system of three particles on a three-dimensional lattice and such type operators were studied in [11, 16–18]. In [16, 17], the sufficient conditions for the finiteness and infiniteness of the discrete spectrum are found. In [11], the Efimov effect for model discrete Schrödinger operator was demonstrated when the parameter function $w(\cdot, \cdot)$ has a special form. In [18], the essential spectrum and the number of eigenvalues of the typical model were studied for the function $w(\cdot, \cdot)$ of the form $w(p, q) = u(p)u(q)$.

Main result of the note is the following lemma.

Lemma 3.1. *Let Assumption* 2.1 *and* 2.2 *be fulfilled.* (i) *For any* $\mu, \lambda > 0$, *the numbers* $2E_\mu^{(1)}$ *and* $2E_\lambda^{(2)}$ *are simple eigenvalues of* $H_{\mu,\lambda}$. *Moreover,*

$$\sigma_{\mathrm{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + m\,; E_\mu^{(1)} + M] \cup [2m\,; 2M] \cup [E_\lambda^{(2)} + m\,; E_\lambda^{(2)} + M];$$

$$\sigma_{\mathrm{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}\,; E_\mu^{(1)} + E_\lambda^{(2)}\,; 2E_\lambda^{(2)}\}.$$

(ii) *For any fixed* $a < m$ *and* $b > M$, *there are two numbers* $\mu_0 = \mu_0(a) > 0$ *and* $\lambda_0 = \lambda_0(b) > 0$, *respectively, such that the numbers* $2a, a + b$ *and* $2b$ *are eigenvalues of* H_{μ_0,λ_0}.

(iii) *For any* $c \in [2m; 2M]$, *there exist two numbers* $\mu_1 > 0$ *and* $\lambda_1 > 0$ *such that the number* c *is an eigenvalue of* H_{μ_1,λ_1}.

Proof. From the definitions of $H_{\mu,\lambda}$ and $h_{\mu,\lambda}$, we obtain the representation

$$H_{\mu,\lambda} = h_{\mu,\lambda} \otimes I + I \otimes h_{\mu,\lambda},$$

where I is the identity operator on $L_2(\mathbb{T})$.

Therefore, by theorem on the spectrum of the tensor sum of two operators, the equality

$$\sigma(H_{\mu,\lambda}) = \sigma(h_{\mu,\lambda}) + \sigma(h_{\mu,\lambda}) \tag{3.2}$$

holds.

(i) By Theorem 2.4, for any $\mu, \lambda > 0$, the operator $h_{\mu,\lambda}$ has two eigenvalues $E_\mu^{(1)} \in (-\infty; m)$ and $E_\lambda^{(2)} \in (M; \infty)$. Hence,

$$\sigma(h_{\mu,\lambda}) = \{E_\mu^{(1)}\} \cup [m; M] \cup \{E_\lambda^{(2)}\}.$$

Then by (3.2), the numbers $2E_\mu^{(1)}$ and $2E_\lambda^{(2)}$ are simple eigenvalues of $H_{\mu,\lambda}$. In addition,

$$\sigma_{\mathrm{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + m; E_\mu^{(1)} + M] \cup [2m; 2M] \cup [E_\lambda^{(2)} + m; E_\lambda^{(2)} + M];$$

$$\sigma_{\mathrm{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}; E_\mu^{(1)} + E_\lambda^{(2)}; 2E_\lambda^{(2)}\}.$$

Moreover, $\sigma_{\mathrm{disc}}(H_{\mu,\lambda}) = \sigma_{\mathrm{pp}}(H_{\mu,\lambda})$ if and only if $E_\mu^{(1)} + E_\lambda^{(2)} \in (E_\mu^{(1)} + M; 2m) \cup (2M; E_\lambda^{(2)} + m)$.

(ii) Let for any $a < m$, $b > M$, $\mu_0 = \mu_0(a) = (I_1(a))^{-1}$, $\lambda_0 = \lambda_0(b) = (-I_2(b))^{-1}$ the equalities $\Delta_{\mu_0}^{(1)}(a) = 0$ and $\Delta_{\lambda_0}^{(2)}(a) = 0$ hold. By Lemma 2.3, the numbers a and b are eigenvalues of the operator h_{μ_0,λ_0}. From equality (3.2) we conclude that the numbers $2a$, $a + b$, $2b$ are eigenvalues of H_{μ_0,λ_0}.

(iii) Let $c \in [2m; 2M]$ be arbitrary. Then for any $b > 2M - m$, we have $c - b < m$. Denote $\mu_1 = (I_1(c - b))^{-1}$ and $\lambda_1 = (-I_2(b))^{-1}$. It is clear that $\Delta_{\mu_1}^{(1)}(c - b) = 0$ and $\Delta_{\lambda_1}^{(2)}(b) = 0$. By Lemma 2.3, the numbers $c - b$ and b are eigenvalues of $h_{\mu_1,\lambda}$ $(\lambda > 0)$ and h_{μ,λ_1} $(\mu > 0)$. From here, we obtain that the numbers $c - b$ and b are eigenvalues of h_{μ_1,λ_1}. From equality (3.2), we obtain that the number c is the eigenvalue of H_{μ_1,λ_1}.

So, in this paper, it was shown that there are two eigenvalues lying, correspondingly, to the left and to the right of the essential spectrum for the Friedrichs model $h_{\mu,\lambda}$. The existence of the threshold eigenvalues and virtual levels (threshold energy resonances) for a generalized Friedrichs model have been studied in [19–21]. We recall also that the symmetric operators of the form $S := A \otimes I + I \otimes T$, where A is symmetric and $T = T^*$ is (in general) unbounded, is considered in [22] and a boundary triplet Π_S for S^* preserving the tensor structure is constructed. This paper presents also an application of the result to 1D Schrödinger operators. Such operators naturally arise in problems of simulation of quantum particle having point contact to reservoirs.

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