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Original article

On the spectrum of the two-particle Schrödinger operator with point potential: one dimensional case

Utkir N. Kuljanov[1,2]

[1]Samarkand State University, Samarkand, Uzbekistan
[2]Samarkand branch of Tashkent State University of Economics, Samarkand, Uzbekistan

uquljonov@bk.ru

ABSTRACT In the paper, a one-dimensional two-particle quantum system interacted by two identical point interactions is considered. The corresponding Schrödinger operator (energy operator) h_ε depending on ε is constructed as a self-adjoint extension of the symmetric Laplace operator. The main results of the work are based on the study of the operator h_ε. First, the essential spectrum is described. The existence of unique negative eigenvalue of the Schrödinger operator is proved. Further, this eigenvalue and the corresponding eigenfunction are found.

KEYWORDS two-particle quantum system, symmetric Laplace operator, eigenvalue, eigenfunction, energy operator

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1. Introduction

The problems of the point interaction of two and three identical quantum particles interacted by point potentials (also called contact potentials and also, occasionally, singular potentials) have been studied in various physical works. In was the works of F. A. Berezin and L. D. Faddeev [1] and R. A. Minlos and L. D. Faddeev [2,3], where a rigorous mathematical description of the point interaction of two and three particles was proposed.

In [2, 3], the Hamiltonian of the system under consideration was investigated using the theory of self-adjoint extensions of symmetric operators. It was introduced as a self-adjoint extension of the symmetric Laplace operator defined on the domain of functions of three variables x_1, x_2, x_3; $x_j \in \mathbb{R}$, $j = 1, 2, 3$ vanishing if any two arguments $x_j = x_k, j \neq k,\ k = 1, 2, 3$ coincide.

The proposed extension is called the Skornyakov-Ter-Martirosyan extension. In [4], on the background of the results of [1, 2], the Hamiltonian of three particles (two fermions and one particle of a different nature) with the same masses interacting as point potentials was studied and it was shown that the Skornyakov-Ter-Martirosyan extensions are self-adjoint and semi-bounded.

In [5], the results of [1–4] were generalized to the case of three arbitrary particles and it was shown that the corresponding Hamiltonian has the discrete spectrum unbounded below. Note that the advantage of one-dimensional models with point perturbations is clear because they are useful for the study of a variety of qualitative properties. For instance, you can see [6–11] for one body problems with delta potentials.

In the discrete case, there were also found conditions for the existence of the eigenvalues as well as their numbers for the Hamiltonian of the system of two particles depending on parameters. For example, in [12–15], the Hamiltonian h of the system of two quantum particles moving on a one and three-dimensional lattices interacting via some attractive potential was considered. Conditions for the existence of eigenvalues of the two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}$, $d = 1, 3$; μ, associated with the Hamiltonian h, were studied depending on the energy of the particle interaction μ and total quasi-momentum $k \in T^d$.

In [2, 3], Faddeev and Minlos studied the energy operator of three identical three-dimensional quantum particles (bosons) interacting in a "pointed way". This operator was defined as a certain self-adjoint extension of the symmetric operator $H_0 = \Delta_{x_1} - \Delta_{x_2} - \Delta_{x_3}$ on the domain of functions of three variables $x_1, x_2, x_3 \in \mathbb{R}^3$ that vanish whenever any two arguments coincide $x_i = x_j, i \neq j$; $i, j = 1, 2, 3$. Minlos and Faddeev found that all nontrivial self-adjoint extensions describing the energy operator have the discrete spectrum unbounded from below and therefore the corresponding quantum system with δ-like pair interactions collapses, i.e. we have a phenomenon of "fall to the center". In recent work [5], Minlos and Melnikov extended these results to the general case of three different particles of different masses. In [2, 5], one can find the mathematical "explanation" for the Thomas effect (unsemiboundedness of the energy operator from below) and also the interpretation of Danilov's "experimental" parameter as the one describing the one-parameter family of self-adjoint extensions of the initial symmetric operator ("three-Hamiltonian").

In this article, following the basic scheme used in [2–5], we study the problem of the point interaction of two arbitrary particles in one-dimensional space. The Laplace operator with domain on variables $x_1, x_2 \in \mathbb{R}$, vanishing as $x_1 = x_2$ is considered. In the momentum representation of the Hamiltonian, after the reduction of the variables, we establish the Skornyakov-Ter-Martirosyan extension h_ε as a self-adjoint operator on its domain. The essential spectrum of h_ε coincides with the interval $[0;\infty)$. It is proved that the operator h_ε has no any eigenvalue as $\varepsilon \le 0$ and if the parameter of the extension is positive, i.e. $\varepsilon > 0$, then h_ε has unique negative eigenvalue.

2. Preliminaries and selection of the extension

The Hamiltonian (energy operator) of the two-particle system under consideration is defined as an extension $\tilde{H}$ of the symmetric operator $\tilde{H}_0$ acting in the Hilbert space $L_2(\mathbb{R}^2) \equiv L_2$ of the form

$$\left(\tilde{H}_0\phi\right)(x_1,x_2) = \left(-\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2}\right)\phi(x_1,x_2),$$

where the domain $D(\tilde{H}_0)$ of $\tilde{H}$ is a manifold of functions $\phi \in L_2$ satisfying condition

$$(\Delta_{x_1} + \Delta_{x_2})\phi \in L_2 \tag{1}$$

with

$$\phi(x,x) = 0. \tag{2}$$

Here Δ_{x_i} is the Laplace operator in the x_i variable $x_i \in \mathbb{R}$, m_i is the mass of the i-th particle, $i = 1,2$.

After the action of the corresponding Fourier transform, the operator $\tilde{H}_0$ transfers to the operator

$$\left(H_0 f\right)(p_1,p_2) = \left(\frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2\right) f(p_1,p_2),$$

defined on the set $D(H_0) \subset L_2$ of functions $f(p_1,p_2)$, satisfying the following conditions:

$$\int\limits_{\mathbb{R}^2} (p_1^4 + p_2^4)|f(p_1,p_2)|^2 dp_1 dp_2 < \infty, \tag{3}$$

with

$$\int\limits_{\Gamma_p} f\left(p_1,p_2\right) d\nu_p = 0, \tag{4}$$

where conditions (1) and (2) are equivalent to conditions (3) and (4), respectively. Here $\Gamma_p = \left\{(p_1,p_2) \in \mathbb{R}^2 : p_1 + p_2 = p\right\},\ p \in \mathbb{R}$ is a family of lines with the natural Lebesgue measure $d\nu_p$.

Making the following change of variables

$$P = p_1 + p_2, \quad p = \frac{m_2}{M}p_1 - \frac{m_1}{M}p_2, \quad M = m_1 + m_2$$

we establish the natural isomorphism between the spaces $L_2(\mathbb{R}) \otimes L_2(\Gamma_p)$ and $L_2\left(\mathbb{R}^2\right)$.

The last space can be identified with the space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$, while the operator H_0 is written as the tensor sum of the following operators

$$H_0 = \left(\frac{1}{2M}P^2 + \frac{1}{2m}h_0\right) \otimes I,$$

where I is the identity operator, $m = m_1 m_2/(m_1+m_2)$, $(1/2M)P^2$ is the operator of multiplication by the number $P^2/(2M)$ in the space $L_2(\mathbb{R})$, and h_0 is a closed non-negative symmetric operator acting in $L_2(\mathbb{R})$ as

$$h_0 f(p) = p^2 f(p).$$

Its domain $D(h_0)$ consists of functions satisfying the conditions:

$$\int p^4 |f(p)|^2 dp < \infty; \quad \int f(p)dp = 0. \tag{5}$$

Further, the integral without indicating limits is understood as integration over $\mathbb{R}$.

The symbol $\Re_z$ denotes the deficiency subspace for the operator h_0, i.e.

$$\Re_z = \left\{g \in L_2(\mathbb{R}) :\ ((h_0 - zI)f, g) = 0,\ f \in D(h_0)\right\}.$$

Lemma 2.1. For any $z \in \mathbf{\Pi_0} = \mathbb{C}^1 \backslash [0,\infty)$, the deficiency subspace $\Re_z \subset L_2(\mathbb{R})$ of h_0 consists of functions of the form

$$g(p) = \frac{c}{p^2 - \bar{z}},\ c \in \mathbb{C}^1.$$

Proof. Let $g \in \Re_z$. Then for any $f \in D(h_0)$, the relation

$$((h_0 - zI)f, g) = \int (p^2 - z) f(p)\overline{g(p)}dp = \int f(p)\overline{(p^2 - \overline{z})g(p)}dp = 0$$

holds.

From the last relation and conditions (5), it follows that

$$(p^2 - \overline{z})g(p) = c$$

or

$$g(p) = \frac{c}{p^2 - \overline{z}}.$$

The lemma is proved. □

It follows from the lemma that for any $z \in \Pi_0$ the deficiency subspace $\Re_z$ of the operator h_0 is determined by the formula

$$\Re_z = \{g \in L_2(\mathbb{R}) : \ ((h_0 - zI)f, g) = 0, \ f \in D(h_0)\}.$$

Therefore, h_0 is a symmetric operator with deficiency indices $(1, 1)$. Using the general extension theory [4], we find that the operator h_0 has one-parameter family of self-adjoint extensions.

Since the operator h_0 is non-negative, as in [2–5], we use the theory of extensions of semibounded operators. The deficiency subspace $\Re_{-1}$ of the operator h_0 consists of functions of the form

$$u_{-1}(p) = \frac{c}{p^2 + 1}, \ c \in \mathbb{C}^1.$$

Using the positivity of H_0 and applying methods of the theory of extensions of semi-bounded operators, as in [2], we prove the following lemma, which allows one to define the adjoint operator h_0^*.

Lemma 2.2. *The domain $D(h_0^*)$ of h_0^* consists of functions of the form*

$$g(p) = f(p) + \frac{c_1}{p^2 + 1} + \frac{c_2}{(p^2 + 1)^2}. \tag{6}$$

where $f \in D(h_0)$, c_1, $c_2 \in \mathbb{C}^1$. The operator h_0^* acts on function g of the form (6) by the following formula

$$(h_0^* g)(p) = p^2 g(p) - c_1,$$

where the constant c_1 is taken from the decomposition (6) of the function g.

Further, we select the extensions of the operator h_0. We define the set

$$D(h_\varepsilon), D(h_0) \subset D(h_\varepsilon) \subset D(h_0^*)$$

as follows:

$$D(h_\varepsilon) = \left\{ g \in D(h_0^*) : g(p) = f(p) + \frac{c}{p^2 + 1} + \frac{(\varepsilon - 2)c}{(p^2 + 1)^2}, \ f \in D(h_0) \right\} \tag{7}$$

The restriction of the operator h_0 to the domain $D(h_\varepsilon)$ is denoted by h_ε and it has the form

$$h_\varepsilon g(p) = p^2 g(p) - c.$$

By definition of h_ε , it is an extension of the operator h_0.

Theorem 2.1. *For any $\varepsilon \in \mathbb{R}$, the extension h_ε is a self-adjoint operator.*

Proof. It is easy to verify that for any $g_1, g_2 \in D(h_\varepsilon)$, the relation $(h_\varepsilon g_1, g_2) = (g_1, h_\varepsilon g_2)$ holds, i.e. h_ε is a symmetric operator. Now, we show that the deficiency indices of the operator h_ε are equal to $(0, 0)$.

Let $\psi \in \Re_{-1}(h_\varepsilon)$. Then the function $\psi(p)$ has the form

$$\psi(p) = \frac{b}{p^2 + 1}, \ b \in \mathbf{C}^1.$$

For any $g \in D(h_\varepsilon)$, the equality $((h_\varepsilon + I)g, \psi) = 0$ holds. Correspondingly, the last equality can be written as

$$((h_\varepsilon + I)g, \psi) = \int \big((p^2 + 1)(f(p) + \frac{c}{p^2 + 1} + \frac{(\varepsilon - 2)c}{(p^2 + 1)^2}) - c\big)\overline{\psi(p)}dp =$$

$$= \int \big((p^2 + 1\big) f(p)\overline{\psi(p)}dp + (\varepsilon - 2)c \int \frac{\overline{b}}{(p^2 + 1)^2} dp = 0. \tag{8}$$

Since

$$\int \big((p^2 + 1\big) f(p)\overline{\psi(p)}dp = 0$$

and $(\varepsilon - 2)c \neq 0$, we have $b = 0$. Hence $\psi(p) = 0$. This proves that the deficiency indices of the operator h_ε are equal to $(0, 0)$. □

3. Spectral properties of the operator h_ε

The main results of the paper are the following theorems.

Theorem 3.1. *For any $\varepsilon \in \mathbb{R}$, the essential spectrum of h_ε coincides with interval $[0,\infty)$. If $\varepsilon \geq 0$ then h_ε has no any negative eigenvalue, and for any $\varepsilon < 0$, the operator h_ε has unique simple eigenvalue $z = -\dfrac{4}{\varepsilon^2}$ and the corresponding eigenfunction has the form $g_\varepsilon(p) = \dfrac{1}{p^2 + \frac{4}{\varepsilon^2}}$.*

Proof. First, we show that the essential spectrum of h_ε equals to $[0;\infty)$. For each $z \geq 0$, let us consider the sequence of cut-off layers:

$$G_n(z) = \left\{ p \in \mathbb{R} : \sqrt{z} + \frac{1}{n+1} < |p| < \sqrt{z} + \frac{1}{n} \right\}, \quad n = 1,2,3,...$$

We split each layer $G_n(z)$ into two half-layers as

$$G_n^+(z) = \{p \in G_n(z) : p \geq 0\}$$

and

$$G_n^-(z) = \{p \in G_n(z) : p < 0\}.$$

By construction, the volumes of these parts are equal and

$$\mu(G_n^+(z)) = \mu(G_n^-(z)) = \frac{1}{2}\mu(G_n(z))$$

. One can see that

$$V_n = \mu\left(G_n(z)\right) = \frac{2}{n\left(n+1\right)}.$$

Let $f_n^{(z)},\ n = 1,2,3,...$ be a sequence of the test functions

$$f_n^{(z)}(p) = \begin{cases} \dfrac{1}{\sqrt{V_n(z)}}, & p \in G_n^+(z); \\ -\dfrac{1}{\sqrt{V_n(z)}}, & p \in G_n^-(z); \\ 0, & p \in \mathbb{R} \setminus G_n(z). \end{cases}$$

Then, it is easy to verify that $f_n^{(z)} \in L_2(\mathbb{R})$, $\left\| f_n^{(z)} \right\| = 1$ and $(f_n^{(z)}, f_m^{(z)}) = 0$ as $n \neq m$.

One can see that

$$\int f_n^{(z)}(p)dp = 0, \quad n = 1,2,3,...$$

i.e. $f_n^{(z)} \in D(h_0)$. Note that

$$\|(h_\varepsilon - zI)f_n^{(z)}\|^2 = \int\limits_{G_n(z)} \frac{1}{V_n(z)} |(p^2 - z)|^2 dp = \frac{2}{V_n} \int\limits_{\sqrt{z}+\frac{1}{n+1}}^{\sqrt{z}+\frac{1}{n}} \left(p^2 - z\right)^2 dp$$

or

$$\|(h_\varepsilon - zI)f_n^{(z)}\|^2 = \frac{2}{V_n} \int\limits_{\sqrt{z}+\frac{1}{n+1}}^{\sqrt{z}+\frac{1}{n}} \left(p^2 - z\right)^2 dp. \tag{9}$$

Since

$$|p| < \sqrt{z} + \frac{1}{n}, \quad p^2 - z < \frac{1}{n}\left(2\sqrt{z} + \frac{1}{n}\right).$$

This gives

$$(p^2 - z)^2 < \left(2\sqrt{z} + \frac{1}{n}\right)^2 \frac{1}{n^2}.$$

Hence, by (9), we have

$$\|(h_\varepsilon - zI)f_n^{(z)}\|^2 < \left(2\sqrt{z} + \frac{1}{n}\right)^2 \frac{1}{n^2}.$$

This shows that

$$\lim_{n\to\infty} \left\| (h_\varepsilon - zI)f_n^{(z)} \right\| = 0.$$

This means that if $z \geq 0$, then $z \in \sigma_{ess}(h_\varepsilon)$, therefore $[0;\infty) \subset \sigma_{ess}(h_\varepsilon)$. In order to show the reverse inclusion $\sigma_{ess}(h_\varepsilon) \subset [0;\infty)$, we construct the resolvent operator of h_ε.

Let

$$(h_\varepsilon - zI)g = \psi.$$

Then

$$(p^2 - z)g(p) - c = \psi(p).$$

If $z < 0$ then $p^2 - z \neq 0$. Hence,

$$g(p) = \frac{\psi(p)}{p^2 - z} + \frac{c}{p^2 - z}. \tag{10}$$

Since $g \in D(h_\varepsilon)$, it represents as

$$g(p) = f(p) + \frac{c}{p^2 + 1} + \frac{(\varepsilon - 2)c}{(p^2 + 1)^2} \tag{11}$$

for some $f \in D(h_\varepsilon)$. Comparing (10) and (11), we obtain the equation for c:

$$f(p) + \left(\frac{1}{p^2 + 1} - \frac{1}{p^2 - z}\right)c + \frac{(\varepsilon - 2)c}{(p^2 + 1)^2} = \frac{\psi(p)}{p^2 - z} \tag{12}$$

where $f \in D(h_0)$. Integrating both sides of (12), taking into account (5) and the identities

$$\int \frac{dp}{p^2 - z}dp = \frac{\pi}{\sqrt{-z}}, \quad z < 0 \tag{13}$$

and

$$\int \frac{dp}{(p^2 + 1)^2} = \frac{\pi}{2}, \tag{14}$$

we have

$$(\varepsilon\sqrt{-z} - 2)\pi c = 2\sqrt{-z}\int \frac{\psi(p)}{p^2 - z}dp$$

or

$$c = \frac{2\sqrt{-z}}{\pi(\varepsilon\sqrt{-z} - 2)}\int \frac{\psi(p)}{p^2 - z}dp.$$

This gives one

$$g(p) = \frac{\psi(p)}{p^2 - z} + \frac{2\sqrt{-z}}{\pi(\varepsilon\sqrt{-z} - 2)} \cdot \frac{1}{p^2 - z}\int \frac{\psi(q)}{q^2 - z}dq.$$

This, if $z \in \Pi_0$ and $\varepsilon\sqrt{-z} - 2 \neq 0$ then the resolvent of the operator h_ε acts in $L_2(\mathbb{R})$ as

$$(R_z(h_\varepsilon)g)(p) = \frac{g(p)}{p^2 - z} + \frac{2\sqrt{-z}}{\pi(\varepsilon\sqrt{-z} - 2)} \cdot \frac{1}{p^2 - z}\int \frac{g(q)}{q^2 - z}dq.$$

This shows that the resolvent of the operator h_ε is a bounded operator for $\varepsilon\sqrt{-z} - 2 \neq 0$ and $z < 0$. It means that $\sigma_{ess}(h_\varepsilon) \subset [0; \infty)$. It follows directly from here that $\sigma_{ess}(h_\varepsilon) = [0; \infty)$. Now, we consider an eigenvalue problem for h_ε. From equation $(h_\varepsilon - zI)g(p) = 0$, we obtain that if $\varepsilon \leq 0$ and $z \in \Pi_0$, then $\varepsilon\sqrt{-z} - 2 \neq 0$. By (12) the resolvent of the operator h_ε is defined on $D(h_\varepsilon)$. Hence, h_ε has no any negative eigenvalue.

Let $\varepsilon > 0$. Then from the equality $\varepsilon\sqrt{-z} - 2 = 0$, we have $z = -\frac{4}{\varepsilon^2}$. The equation

$$(h_\varepsilon - zI)g(p) = 0$$

gives one

$$g(p) = \frac{c}{p^2 - z}. \tag{15}$$

We show that $g \in D(h_\varepsilon)$. To obtain this, g should be represented in the form (11) for some $f \in D(h_0)$. Assume that g is represented as (11). Comparing (11) with (15), we obtain

$$f(p) + \frac{c}{p^2 + 1} + \frac{(\varepsilon - 2)c}{(p^2 + 1)^2} = \frac{c}{p^2 - z},$$

i.e.

$$f(p) = \frac{c(1 - \frac{4}{\varepsilon^2})}{(p^2 + \frac{4}{\varepsilon^2})(p^2 + 1)} - \frac{c(\varepsilon - 2)}{(p^2 + 1)^2}.$$

Taking into account the identities (13) and (14), one can see that $\int f(p)dp = 0$. This gives one that $f \in D(h_0)$. Theorem 3.1 is proved. □

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Information about the authors:

Utkir Nematovich Kuljanov – Samarkand State University, 140104, University boulvare. 15, Samarkand, Uzbekistan; Samarkand branch of Tashkent State University of Economics, Professorlar street 51, Samarkand, Uzbekistan;
ORCID 0009-0003-4286-4704; uquljonov@bk.ru