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Original article

A model of charged particle on the flat Möbius strip in a magnetic field

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ABSTRACT The spectral problem for the Schrödinger operator with a magnetic field on the flat Möbius strip is considered. The model construction is described. It is compared with the case of the Laplace operator.

KEYWORDS Landau operator, flat Möbius strip, spectrum

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1. Introduction

The quantum Hall effect discovered at the end of the twentieth century (see, e.g., [1–5]) is intensively used in nanoelectronics. From the mathematical point of view the problem is related to the investigation of two-dimensional magnetic Schrödinger operator H_L [3–5]. The magnetic Schrödinger operator in a strip on the plane was studied in many papers [6–9]. There are papers devoted to the spectral problem for three-dimensional Hamiltonian with a magnetic field (see, e.g., [10, 11]).

We use the results of work [12] which studied the magnetic Schrödinger operator in an infinite strip on the plane.

Last time, curved nanostructures attract a special attention. Physicists investigate the properties of nanosystems caused by the nanostructure curvature (see, e.g., [13–18]). We can mention also a model based on quantum mechanics in spaces of constant curvature [19,20]. Hamiltonians on curved manifolds are especially interesting. In the present paper we deal with the Möbius strip. Recently, a work [21] appeared which considers the Dirichlet Laplacian on the Möbius strip. The authors deal with Courant-sharp property for Dirichlet eigenfunctions on the flat Möbius strip. In the present paper we consider the Dirichlet eigenfunctions for the magnetic Schrödinger operator (Landau operator) on the flat Möbius strip.

Let us describe the flat Möbius strip. Usually, it was made by the following way [21]. We start with the infinite strip $S_\infty = (-a, a) \times (-\infty, \infty)$ with width $2a$, equipped with the flat metric $dx^2 + dy^2$ of $\mathbb{R}^2$. Given $b > 0$, define the following isometry of S_{infty}:

$$\sigma_b : (x, y) \to (-x, y + b).$$

Define the groups

$$G = \{\sigma_b^k | k \in \mathbb{Z}\}, \quad G_2 = \{\sigma_b^k | k \in 2\mathbb{Z}\}.$$

The group G_2 is a subgroup of G, of index 2, generated by σ_b^2. The action of G on S_∞ is smooth, isometric, totally discontinuous, without fixed points. Correspondingly, we can consider the quotient manifolds with boundary

$$C_b = S_\infty/G_2, \quad M_b = S_\infty/G,$$

equipped with the flat metric induced from the metric of S_∞. Here C_b is the cylinder and M_b is the flat Möbius strip.

This construction is convenient for the authors of [21] because they deal with the Laplacian which “doesn't feel” a direction, i. e. it is invariant in respect to the map $(x, y) \to (-x, y)$. We will deal with the magnetic Schrödinger operator (Landau operator) which “feels” the direction. That is why, we will use another construction of the flat Möbius strip related to gluing of rectangles.

In the present short rapid note we present the main theorem only. Detailed proof, description of the model and analysis of the result will be published in the next paper.

2. Flat Möbius strip in the magnetic field

For the case of the magnetic Schrödinger operator (Landau operator), the situation is more complicated than for the Laplace operator. Consider four copies Ω_j, $j = 1, 2, 3, 4$, of the rectangle $\Omega = (-a, a) \times (-b, b)$. The initial operator is the orthogonal sum of operators H_j^M defined in $L_2(\Omega_j)$: $H^M = H_1^M \oplus H_2^M \oplus H_3^M \oplus H_4^M$ where

$$H_{1,2}^M = -\frac{\partial^2}{\partial x^2} - (\frac{\partial}{\partial y} - 2i\pi\xi x)^2,$$
$$H_{3,4}^M = -\frac{\partial^2}{\partial x^2} - (\frac{\partial}{\partial y} + 2i\pi\xi x)^2. \tag{1}$$

It is the Hamiltonian of free two-dimensional particle with charge e in the homogeneous magnetic field $\mathbf{B}$ orthogonal to the plane of the particle confinement. Here the vector potential of the magnetic field is chosen in the Landau gauge. Let $\Phi_0 = 2\pi\hbar c/|e|$ be the magnetic flux quantum playing a role of a unit for the magnetic flux in the system, $\xi = |\mathbf{B}|/\Phi_0$ is the density of the magnetic flux, i.e. the number of the magnetic flux quanta through the unit area on the plane of the system, x, y are the Cartesian coordinates on the plane. The system of physical units is chosen in such a way that the charge of the particle e, the speed of light c and the Planck constant $\hbar$ equal 1, the mass of the particle is one half.

We include in the domain of H^M functions $(u_1, u_2, u_3, u_4) \in \sum_{j=1}^{j=4} \oplus W_2^2(\Omega_j)$, $W_2^2(\Omega_j)$ is the Sobolev space in Ω_j, satisfying the following conditions:

$$D(H^M) : \begin{cases} u_j(-a, y) = u_j(a, y) = 0, \quad j = 1, 2, 3, 4 \\ u_1(x, b) = u_2(-x, b), \quad \frac{\partial u_1}{\partial y_1}(x, b) = -\frac{\partial u_2}{\partial y_2}(-x, b), \\ u_2(x, -b) = u_3(x, -b), \quad \frac{\partial u_2}{\partial y_2}(x, -b) = -\frac{\partial u_3}{\partial y_3}(x, -b), \\ u_3(x, b) = u_4(-x, b), \quad \frac{\partial u_3}{\partial y_3}(x, b) = -\frac{\partial u_4}{\partial y_4}(-x, b), \\ u_4(x, -b) = u_1(x, -b), \quad \frac{\partial u_4}{\partial y_4}(x, -b) = -\frac{\partial u_1}{\partial y_1}(x, -b). \end{cases} \tag{2}$$

Remark. Each rectangle Ω_j presents, actually, one side of the rectangular sheet. The magnetic flux is related to the side of the surface. Correspondingly, if the sheet is turned over, then the sign of the flux (ξ) changes. That is why, there are different signs in expressions for $H_{1,2}^M$ and $H_{3,4}^M$ in (1). There is no change of the sign between H_1^M and H_2^M (H_3^M and H_4^M) because when one glues Ω_1 to Ω_2 (Ω_3 to Ω_4) in accordance with (2), there is, simultaneously, a replacement $x \to -x$.

Solving equations $H^M\Psi = E\Psi$ at each rectangle and satisfying the gluing and boundary conditions (2), one obtains the spectral equation and the eigenfunctions.

The spectral equation is as follows:

$$\begin{vmatrix} \Phi_{1,n}(-a) & \Phi_{2,n}(-a) \\ \Phi_{1,n}(a) & \Phi_{2,n}(a) \end{vmatrix} = 0. \tag{3}$$

Here

$$\Phi_{1,n}(x;\xi) = e^{-\pi|\xi|(x-\frac{n}{T\xi})^2} \Phi\left(-\frac{E}{8\pi|\xi|} + \frac{1}{4}, \frac{1}{2}; \left(x - \frac{n}{T\xi}\right)^2 2\pi|\xi|\right), \tag{4}$$

$$\Phi_{2,n}(x;\xi) = e^{-\pi|\xi|^2} \left(x - \frac{n}{T\xi}\right) \sqrt{2\pi|\xi|}\Phi\left(-\frac{E}{8\pi|\xi|} + \frac{3}{4}, \frac{3}{2}; \left(x - \frac{n}{T\xi}\right)^2 2\pi|\xi|\right), \tag{5}$$

where $T = 8b$, $\Phi(\tilde{a}, \frac{1}{2}; z)$ is the Kummer function:

$$\Phi(\tilde{a}, \frac{1}{2}; z) = 1 + \sum_{k=1}^{\infty} \frac{(\tilde{a})_k}{(1/2)_k} \frac{z^k}{k!}, \tag{6}$$

$$(\tilde{a})_k = \tilde{a}(\tilde{a}+1)...(\tilde{a}+k-1), \quad (\tilde{a})_0 = 1.$$

Roots $E_{n,m}$ of equation (3) gives us the eigenvalues of the operator. It is known [12] that the roots of equation (3) can be ordered increasingly. Correspondingly, $E_{n,m}$ is the $m-$th root of $n-$th equation (3).

The main result is the following theorem.

Theorem 2.1. *The eigenvalues E_{nm} of the operator H^M are the roots of equation* (3) *with $T = 8b$. The corresponding eigenfunctions have the following form:*

$$\begin{cases} \Psi^{(1)}_{n,m} = A_{n,m} e^{i\frac{\pi n y}{4b}} \phi_{n,m}(x;\xi), \quad (x,y) \in \Omega_1 \\ \Psi^{(2)}_{n,m} = i^n A_{n,m} e^{-i\frac{\pi n y}{4b}} \phi_{n,m}(x;\xi), \quad (x,y) \in \Omega_2, \\ \Psi^{(3)}_{n,m} = (-1)^n A_{n,m} e^{i\frac{\pi n y}{4b}} \phi_{n,m}(x;\xi), \quad (x,y) \in \Omega_3 \\ \Psi^{(2)}_{n,m} = (-i)^n A_{n,m} e^{-i\frac{\pi n y}{4b}} \phi_{n,m}(x;\xi), \quad (x,y) \in \Omega_4, \end{cases} \tag{7}$$

where $A_{n,m}$ is some constant, $\phi_{n,m}(x;\xi)$ is given by (8).

$$\phi_{n,m}(x;\xi) = \Phi_{2,n,m}(a;\xi)\Phi_{1,n,m}(x;\xi) + \Phi_{1,n,m}(a;\xi)\Phi_{2,n,m}(x;\xi), \tag{8}$$

where $\Phi_{j,n,m}(x;\xi)$ is $\Phi_{j,n}(x;\xi)$ for $E = E_{nm}$.

Functions $\Phi_{1,n}(x;\xi)$ and $\Phi_{2,n}(x;\xi)$ are two linearly independent solutions of the following equation

$$\psi"(x) - \left((2\pi\xi)^2 \left(x - \frac{n}{T\xi} \right)^2 - E \right) \psi(x) = 0. \tag{9}$$

One can note that function $\phi_{n,m}(x;\xi)$ satisfies the following property:

$$\phi_{n,m}(-x;\xi) = \phi_{n,m}(x;-\xi) = \phi_{-n,m}(x;\xi). \tag{10}$$

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