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Original article

Determination of the coefficient function in a Whitham type nonlinear differential equation with impulse effects

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ABSTRACT In the article, the problems of unique solvability and determination of the redefinition coefficient function in the initial inverse problem for a nonlinear Whitham type partial differential equation with impulse effects are studied. The modified method of characteristics allows partial differential equations of the first order to be represented as ordinary differential equations that describe the change of unknown function along the line of characteristics. The unique solvability of the initial inverse problem is proved by the method of successive approximations and contraction mappings. The determination of the unknown coefficient is reduced to solving the nonlinear integral equation.

KEYWORDS inverse problem, Whitham type equations, determination of the coefficient function, method of successive approximations, unique solvability.

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1. Problem statement

It is known that the dynamics of evolving processes sometimes undergoes abrupt changes. Often, such short-term perturbations are interpreted as impulses. That is, we actually have a dynamic system with impulse effects, the solutions of which are functions with first kind "discontinuities". Differential and integro-differential equations with impulse effects have applications in biological, chemical and physical sciences, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, etc. [1–5]. In particular, such kind of problems appears in biophysics at micro- and nanoscales [6–10]. A lot of publications of studying the differential equations with impulse effects related to various natural and technical processes are appearing [11–20].

Partial differential equations of the first order can be locally solved by methods of the theory of ordinary differential equations by reducing them to a characteristic system. The application of the method of characteristics to the solution of partial differential equations of the first order makes it possible to reduce the study of wave evolution [21]. In [22, 23], methods for integrating nonlinear partial differential equations of the first order were developed. Further, many papers appeared devoted to the study of questions of the unique solvability of the Cauchy problem for different types of partial differential equations of the first order (see, for example, [24–33]). The issues of determining the coefficient in various inverse problems have been considered by many authors, in particular, in [34–39].

In this paper, we consider the problems of unique solvability and determination of the redefinition coefficient function in the nonlinear inverse problem for a Whitham type partial differential equation with nonlinear initial value and nonlinear impulse conditions. So, in the domain $\Omega \equiv [0; T] \times \mathbb{R}$ for $t \neq t_i$, $i = 1, 2, ..., p$, we study the following quasilinear equation

$$\frac{\partial u(t,x)}{\partial t} + u(t,x)\frac{\partial u(t,x)}{\partial x} = \alpha(t)\,\beta(x) + F\left(t, x, u\left(t, x\right)\right) \tag{1}$$

with nonlinear initial value condition

$$u(t,x)_{|t=0} = \varphi\left(x, \int\limits_0^T K(\xi) u(\xi, x) d\xi\right), \quad x \in \mathbb{R} \tag{2}$$

and nonlinear impulsive condition

$$u\left(t_i^+, x\right) - u\left(t_i^-, x\right) = G_i\left(u\left(t_i, x\right)\right), \quad i = 1, 2, ..., p, \tag{3}$$

where $u(t,x)$ is the desired function, $\alpha(t)$ is unknown coefficient function, $t \neq t_i$, $i = 1, 2, ..., p$, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T < \infty$, $0 \neq \beta(x) \in C^1(\mathbb{R})$, $\mathbb{R} \equiv (-\infty, \infty)$, $F(t,x,u) \in C^{0,1,0}(\Omega \times \mathbb{R})$, $\varphi(x,u) \in C^1(\mathbb{R}^2)$,

$K(t) \in C[0,T]$, $u\left(t_i^+, x\right) = \lim\limits_{\nu \to 0^+} u\left(t_i + \nu, x\right)$, $u\left(t_i^-, x\right) = \lim\limits_{\nu \to 0^-} u\left(t_i - \nu, x\right)$ are the right-hand side and the left-hand side limits of function $u(t,x)$ at the point $t = t_i$, respectively.

We use the following Banach spaces: the space $C\left(\Omega, \mathbb{R}\right)$ which consists of continuous functions $u(t,x)$ with the norm

$$\| u \|_C = \sup_{(t,x)\in\Omega} | u(t,x) |$$

and the space

$$PC\left(\Omega, \mathbb{R}\right) = \left\{u : \Omega \to \mathbb{R};\ u(t,x) \in C\left(\Omega_{i,i+1}, \mathbb{R}\right),\ i = 1,...,p\right\}$$

with the following norm

$$\| u \|_{PC} = \max\left\{\| u \|_{C(\Omega_{i,i+1})},\ i = 1,2,...,p\right\},$$

where $\Omega_{i,i+1} = (t_i, t_{i+1}] \times \mathbb{R}$, $u\left(t_i^+, x\right)$ and $u\left(t_i^-, x\right)$ $(i = 0,1,...,p)$ exist and are bounded; $u\left(t_i^-, x\right) = u\left(t_i, x\right)$.

To determine the redefinition coefficient function $\alpha(t)$ in the initial value problem (1)–(3), we use the following nonlinear condition

$$u(t, x_0) = \psi\left(t, \int_0^T \gamma(\xi)\,\alpha(\xi) d\xi\right), \tag{4}$$

where $x_0 \in \mathbb{R}$, $\psi(t,u) \in C^{1,0}([0;T], \mathbb{R})$, $\gamma(t) \in C[0,T]$,

$$\varphi\left(x_0, \int_0^T K(\xi) \int_0^T \gamma(\theta)\,\alpha(\theta) d\theta d\xi\right) = \psi\left(0^+, \int_0^T \gamma(\xi)\,\alpha(\xi) d\xi\right).$$

Direct problem. Find unknown function $u(t,x) \in PC(\Omega, \mathbb{R})$ such that the function $u(t,x)$ for all $(t,x) \in \Omega,\ t \neq t_i,\ i = 1,2,...,p$ satisfies the differential equation (1), initial value condition (2) and for $(t,x) \in \Omega,\ t = t_i,\ i = 1,2,...,p$, satisfies the nonlinear limit condition (3).

Inverse problem. Find a pair of unknown functions $u(t,x) \in PC(\Omega, \mathbb{R})$ and $\alpha(t) \in C([0,T], \mathbb{R})$ such that the function $u(t,x)$ for all $(t,x) \in \Omega,\ t \neq t_i,\ i = 1,2,...,p$ satisfies the differential equation (1), initial value condition (2), for $(t,x) \in \Omega,\ t = t_i,\ i = 1,2,...,p$ satisfies the nonlinear limit condition (3) and nonlinear additional condition (4).

2. Reducing the direct problem to a functional-integral equation

We show that the direct initial value problem (1)–(3) with impulse effects is reduced to solving the following nonlinear functional-integral equation

$$u(t,x) = \Theta(t,x;u) \equiv \varphi\left(p(t,0,x), \int_0^T K(\xi) u(\xi, p(t,\xi,x)) d\xi\right) +$$

$$+ \int_0^t \left[\alpha(s)\,\beta\left(p\left(t,s,x\right)\right) + F\left(s, p(t,s,x), u\left(s, p(t,s,x)\right)\right)\right] ds +$$

$$+ \sum_{0<t_i<t} G_i\left(u\left(t_i, p(t,t_i,x)\right)\right), \tag{5}$$

where $p(t,s,x)$ is defined from the integral equation

$$p(t,s,x) = x - \int_s^t u\left(\theta, p(t,\theta,x)\right) d\theta, \quad p(t,t,x) = x, \tag{6}$$

$x \in \mathbb{R}$ plays the role of a parameter.

Let the function $u(t,x) \in PC(\Omega, \mathbb{R})$ be a solution of the direct problem (1)–(3). We present the domain Ω as follows $\Omega = \Omega_{0,1} \cup \Omega_{1,2} \cup \cdots \cup \Omega_{p,p+1}$, where $\Omega_{i,i+1} = (t_i, t_{i+1}] \times \mathbb{R}$. On the first domain $\Omega_{0,1}$, the equation (1) is rewritten as

$$D_u[u] = \alpha(t)\,\beta(x) + F(t,x,u(t,x)), \tag{7}$$

where $D_u = \left(\dfrac{\partial}{\partial t} + u(t,x)\dfrac{\partial}{\partial x}\right)$ is the Whitham operator.

Now we introduce the extended characteristics which is defined as follows:

$$p(t,s,x) = x - \int_s^t u(\theta,x) d\theta, \quad p(t,t,x) = x.$$

We introduce a function of three dimensional argument $w(t,s,x)=u\left(s,p(t,s,x)\right)$, such that for $t=s$, it takes the form $w(t,t,x)=u(t,p(t,t,x))=u(t,x)$. Let us differentiate the function $w(t,s,x)$ with respect to the new argument s

$$w_s(t,s,x)=u_s\left(s,p(t,s,x)\right)+u_p\left(s,p(t,s,x)\right)\cdot p_s(t,s,x).$$

Then, taking into account the last relation, we rewrite equation (7) in the following extended form

$$\frac{\partial}{\partial s}w(t,s,x)=\alpha(s)\,\beta\left(p(t,s,x)\right)+F\left(s,p(t,s,x),w(t,s,x)\right). \tag{8}$$

Integrating equations (8) along the extended characteristics, we obtain

$$\int\limits_0^{t_1}\left[\alpha(s)\,\beta\left(p(t,s,x)\right)+F\left(s,p(t,s,x),w(t,s,x)\right)\right]ds=w(t,t_1^-,x)-w(t,0^+,x),\quad t\in(0,t_1], \tag{9}$$

$$\int\limits_{t_1}^{t_2}\left[\alpha(s)\,\beta\left(p\left(t,s,x\right)\right)+F\left(s,p(t,s,x),w\left(t,s,x\right)\right)\right]ds=w(t,t_2^-,x)-w(t,t_1^+,x),\quad t\in(t_1,t_2], \tag{10}$$

$$\int\limits_{t_p}^{t_{p+1}}\left[\alpha(s)\,\beta\left(p(t,s,x)\right)+F\left(s,p(t,s,x),w\left(t,s,x\right)\right)\right]ds=w(t,t_{p+1}^-,x)-w(t,t_p^+,x),\quad t\in(t_p,t_{p+1}],\quad t_{p+1}=T. \tag{11}$$

Taking into account the relations $w(t,0^+,x)=w(t,0,x),\ w(t,t_{p+1}^-,x)=w(t,s,x)$ issued from integral relations (9)–(11) on the interval $(0,T]$, we have

$$\int\limits_0^{s}\left[\alpha(\varsigma)\,\beta\left(p(t,\varsigma,x)\right)+F\left(\varsigma,p(t,\varsigma,x),w\left(t,\varsigma,x\right)\right)\right]d\varsigma=$$

$$=\left[w\left(t,t_1,x\right)-w\left(t,0^+,x\right)\right]+\left[w\left(t,t_2,x\right)-w\left(t,t_1^+,x\right)\right]+...+\left[w(t,s,x)-w\left(t,t_p^+,x\right)\right]=$$

$$=-w(t,0,x)-\left[w\left(t,t_1^+,x\right)-w\left(t,t_1,x\right)\right]-\left[w\left(t,t_2^+,x\right)-w\left(t,t_2,x\right)\right]-...-$$

$$-\left[w\left(t,t_p^+,x\right)-w\left(t,t_p,x\right)\right]+w(t,s,x). \tag{12}$$

Taking into account the impulsive condition (3), we rewrite the last equality (12) as follows

$$w(t,s,x)=w(t,0,x)+\int\limits_0^{s}\left[\alpha(\varsigma)\,\beta\left(p(t,\varsigma,x)\right)+F\left(\varsigma,p(t,\varsigma,x),w\left(t,\varsigma,x\right)\right)\right]d\varsigma+\sum_{0<t_i<s}G_i\left(w\left(t,t_i,x\right)\right), \tag{13}$$

where $w(t,0,x)$ is arbitrary constant along the characteristics, which should be determined. The initial value condition (2) for equation (13) takes the form

$$w(t,0,x)=\varphi\left(p(t,0,x),\int\limits_0^{T}K(\xi)w(t,\xi,x)d\xi\right).$$

Then, taking into account this initial value condition, from (13), we obtain that

$$w(t,s,x)=\varphi\left(p(t,0,x),\int\limits_0^{T}K(\xi)w(t,\xi,x)d\xi\right)+$$

$$+\int\limits_0^{s}\left[\alpha(\varsigma)\,\beta\left(p(t,\varsigma,x)\right)+F\left(\varsigma,p(t,\varsigma,x),w\left(t,\varsigma,x\right)\right)\right]d\varsigma+\sum_{0<t_i<s}G_i\left(w\left(t,t_i,x\right)\right). \tag{14}$$

For $t=s$, from (14), we arrive at the nonlinear functional-integral equation (5) together with the integral equation (6).

3. Solvability of the functional-integral equation

For fixed values of redefinition function $\alpha(t)$, we study the functional-integral equation (5).

Theorem 1. Let the following conditions be satisfied:

1. $0 < \sup\limits_{x\in\mathbb{R}} |\,\varphi(x,0)\,| \le \Delta_\varphi < \infty$;
2. $0 < \sup\limits_{x\in\mathbb{R}} |\,\beta(x)\,| \le \Delta_\beta < \infty$;
3. $\sup\limits_{x\in\mathbb{R}} |\,F(t,x,0)\,| \le \Delta_f(t),\ 0 < \Delta_f(t) \in C[0;T]$;
4. $0 < |\,G_i(0)\,| \le \Delta_{G_i} < \infty,\ \ i = 1,2,...,p$;
5. $|\varphi(x_1,u_1) - \varphi(x_2,u_2)| \le \chi_1\left(|x_1 - x_2| + |u_1 - u_2|\right),\ 0 < \chi_1 = \text{const}$;
6. $|\,\beta(x_1) - \beta(x_2)\,| \le \chi_2\,|\,x_1 - x_2\,|\,,\ 0 < \chi_2 = \text{const}$;
7. $|\,G_i(u_1) - G_i(u_2)\,| \le \chi_{3i}\,|\,u_1 - u_2\,|\,,\ 0 < \chi_{3i} = \text{const}$;
8. $|\,F(t,x_1,u_1) - F(t,x_2,u_2)\,| \le Q(t)\,|\,x_1 - x_2\,| + P(t)\,|\,u_1 - u_2\,|$;
9. $0 < Q(t), P(t) \in C[0;T],\ 0 < \max\limits_{t\in[0;T]} \int\limits_0^t \left[Q(s)(t-s) + P(s)\right] ds < \infty$;
10. $\rho_1 = \max\limits_{t\in[0;T]} \int\limits_0^t H(t,s)ds + \sum\limits_{i=1}^{p} \chi_{3i} < 1$, where

$$H(t,s) = \chi_1\left(1 + |\,K(s)\,|\right) + \left(Q(s) + \chi_2\,|\alpha(s)|\right)(t-s) + P(s).$$

Then, for fixed values of $\alpha(t)$, the functional-integral equation (5) has unique solution in the domain Ω. This solution can be founded by the following successive approximations:

$$u_0(t,x) = 0,\ \ u_{k+1}(t,x) \equiv \Theta(t,x;u_k,p_k),\ \ k = 0,\ 1,\ 2,\ \ldots, \tag{15}$$

where $p_k(s,t,x)$ is defined from the following iteration

$$p_0(t,t,x) = x,\ \ p_k(t,s,x) = x - \int\limits_s^t u_{k-1}\left(\theta, p_{k-1}(t,\theta,x)\right) d\theta.$$

Proof. By virtue of the conditions of the theorem, we obtain that the following estimate holds for the first difference of approximation (15):

$$|\,u_1(t,x) - u_0(t,x)\,| \le \sup_{x\in\mathbb{R}} |\,\varphi(x,0)\,| + \sup_{(t,x)\in\Omega} \int\limits_0^t |\,\alpha(s)\,\beta(x)\,|\,ds +$$

$$+ \sum_{0<t_i<T} |\,G_i\,(0)\,| + \max_{t\in[0;T]} \int\limits_0^t \Delta_f(s)ds \le \Delta_\varphi + \sum_{i=1}^{p} \Delta_{G_i} + \Delta_1 + \Delta_2 < \infty, \tag{16}$$

where

$$\Delta_1 = \max_{t\in[0;T]} \int\limits_0^t \Delta_f(s)ds < \infty,\ \Delta_2 = \Delta_\beta \max_{t\in[0;T]} \int\limits_0^t |\,\alpha(s)\,|\,ds < \infty.$$

Taking into account estimate (16) and the conditions of the theorem, we obtain that for arbitrary difference of approximation (15), the following estimate holds:

$$|\,u_{k+1}(t,x) - u_k(t,x)\,| \le \left|\varphi\left(p_{k+1}(t,0,x), \int\limits_0^T K(\xi)u_k(\xi, p_k(t,\xi,x))d\xi\right) - \right.$$

$$\left. -\varphi\left(p_k(t,0,x), \int\limits_0^T K(\xi)u_{k-1}(\xi, p_{k-1}(t,\xi,x))d\xi\right)\right| +$$

$$+ \int\limits_0^t |\,\alpha(s)\,| \cdot |\,\beta\left(p_{k+1}(t,s,x)\right) - \beta\left(p_k(t,s,x)\right)|\,ds +$$

$$+ \int\limits_0^t |\,F\left(s, p_{k+1}(t,s,x), u_k\left(s, p_k(t,s,x)\right)\right) - F\left(s, p_k(t,s,x), u_{k-1}\left(s, p_{k-1}(t,s,x)\right)\right)|\,ds +$$

$$+\sum_{0<t_i<t} |\, G_i\left(u_k\left(t_i, p_k(t,t_i,x)\right)\right) - G_i\left(u_{k-1}\left(t_i, p_{k-1}(t,t_i,x)\right)\right) | \leq$$

$$\leq \chi_1 \left[\int\limits_0^t |\, u_k(s,x) - u_{k-1}(s,x)\,|\, ds + \int\limits_0^T |\, K(s)\,| \cdot |\, u_k(s,x) - u_{k-1}(s,x)\,|\, ds \right] +$$

$$+\int\limits_0^t \left[\left(Q(s) + \chi_2\, |\alpha(s)|\right) \int\limits_s^t |\, u_k(\theta,x) - u_{k-1}(\theta,x)\,|\, d\theta + P(s)\,|\, u_k(s,x) - u_{k-1}(s,x)\,| \right] ds +$$

$$+\sum_{0<t_i<t} \chi_{3i}\,|\, u_k\left(t_i,x\right) - u_{k-1}\left(t_i,x\right) | \leq$$

$$\leq \max_{t\in[0;T]} \int\limits_0^t H(t,s)\,|\, u_k(s,x) - u_{k-1}(s,x)\,|\, ds + \sum_{i=1}^p \chi_{3i}\,|\, u_k(t,x)) - u_{k-1}(t,x)\,| \,, \tag{17}$$

where

$$H(t,s) = \chi_1\left(1 + |\, K(s)\,|\right) + \left(Q(s) + \chi_2\, |\alpha(s)|\right)(t-s) + P(s).$$

In estimation (17), we pass to the norm in the space $PC\left(\Omega, \mathbb{R}\right)$ and arrive at the estimate

$$\|\, u_{k+1}(t,x) - u_k(t,x)\,\|_{PC} \leq \rho_1 \cdot \|\, u_k(t,x) - u_{k-1}(t,x)\,\|_{PC}\,, \tag{18}$$

where

$$\rho_1 = \max_{t\in[0;T]} \int\limits_0^t H(t,s)ds + \sum_{i=1}^p \chi_{3i}.$$

Since $\rho_1 < 1$, it follows from estimate (18) that the sequence of functions $\{u_k(t,x)\}_{k=1}^{\infty}$, defined by formula (15), converges absolutely and uniformly in the domain Ω. In addition, it follows from the existence of the unique fixed point of the operator $\Theta(t,x;u)$ on the right side of (5) that the functional-integral equation (5) has unique solution in the domain Ω. The theorem has been proven.

Corollary. *Let all the conditions of Theorem 1 be satisfied. Then, for fixed values of the function $\alpha(t)$, the direct initial value problem* (1)–(3) *with impulse effects has unique solution in the domain Ω.*

Remark. *Functional-integral equation* (5) *contains four nonlinear functions. So, in the formulation of the theorem, we required that for each nonlinear function the boundedness condition and the Lipschitz condition be satisfied.*

4. Determination of the redefinition coefficient function

Using the nonlinear additional condition (4), from the functional-integral equation (5), we obtain the nonlinear integral equation

$$\int\limits_0^t \beta(t,s)\,\alpha(s)ds + \varphi\left(p(t,0,x_0), \int\limits_0^T K(\xi)\psi\left(\xi, \int\limits_0^T \gamma(\theta)\alpha(\theta)d\theta\right)d\xi\right) +$$

$$+\int\limits_0^t F\left(s, p(t,s,x_0), \psi\left(s, \int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right)\right)ds +$$

$$+\sum_{0<t_i<t} G_i\left(\psi\left(t_i, \int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right)\right) = \psi\left(t, \int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right), \tag{19}$$

where $\beta(t,s) = \beta\left(p(t,s,x_0)\right)$. Integral equation (19) is a very complex equation, because $p(t,s,x_0)$ contains redefinition function in the nonlinear form. But, for $t = s$ in $p(t,s,x_0)$, from equation (19), we come to simpler integral equation with respect to redefinition function $\alpha(t)$:

$$\beta(x_0)\int\limits_0^t \alpha(s)ds + \varphi\left(x_0, \int\limits_0^T K(\xi)\psi\left(\xi, \int\limits_0^T \gamma(\theta)\alpha(\theta)d\theta\right)d\xi\right) +$$

$$+\int\limits_0^t F\left(s, x_0, \psi\left(s, \int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right)\right)ds +$$

$$+\sum_{0<t_i<t} G_i\left(\psi\left(t_i,\int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right)\right)=\psi\left(t,\int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right). \tag{20}$$

Here the following holds true.

Theorem 2. Let all conditions of Theorem 1 be satisfied. Let the following conditions be fulfilled:

1. $\max\limits_{t\in[0;T]} |\,\psi(t,0)\,| \le \Delta_\psi < \infty;\ 0<\Delta_f(t)\in C[0;T];$
2. $|\,\psi(t,u_1)-\psi(t,u_2)\,| \le \chi_4\,|\,u_1-u_2\,|,\quad 0<\chi_4=\text{const};$
3. $\rho_2=\dfrac{\chi_0}{T\,|\beta(x_0)|}\int\limits_0^T |\gamma(\xi)|\,d\xi<1$, where

$$\chi_0=\chi_4\left[1+\chi_1\int\limits_0^T |K(\xi)|\,d\xi+\int\limits_0^t |\Delta_F(s)|\,ds+\sum_{i=1}^n \chi_{3i}\right].$$

Then the inverse problem (1)–(4) with impulse effects has unique pair of solutions $\{u(t,x),\alpha(t)\}$.

Proof. The method of successive approximations can be applied to equation (20). This equation can be reduced to a functional-integral equation by differentiation. The iteration process for equation (20) can be described as follows

$$\alpha_0(t)=0,\ \ \beta(x_0)\int\limits_0^t \alpha_{k+1}(s)ds=$$

$$=\psi\left(t,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)-\varphi\left(x_0,\int\limits_0^T K(\xi)\psi\left(\xi,\int\limits_0^T \gamma(\theta)\alpha_k(\theta)d\theta\right)d\xi\right)-$$

$$-\int\limits_0^t F\left(s,x_0,\psi\left(s,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)\right)ds-\sum_{0<t_i<t} G_i\left(\psi\left(t_i,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)\right). \tag{21}$$

By virtue of the conditions of the theorem, for the first difference from the approximations (21), we obtain

$$|\beta(x_0)|\int\limits_0^t |\alpha_1(s)-\alpha_0(s)|\,ds \le \left|\psi\left(t,\int\limits_0^T \gamma(\xi)\alpha_0(\xi)d\xi\right)\right|+$$

$$+\left|\varphi\left(x_0,\int\limits_0^T K(\xi)\psi\left(\xi,\int\limits_0^T \gamma(\theta)\alpha_0(\theta)d\theta\right)d\xi\right)\right|+$$

$$+\int\limits_0^t \left|F\left(s,x_0,\psi\left(s,\int\limits_0^T \gamma(\xi)\alpha_0(\xi)d\xi\right)\right)\right|ds+\sum_{0<t_i<t}\left|G_i\left(\psi\left(t_i,\int\limits_0^T \gamma(\xi)\alpha_0(\xi)d\xi\right)\right)\right|\le$$

$$\le \Delta_\psi+\Delta_\varphi+\Delta_1+\sum_{i=1}^p \Delta_{G_i}<\infty. \tag{22}$$

Now for the arbitrary difference from the approximations (21), we obtain

$$|\beta(x_0)|\int\limits_0^t |\alpha_{k+1}(s)-\alpha_k(s)|\,ds \le \left|\psi\left(t,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)-\psi\left(t,\int\limits_0^T \gamma(\xi)\alpha_{k-1}(\xi)d\xi\right)\right|+$$

$$+\left|\varphi\left(x_0,\int\limits_0^T K(\xi)\psi\left(\xi,\int\limits_0^T \gamma(\theta)\alpha_k(\theta)d\theta\right)d\xi\right)-\varphi\left(x_0,\int\limits_0^T K(\xi)\psi\left(\xi,\int\limits_0^T \gamma(\theta)\alpha_{k-1}(\theta)d\theta\right)d\xi\right)\right|+$$

$$+\int\limits_0^t \left|F\left(s,x_0,\psi\left(s,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)\right)-F\left(s,x_0,\psi\left(s,\int\limits_0^T \gamma(\xi)\alpha_{k-1}(\xi)d\xi\right)\right)\right|ds+$$

$$+\sum_{0<t_i<t}\left|G_i\left(\psi\left(t_i,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)\right)-G_i\left(\psi\left(t_i,\int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right)\right)\right|\le$$

$$\leq \chi_4 \int\limits_0^T |\gamma(\xi)| \cdot |\alpha_k(\xi) - \alpha_{k-1}(\xi)| \, d\xi +$$

$$+\chi_1 \int\limits_0^T |K(\xi)| \cdot \left| \psi\left(\xi, \int\limits_0^T \gamma(\theta)\alpha_k(\theta)d\theta\right) - \psi\left(\xi, \int\limits_0^T \gamma(\theta)\alpha_{k-1}(\theta)d\theta\right) \right| d\xi +$$

$$+\int\limits_0^t |\Delta_F(s)| \cdot \left| \psi\left(s, \int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right) - \psi\left(s, \int\limits_0^T \gamma(\xi)\alpha_{k-1}(\xi)d\xi\right) \right| ds +$$

$$+\sum_{0<t_i<t} \chi_{3i} \left| \psi\left(t_i, \int\limits_0^T \gamma(\xi)\alpha_k(\xi)d\xi\right) - \psi\left(t_i, \int\limits_0^T \gamma(\xi)\alpha_{k-1}(\xi)d\xi\right) \right| \leq$$

$$\leq \chi_0 \int\limits_0^T |\gamma(\xi)| \cdot |\alpha_k(\xi) - \alpha_{k-1}(\xi)| \, d\xi,$$

where

$$\chi_0 = \chi_4 \left[1 + \chi_1 \int\limits_0^T |K(\xi)| \, d\xi + \int\limits_0^t |\Delta_F(s)| \, ds + \sum_{i=1}^n \chi_{3i} \right].$$

Hence, we pass to the norm in the space $PC\left([0;T], \mathbb{R}\right)$ and arrive at the estimate

$$\|\alpha_{k+1}(t) - \alpha_k(t)\|_{PC} \leq \rho_2 \|\alpha_k(t) - \alpha_{k-1}(t)\|_{PC}, \tag{23}$$

where

$$\rho_2 = \frac{\chi_0}{|\beta(x_0)| \, T} \int\limits_0^T |\gamma(\xi)| \, d\xi.$$

According to the last condition of the theorem, one has the inequality $\rho_2 < 1$. We take into account that

$$\varphi\left(x_0, \int\limits_0^T K(\xi) \int\limits_0^T \gamma(\theta)\alpha(\theta)d\theta d\xi\right) = \psi\left(0^+, \int\limits_0^T \gamma(\xi)\alpha(\xi)d\xi\right).$$

So, from estimates (22) and (23), it follows that integral equation (20) has the unique solution on the interval $[0;T]$.

We substitute the solution of the nonlinear integral equation (20) into the functional-integral equation (5) and obtain the desired solution $u(t,x)$ by the method of successive approximations. The proof of the second theorem is completed.

5. Conclusion

In this paper, the problems of unique solvability and determination of the redefinition coefficient function $\alpha(t)$ in the initial inverse problem (1)–(4) for a nonlinear Whitham type partial differential equation with impulse effects are studied. The modified method of characteristics allows partial differential equations of the first order to be represented as ordinary differential equations that describe the change of unknown function along the line of characteristics. The nonlinear functional-integral equation (5) is obtained. The unique solvability of the initial inverse problem (1)–(4) is proved by the method of successive approximations and contraction mappings. The determination of the unknown coefficient function $\alpha(t)$ is reduced to solving the integral equation (20). After solving nonlinear integral equation (20) by iteration process, we substitute its solution into the functional-integral equation (5) and obtain the desired solution $u(t,x)$ by the method of successive approximations.

The results of this work allows one to investigate other type partial differential equations of the first order with impulse effects. In our present work, we studied the given differential equation (1) with initial value condition with respect to first argument t. Next step is that we will study this equation (1) with initial value condition with respect to the second argument x. Moreover, we would like to study inverse problem with redefinition function, which is part of the initial condition. So, we hope that our work will stimulate the study of various kind of inverse boundary value problems for impulsive partial differential and integro-differential equations with many redefinition functions and results of investigations find applications in mechanics, technology and in nanotechnology.

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