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Irreducible characters of the icosahedral group

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Dedicated to Professor Dr. Vladimir N. Chubarikov

ABSTRACT To study point groups, their irreducible characters are essential. The table of irreducible characters of the icosahedral group A_5 is usually obtained by using its duality to the dodecahedral group. It seems that there is no literature which gives a routine computational way to complete it. In the works of Harter and Allen, a computational method is given and the character table up to the tetrahedral group A_4 using the group algebra table and linear algebra. In this paper, we employ their method with the aid of computer programming to complete the table. The method is applicable to any other more complicated groups.

KEYWORDS icosahedral group, irreducible representation, simple characters, regular representation, eigenvalues.

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1. Introduction and irreducible characters of S_3

The icosahedral group A_5 also denoted C_{60} is important in the light of recent developments of fullerene structures, cf. [1–4]. The character table of the icosahedral group A_5 is usually obtained by using its duality to the dodecahedral group [5](pp. 216–219). Alternatively, it is simply stated without any indication of proof, cf. e.g. [6, 7]. It seems that there is no literature which gives a routine computational way to find it with the aid of computers. We describe the method of Harter [8, 9] and Allen [10] and determine the character table of the icosahedral group. The method is rather a light-hearted one and without much knowledge, one can construct character tables. The procedure is described in the following subsections. For some algebraic preliminaries, see, e.g. [11].

1.1. Algebra table and regular representation matrices

[10](p. 27) gives one the table of irreducible characters up to the tetrahedral group T_d and we shall give one for the icosahedral group. We need to form the table of conjugate classes and their algebra table. We illustrate the procedure by the 3rd symmetric group (Table 1).

TABLE 1. Conjugate classes of S_3

label	representative	type	cardinality
C_1	$(1)(2)(3)$	$(1,0,0)$	1
C_2	$(1,2)(3)$	$(0,1,0)$	3
C_3	$(1,2,3)$	$(0,0,1)$	2

Here C_i^{-1} is the conjugate class consisting of all the inverses of elements of C_i (Table 2).

The (right) regular representation matrix $R(C_\alpha)$ has the (i,j)-entry $c_{i\alpha}^j$, which are the structure constants defined by

$$C_iC_\alpha = \sum_{j=1}^{n} c_{i\alpha}^j C_j, \tag{1}$$

where n is the number of conjugate classes of G.

TABLE 2. Class algebra table for S_3

classes	C_1	C_2	C_3
C_1	C_1	C_2	C_3
$C_2^{-1} = C_2$	C_2	$3C_1 + 3C_3$	$2C_2$
$C_3^{-1} = C_3$	C_3	$2C_2$	$2C_1 + C_3$

Looking at each column in Table 2, we see immediately that

$$R(C_2) = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}, \quad R(C_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}. \tag{2}$$

Note that we always have $R(C_1) = E$, where E is the identity matrix.

1.2. Eigenvalues and eigenspaces of regular representations

$R(C_2)$ has eigenvalues $0, \pm 3$ with the following eigenspaces

$$E_{R(C_2)}(0) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad E_{R(C_2)}(\pm 3) = \mathbb{R}\begin{pmatrix} 1 \\ \pm 3 \\ 2 \end{pmatrix} \tag{3}$$

and $R(C_3)$ has eigenvalues $-1, 2, 2$ with the following eigenspaces

$$E_{R(C_3)}(-1) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad E_{R(C_3)}(2) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbb{R}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}. \tag{4}$$

Remark 1. *To find eigenvalues of $R(C_j)$, $j = 2, 3$ Allen uses the method of raising-to-powers.*

$$C_2^0 = C_1, \quad C_2^1 = C_2, \quad C_2^2 = 3C_1 + 3C_3, \quad C_2^3 = 3C_1 + 3C_3 = 3C_2 + 3C_3C_2 = 3C_2 + 6C_2 = 9C_2, \tag{5}$$

whence the Cayley-Hamilton equation resp. the characteristic equation

$$C_2^3 - 9C_2 = 0, \quad \lambda^3 - 9\lambda = 0 \tag{6}$$

and the eigenvalues are $0, \pm 3$.

$$C_3^0 = C_1, \quad C_3^1 = C_3, \quad C_3^2 = 2C_1 + C_3, \quad C_3^3 = 2C_1 + 3C_3, \tag{7}$$

whence the Cayley-Hamilton equation $C_3^3 - 4C_3^2 + 4C_2 = 0$, But we already have a lower order equation resp. the characteristic equation

$$C_3^2 - C_3 - 2C_1 = 0, \quad \lambda^2 - \lambda - 2 = 0 \tag{8}$$

and the eigenvalues are $2, -1$. *But this is practical only for lower degree matrices. This process may be automated.*

1.3. Matching the eigenvalues

This process may remain manual and depends on inspection.

A character table (CT) is in effect a collection of traces of IR's (Irreducible representation) of the group. As such, all of the entries in a given row of a CT belong to the same IR. Up to now the eigenvalues are arranged in sets according to classes C_i. For a specific IR, $\mathcal{P}$, say, the character $\chi_l^{(\alpha)}$ assigned to class C_i is associated with a specific member of the set $\{\lambda_i\}$. It is therefore required that for a given $\mathcal{P}^{(\alpha)}$, a single eigenvalue be picked from each of the n sets λ_i and that these eigenvalues be arranged in a new set

$$\{\lambda^{(\alpha)}\} = \lambda_1^{(\alpha)}, \cdots, \lambda_n^{(\alpha)} \tag{9}$$

all of which are associated with the given $\mathcal{P}^{(\alpha)}$. This procedure is called *matching the eigenvalues*.

The collection of eigenvalues $\lambda^{(\alpha)}$ has a single column vector $\boldsymbol{v}^{(\alpha)}$ associated with it, which has the property

$$R[C_i]\boldsymbol{v}^{(\alpha)} = \lambda_i^{(\alpha)}\boldsymbol{v}^{(\alpha)}, \quad i = 1, \cdots, n. \tag{10}$$

The vector $\boldsymbol{v}^{(\alpha)}$ is, simultaneously, an eigenvector of every $R(C_i)$. When this property is used in conjunction with (3) and (4), we see that $\lambda = -1$ from $R(C_2)$, and $\lambda = 0$ from $R(C_3)$ belong to the same set $\{\lambda^{(1)}\}$, where the common eigenvectors are

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}^{(3)} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}. \tag{11}$$

Here $\boldsymbol{v}^{(2)}$ is a common eigenvector of $R(C_2)$ and $R(C_3)$ belonging to the eigenvalue $\lambda_2^{(2)} = 3$ resp. $\lambda_3^{(2)} = 2$. Similarly, $\boldsymbol{v}^{(3)}$ is a common eigenvector belonging to the eigenvalue $\lambda_2^{(2)} = -3$ resp. $\lambda_3^{(3)} = 2$. Every set $\{\lambda^{(\alpha)}\}$ contains the n-fold multiple eigenvalues $\lambda = 1$ from $R(C_1)$, so that the complete set found is represented in Table 3.

TABLE 3. Eigenvalues arranged

eigenvalue set	C_1	C_2	C_3	eigenvector
$\{\lambda^{(1)}\}$	$\lambda_1^{(1)} = 1$	$\lambda_2^{(1)} = 0$	$\lambda_2^{(1)} = -1$	$\boldsymbol{v}^{(1)}$
$\{\lambda^{(2)}\}$	$\lambda_1^{(2)} = 1$	$\lambda_2^{(2)} = 3$	$\lambda_3^{(2)} = 2$	$\boldsymbol{v}^{(2)}$
$\{\lambda^{(3)}\}$	$\lambda_1^{(3)} = 1$	$\lambda_2^{(3)} = -3$	$\lambda_3^{(3)} = 2$	$\boldsymbol{v}^{(3)}$

1.4. Finding values of irreducible characters

To find CT we accommodate the values of $\lambda_l^{(\alpha)}$ and arrange the characters in the order of increasing dimension of IR. The following formula appears as the coefficients of (49') in [9](p. 747, l. 2):

$$\chi_j^{(\alpha)} = \frac{\ell^{(\alpha)}}{\mathrm{card}(C_j)} \lambda_j^{(\alpha)}, \tag{12}$$

where $\chi_l^{(\alpha)}$ is the character value $\chi^{(\alpha)}(ord(C_j))$ of the jth class in the αth irreducible representation (IR), $\ell^{(\alpha)}$ the dimension of the αth IR and $ord(C_j)$ is the order of the jth class.

We appeal to the formula ([9](p. 747))

$$\frac{1}{|G|} \sum_j \frac{(\lambda_j^{(\alpha)})^2}{\mathrm{card}(C_j)} = \frac{1}{(\ell^{(\alpha)})^2}. \tag{13}$$

It follows that $\ell^{(2)} = 2$ and other two are 1. We rearrange Table 3 in the order of dimensions and label them as follows (Table 4) (so as to compare with [10](p. 23)).

TABLE 4. Eigenvalues arranged

IR	C_1	C_2	C_3	dim
$\mathcal{P}^{(0)}$	$\lambda_1^{(0)} = 1$	$\lambda_2^{(0)} = 3$	$\lambda_3^{(0)} = 2$	$\ell^{(0)} = 1$
$\mathcal{P}^{(1)}$	$\lambda_1^{(1)} = 1$	$\lambda_2^{(1)} = -3$	$\lambda_2^{(1)} = 2$	$\ell^{(1)} = 1$
$\mathcal{P}^{(2)}$	$\lambda_1^{(2)} = 1$	$\lambda_2^{(2)} = 0$	$\lambda_3^{(2)} = -1$	$\ell^{(2)} = 2$

We stretch the interpretation of (12) to mean

$$\chi_j^{(\alpha)} = \ell^{(\alpha)} \left(\frac{\lambda_j^{(\alpha)}}{\mathrm{card}(C_j)} \right). \tag{14}$$

Then

$$\chi_j^{(2)} = 2 \left(\frac{1}{\mathrm{card}(C_1)}, \frac{0}{\mathrm{card}(C_2)}, \frac{-1}{\mathrm{card}(C_3)} \right) = \left(\frac{2}{1}, \frac{0}{3}, -\frac{2}{2} \right) = (2, 0, -1). \tag{15}$$

Similarly,

$$\chi_j^{(0)} = \left(\frac{1}{\mathrm{card}(C_1)}, \frac{3}{\mathrm{card}(C_2)}, \frac{2}{\mathrm{card}(C_3)} \right) = (1, 1, 1), \quad \chi_j^{(1)} = (1, -1, 1). \tag{16}$$

TABLE 5. Values of irreducible characters

IR	C_1	C_2	C_3
$\mathcal{P}^{(0)}$	$\chi_1^{(0)} = 1$	$\chi_2^{(0)} = 1$	$\chi_3^{(0)} = 1$
$\mathcal{P}^{(1)}$	$\chi_1^{(1)} = 1$	$\chi_2^{(1)} = -1$	$\chi_2^{(1)} = 1$
$\mathcal{P}^{(2)}$	$\chi_1^{(2)} = 2$	$\chi_2^{(2)} = 0$	$\chi_3^{(2)} = -1$

TABLE 6. Conjugate classes of S_5

label	representative	type	cardinality
C_1	$(1)(2)(3)(4)(5)$	$(5,0,0,0,0)$	1
C_2	$(1,2)(3)(4)(5)$	$(3,1,0,0,0)$	10
C_2	$(1,2)(3,4)(5)$	$(1,2,0,0,0)$	15
C_4	$(1,2,3)(4)(5)$	$(2,0,1,0,0)$	20
C_5	$(1,2,3)(4,5)$	$(0,1,1,0,0)$	20
C_6	$(1,2,3,4)(5)$	$(1,0,0,1,0)$	30
C_7	$(1,2,3,4,5)$	$(0,0,0,0,1)$	24

TABLE 7. Conjugacy classes of A_5

class	type	representative	k_j
C_1	$(5,0,0,0,0)$	(1)	1
C_2	$(2,0,1.0,0)$	$(1,2,3)$	20
C_3	$(1,2,0,0,0)$	$(1,2)(3,4)$	15
C_4	$(0,0,0,0,1)$	$(1,2,3,4,5)$	12
C_5	$(0,0,0,0,1)$	$(2,1,3,4,5)$	12

2. Irreducible characters of A_5

This will be much harder. We need to prepare the class algebra table (Table 6).
The goal is to establish the following theorem.

Theorem 1. *All simple characters of A_5 are given by Table 8:*

TABLE 8. All simple characters of A_5 (τ indicates $\dfrac{1+\sqrt{5}}{2}$ – the golden ratio)

	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	3	0	-1	τ	$-\tau^{-1}$
χ_3	3	0	-1	$-\tau^{-1}$	τ
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

2.1. Class algebra table and regular representation matrices

For this, we need the class algebra table (Table 9)

TABLE 9. Class algebra table for A_5

classes	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2^{-1}=C_2	C_2	$20C_1$+$7C_2$+$8C_3$+$5C_4$+$5C_5$	$6C_2$+$4C_3$+$5C_4$+$5C_5$	$3C_2$+$4C_3$+$5C_4$+$5C_5$	$3C_2$+$4C_3$+$5C_4$+$5C_5$
C_3^{-1}=C_3	C_3	$6C_2$+$4C_3$+$5C_4$+$5C_5$	$15C_1$+$3C_2$+$2C_3$+$5C_4$+$5C_5$	$3C_2$+$4C_3$+$5C_5$	$3C_2$+$4C_3$+$5C_4$
C_4^{-1}=C_4	C_4	$3C_2$+$4C_3$+$5C_4$+$5C_5$	$3C_2$+$4C_3$+$5C_5$	$12C_1$+$3C_2$+$5C_4$+C_5	$3C_2$+$4C_3$+C_4+C_5
C_5^{-1}=C_5	C_5	$3C_2$+$4C_3$+$5C_4$+$5C_5$	$3C_2$+$4C_3$+$5C_4$	$3C_2$+$4C_3$+C_4+C_5	$12C_1$+$3C_2$+C_4+$5C_5$

$$R(C_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 20 & 7 & 8 & 5 & 5 \\ 0 & 6 & 4 & 5 & 5 \\ 0 & 3 & 4 & 5 & 5 \\ 0 & 3 & 4 & 5 & 5 \end{pmatrix}. \tag{17}$$

$$R(C_3) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 4 & 5 & 5 \\ 15 & 3 & 2 & 5 & 5 \\ 0 & 3 & 4 & 0 & 5 \\ 0 & 3 & 4 & 5 & 0 \end{pmatrix}. \tag{18}$$

$$R(C_4) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 4 & 5 & 5 \\ 0 & 3 & 4 & 0 & 5 \\ 12 & 3 & 0 & 5 & 1 \\ 0 & 3 & 4 & 1 & 1 \end{pmatrix}. \tag{19}$$

$$R(C_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 4 & 5 & 5 \\ 0 & 3 & 4 & 5 & 0 \\ 0 & 3 & 4 & 1 & 1 \\ 12 & 3 & 0 & 1 & 5 \end{pmatrix}. \tag{20}$$

2.2. Finding eigenvalues and eigenvectors by a computer

(1) The eigenvalues of the matrix $R(C_2)$ are obtained by a python program:

$$20, 5, -4, 0, 0$$

. Their corresponding eigenvectors are as follows

$$v_1 = (0.03, -0.15, -0.196, -0.012, -0.016)$$
$$v_2 = (0.661, -0.753, -0.784, 0, 0)$$
$$v_3 = (0.496, 0, 0.588, 0.063, 0.849)$$
$$v_4 = (0.396, 0.452, 0, -0.73, -0.437)$$
$$v_5 = (0396, 0.452, 0, 0.679, -0.241)$$

and the polynomial is

$$\lambda^5 - 21\lambda^4 + 400\lambda^2.$$

(2) The eigenvalues of the matrix $R(C_3)$ obtained by a python program are:

$$15, -5, 3, 0, -5$$

and the their corresponding eigenvectors are

$$\begin{aligned} v_1 &= (-0.033, -0.171, 0.196, 0.15, 0.016) \\ v_2 &= (-0.661, 0, -0.784, 0.753, 0) \\ v_3 &= (-0.496, 0.857, 0.588, 0, -0.08) \\ v_4 &= (-0.396, -0.342, 0, -0.452, -0.671) \\ v_5 &= (-0.396, -0.342, 0, -0.452, 0.736) \end{aligned}$$

and the polynomial is

$$\lambda^5 - 8\lambda^4 - 110\lambda^3 + 1125\lambda.$$

(3) The eigenvalues of the matrix $R(C_4)$ obtained by a python program are:

$$12, 6.47, 0, -3, -2.47$$

and the their corresponding eigenvectors are

$$\begin{aligned} v_1 &= (0.033, 0.116, -0.196, -0.15, 0.116) \\ v_2 &= (0.661, 0, 0.784, -0.753, 0) \\ v_3 &= (0.496, -0.581, -0.588, 0, -0.581) \\ v_4 &= (0.396, 0.752, 0, 0.452, -0.287) \\ v_5 &= (0.396, -0.287, 0, 0.452, 0.752) \end{aligned}$$

and the polynomial is

$$\lambda^5 - 13\lambda^4 - 16\lambda^3 + 288\lambda^2 + 576\lambda.$$

(4) The eigenvalues of the matrix $R(C_5)$ obtained by a python program are:

$$12, 6.47, 0, -3, -2.47$$

and the their corresponding eigenvectors are

$$\begin{aligned} v_1 &= (0.033, 0.116, -0.196, 0.15, 0.116) \\ v_2 &= (0.661, 0, 0.784, 0.753, 0) \\ v_3 &= (0.496, -0.581, -0.588, 0, -0.581) \\ v_4 &= (0.396, -0.287, 0, -0.452, 0.752) \\ v_5 &= (0.396, 0.752, 0, -0.452, -0.287) \end{aligned}$$

and the polynomial is

$$\lambda^5 - 13\lambda^4 - 16\lambda^3 + 288\lambda^2 + 576\lambda = \lambda(\lambda - 12)(\lambda + 3)(\lambda^2 - 4\lambda - 16)$$

The eigenvalues of $R(C_5)$ are:

$$12, 4\tau, 0, -3, -4\tau^{-1}.$$

2.3. Matched eigenvalues

This section combines §1.2 and §1.3 to give the table corresponding to Table 4.

TABLE 10. Eigenvalues arranged

	C_1	C_2	C_3	C_4	C_5	eigenvectors	dim
$\{\lambda^{(1)}\}$	1	20	15	12	12	$\boldsymbol{v}^{(1)}$	$\ell^{(1)} = 1$
$\{\lambda^{(2)}\}$	1	0	-5	4τ	$-4\tau^{-1}$	$\boldsymbol{v}^{(2)}$	$\ell^{(2)} = 3$
$\{\lambda^{(3)}\}$	1	0	-5	$-4\tau^{-1}$	4τ	$\boldsymbol{v}^{(3)}$	$\ell^{(3)} = 3$
$\{\lambda^{(4)}\}$	1	5	0	-1	-3	$\boldsymbol{v}^{(4)}$	$\ell^{(4)} = 4$
$\{\lambda^{(2)}\}$	1	-4	3	0	0	$\boldsymbol{v}^{(5)}$	$\ell^{(5)} = 5$

Here

$$\begin{aligned}
\boldsymbol{v}^{(1)} &= {}^t(0.03, -0.15, -0.196, -0.012, -0.016)\\
\boldsymbol{v}^{(2)} &= {}^t(0396, 0.452, 0, 0.679, -0.241)\\
\boldsymbol{v}^{(3)} &= {}^t(0.661, -0.753, -0.784, 0, 0)\\
\boldsymbol{v}^{(4)} &= {}^t(0.396, 0.452, 0, -0.73, -0.437)\\
\boldsymbol{v}^{(5)} &= {}^t(0.496, 0, 0.588, 0.063, 0.849).
\end{aligned}$$

2.4. Proof of Theorem 1

Using the method in §1.4, we find the values of all ICs.

$$\begin{aligned}
\chi_j^{(1)} &= 1\left(\frac{1}{\mathrm{card}(C_1)}, \frac{20}{\mathrm{card}(C_2)}, \frac{15}{\mathrm{card}(C_3)}, \frac{12}{\mathrm{card}(C_4)}, \frac{12}{\mathrm{card}(C_5)}\right)\\
&= \left(\frac{1}{1}, \frac{20}{20}, \frac{15}{15}, \frac{12}{12}, \frac{12}{12}\right) = (1, 1, 1, 1, 1).
\end{aligned} \tag{21}$$

$$\begin{aligned}
\chi_j^{(2)} &= 3\left(\frac{1}{\mathrm{card}(C_1)}, \frac{0}{\mathrm{card}(C_2)}, \frac{-5}{\mathrm{card}(C_3)}, \frac{4\tau}{\mathrm{card}(C_4)}, \frac{-4\tau^{-1}}{\mathrm{card}(C_5)}\right)\\
&= (1, 0, -1, \tau, -\tau^{-1}).
\end{aligned} \tag{22}$$

$$\begin{aligned}
\chi_j^{(3)} &= 3\left(\frac{1}{\mathrm{card}(C_1)}, \frac{0}{\mathrm{card}(C_2)}, \frac{-5}{\mathrm{card}(C_3)}, \frac{-4\tau^{-1}}{\mathrm{card}(C_4)}, \frac{4\tau}{\mathrm{card}(C_5)}\right)\\
&= (1, 0, -1, -\tau^{-1}, \tau).
\end{aligned} \tag{23}$$

$$\begin{aligned}
\chi_j^{(4)} &= 4\left(\frac{1}{\mathrm{card}(C_1)}, \frac{5}{\mathrm{card}(C_2)}, \frac{0}{\mathrm{card}(C_3)}, \frac{-1}{\mathrm{card}(C_4)}, \frac{-3}{\mathrm{card}(C_5)}\right)\\
&= (4, 1, 0, -1, -1).
\end{aligned} \tag{24}$$

$$\begin{aligned}
\chi_j^{(5)} &= 5\left(\frac{1}{\mathrm{card}(C_1)}, \frac{-4}{\mathrm{card}(C_2)}, \frac{3}{\mathrm{card}(C_3)}, \frac{0}{\mathrm{card}(C_4)}, \frac{0}{\mathrm{card}(C_5)}\right)\\
&= 5\left(\frac{1}{1}, \frac{-4}{20}, \frac{3}{15}, 0, 0\right) = (5, -1, 1, 0, 0).
\end{aligned} \tag{25}$$

as in Table 7. This proves Theorem 1. □

3. Conclusion

The method described here of Harter and Allen may be applied to any other interesting finite groups which will be conducted elsewhere.

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