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Original article

Boundary value problem for a degenerate equation with a Riemann–Liouville operator

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ABSTRACT In the article, the uniqueness and solvability of one boundary value problem for a high-order equation with two lines of degeneracy with a fractional Riemann–Liouville derivative in a rectangular domain is studied by the Fourier method. Sufficient conditions for the well-posedness of the problem posed are obtained.

KEYWORDS high order equation, initial-boundary value problem, fractional derivative in the sense of Riemann–Liouville, eigenvalue, eigenfunction, Kilbas-Saigo function, series, convergence, existence, uniqueness

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1. Introduction and problem statement

Fractional partial differential equations underlie the mathematical modeling of various physical processes and environmental phenomena that demonstrate a fractal nature [1, 2]. Model equations with fractional derivatives with constant coefficients are well studied. Recently, specialists have intensively studied equations with variable coefficients. Degenerate equations are among such equations. The number of works on degenerate equations with fractional derivatives is relatively small. Many of these papers deal with ordinary differential equations. In work [2] the following equation was considered:

$$D_{0x}^{\alpha} t^{\beta} u(t) = \lambda u(x), \quad 0 < x < b,$$

where $0 < \alpha < 1$, λ is the spectral parameter, $\beta = \text{const} \geq 0$. In the same work, it was noted that the equation

$$D_{0x}^{\alpha} t^{\beta} u(t) + \sum_{j=1}^{n} a_j(x) D_{0x}^{\alpha_j} u(t) + b(x) u = c(x), \quad 0 \neq \alpha_j < \alpha,$$

plays an important role in the theory of inverse problems for degenerate equations of hyperbolic type. In article [3], the following problem was investigated for solvability

$$\begin{cases} \partial_0^{\nu} (k(t) y(t)) + c(y(t)) = f(t), \\ k(0) y(0) = 0. \end{cases}$$

Here $\partial_0^{\nu}-$ is the fractional differentiation operator in the Caputo sense, $0 < \nu < 1$, $k(t) \in C^1[0, T]$, $k(t) \geq 0$, for $t \in [0; T]$, $c(\eta) \in C(\mathbb{R})$, $c(0) = 0$. In paper [4], solutions in the closed form to the fractional order equations were found

$$\left(D_{0+}^{\alpha} y\right)(x) = a x^{\beta} y(x) + f(x) \quad (0 < x < d \leq \infty, \ \alpha > 0, \ \beta \in \mathbb{R}, \ a \neq 0),$$

$$\left(D_{-}^{\alpha} y\right)(x) = a x^{\beta} y(x) + f(x) \quad (0 \leq d < x < \infty, \ \alpha > 0, \ \beta \in \mathbb{R}, \ a \neq 0),$$

with the fractional Riemann–Liouville derivatives on the semiaxis $(0, \infty)$ [5]:

$$\left(D_{0+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1 - \{\alpha\})} \int_0^x \frac{y(t)\, dt}{(x - t)^{\{\alpha\}}}, \quad (x > 0; \ \alpha > 0),$$

$$\left(D_{-}^{\alpha} y\right)(x) = \left(-\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1 - \{\alpha\})} \int_x^{\infty} \frac{y(t)\, dt}{(t - x)^{\{\alpha\}}}, \quad (x > 0; \ \alpha > 0),$$

($[\alpha]$ and $\{\alpha\}$ mean the integer and fractional parts of the real number α, respectively). Applied problems lead to such equations [6]. One can find an example of such equation in the theory of polarography [7]:

$$\left(D_{0+}^{1/2} y\right)(x) = a x^{\beta} y(x) + x^{-1/2}, \quad (0 < x, \ -1/2 < \beta \leq 0),$$

which arises for $a = -1$ from diffusion problems [7]. As for the degenerate partial differential equations involving fractional derivatives, it should be noted that the researches in this area are quite new. We mention here papers [8–15], in which various boundary value problems for degenerate equations involving fractional derivatives were studied.

It should also be noted that fractional-order integro-differential operators have been recently actively used in nanosystem studies. We note in this field, for example, works [16–18].

In this paper, we study a boundary value problem in a rectangular domain for an equation of high even order involving a fractional derivative, which has degeneracy in both variables.

In the region $\Omega = \Omega_x \times \Omega_y$, $\Omega_x = \{x : \ 0 < x < 1\}$, $\Omega_y = \{y : \ 0 < y < 1\}$, let us consider the equation

$$D_{0x}^{\alpha} u(x,y) + x^{\beta} K(y)\, l(u(x,y)) = 0, \tag{1}$$

where

$$l(u(x,y)) = (-1)^s \frac{\partial^{2s} u(x,y)}{\partial y^{2s}} + \frac{\partial^{s-1}}{\partial y^{s-1}}\left((-1)^{s-1} p_{s-1}(y) \frac{\partial^{s-1} u(x,y)}{\partial y^{s-1}}\right) + \ldots$$
$$+ \frac{\partial}{\partial y}\left(-p_1(y)\frac{\partial u(x,y)}{\partial y}\right) + p_0(y)\, u(x,y),$$
$$p_j(y) \in C^j\left(\overline{\Omega}_y\right), \quad j = 0,1,\ldots,s-1,\ s \in \mathbb{N},$$

$K(y) \in C[0;1]$; $K(y) \in C^{(2s)}$ for $y \in (0,1]$;

$$K(y) > 0, \quad y \in (0,1], \quad K(0) = 0, \quad K^{(i)}(y) = O\left(y^{m-i}\right), \quad y \to +0, \quad 0 \le m < s, \quad i = 0,1,\ldots 2s,$$

$0 < \alpha < 1, -\alpha < \beta \in \mathbb{R}$, D_{0x}^{α} is the Riemann–Liouville fractional differentiation operator of order α

$$D_{0x}^{\alpha} u(x,y) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_0^x \frac{u(\tau,y)\, d\tau}{(x-\tau)^{\alpha}}.$$

Let us consider the following problem for equation (1).

Problem A. Find a solution to equation (1) belonging to the class

$$D_{0x}^{\alpha} u(x,y) \in C(\Omega), \quad x^{1-\alpha} u(x,y) \in C\left(\overline{\Omega}\right),$$
$$\frac{\partial^{2s-1} u(x,y)}{\partial y^{2s-1}} \in C\left(\Omega_x \times \bar{\Omega}_y\right), \quad \frac{\partial^{2s} u(x,y)}{\partial y^{2k}} \in C\left(\Omega_x \times \Omega_y\right),$$

satisfying the conditions

$$\frac{\partial^j u(x,0)}{\partial y^j} = \frac{\partial^j u(x,1)}{\partial y^j} = 0, \quad 0 < x \le 1,\ j = 0,1,\ldots,s-1, \tag{2}$$

$$\lim_{x \to 0} x^{1-\alpha} u(x,y) = \frac{\varphi(y)}{\Gamma(\alpha)}, \tag{3}$$

where $\varphi(y) \in C^{(2s)}$, $y \in [0,1]$.

2. Searching a solution

We are looking for a solution in the form

$$u(x,y) = X(x)\, Y(y).$$

Then, with respect to the variable y, taking into account condition (2), we obtain the following spectral problem:

$$\begin{cases} l(Y(y)) = \lambda \dfrac{Y(y)}{K(y)}, \\ Y^{(j)}(0) = Y^{(j)}(1) = 0, \quad j = 0,1,\ldots,s-1. \end{cases} \tag{4}$$

We reduce problem (4) to an integral equation using Green's function and obtain the necessary estimates for the eigenfunctions. Further, we will assume that $\lambda > 0$, this condition is valid, for example, in the case when $0 \le p_j(y), j = 0,1,\ldots,s-1$. This implies the existence of continuous symmetric Green's function, which will be denoted by $G(y,\xi)$. Taking into account the boundary conditions at the point $y = 0$ and applying the Lagrange theorem on finite increments (the mean-value theorem (for derivatives)), we have

$$Y(y) = O(y^s), \quad y \to +0.$$

This relation is also valid for Green's function. It remains to show the existence of eigenvalues and eigenfunctions of problem (4). The integral equation equivalent to the problem (4) has the following form

$$Y(y) = \lambda \int_0^1 \frac{G(y,\xi)\, Y(\xi)\, d\xi}{K(\xi)}, \tag{5}$$

we rewrite (5) in the form

$$\frac{Y(y)}{\sqrt{K(y)}} = \lambda \int_0^1 \frac{G(y,\xi)}{\sqrt{K(\xi)}\sqrt{K(y)}} \frac{Y(\xi)}{\sqrt{K(\xi)}} d\xi.$$

Let us introduce the notation

$$\overline{Y}(y) = \frac{Y(y)}{\sqrt{K(y)}}, \quad \overline{G}(y,\xi) = \frac{G(y,\xi)}{\sqrt{K(\xi)}\sqrt{K(y)}}.$$

From this, we have

$$\bar{Y}(y) = \lambda \int_0^1 \bar{G}(y,\xi)\, \bar{Y}(\xi)\, d\xi. \tag{6}$$

Relation (6) is an integral equation with symmetric and continuous, in both variables, kernel. According to the theory of equations with symmetric kernels, equation (6) has at most a countable number of eigenvalues and eigenfunctions. Thus, problem (4) has eigenvalues $\lambda_n > 0$, $n = 1, 2, ...$, and the corresponding eigenfunctions are $Y_n(y)$. Let's order the eigenvalues increasingly: $0 < \lambda_1 \leq \lambda_2 \leq ....$ Further, we assume that

$$\|Y_n(y)\|^2 = \int_0^1 \frac{Y_n^2(y)dy}{K(y)} = 1.$$

From here, using (5), we come to the Bessel inequality

$$\sum_{n=1}^{\infty} \left(\frac{Y_n(y)}{\lambda_n}\right)^2 \leq \int_0^1 \frac{G^2(y,\xi)\, d\xi}{K(\xi)} < \infty. \tag{7}$$

Let us find the conditions ensuring that given function is expanded into a Fourier series in terms of the system of eigenfunctions of problem (4). The following theorem holds.

Theorem 1. If the function $\varphi(y)$ satisfies the following conditions:
1). $\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, j = \overline{0, s-1}$;
2). $\varphi^{(2s)}(y)$ is continuous on $[0,1]$,
then it can be expanded in terms of the eigenfunctions of the problem (4), which converges uniformly and absolutely.

Proof. Let $l(\varphi(y)) = g(y)$, then

$$\varphi(y) = \int_0^1 G(y,\xi)\, g(\xi) d\xi.$$

Correspondingly,

$$\frac{\varphi(y)}{\sqrt{K(y)}} = \int_0^1 \frac{G(y,\xi)}{\sqrt{K(y)\,K(\xi)}} \sqrt{K(\xi)} g(\xi)\, d\xi,$$

or

$$\frac{\varphi(y)}{\sqrt{K(y)}} = \int_0^1 \overline{G}(y,\xi) \sqrt{K(\xi)} g(\xi)\, d\xi.$$

Let us use the Hilbert-Schmidt theorem, then one has

$$\frac{\varphi(y)}{\sqrt{K(y)}} = \sum_{n=1}^{\infty} c_n \frac{Y_n(y)}{\sqrt{K(y)}},$$

where

$$c_n = \int_0^1 \frac{\varphi(y)\, Y_n(y)}{K(y)} dy.$$

Further, after reduction by $\dfrac{1}{\sqrt{K(y)}}$, we obtain

$$\varphi(y) = \sum_{n=1}^{\infty} c_n Y_n(y).$$

Theorem 1 is proved.

Let us fix n and start solving the problem of searching the function $X_n(x)$. We assume that the initial function $\varphi(y)$ satisfies the conditions of Theorem 1. Taking into account condition (3), we obtain the following initial problem:

$$\begin{cases} D_{0x}^{\alpha} X_n(x) = -\lambda_n x^{\beta} X_n(x), \\ \lim\limits_{x\to 0} x^{1-\alpha} X_n(x) = \dfrac{\varphi_n}{\Gamma(\alpha)}, \end{cases} \tag{8}$$

where

$$\varphi_n = \int_0^1 \frac{\varphi(y) Y_n(y)\, dy}{K(y)}, \quad n = 1, 2, \ldots.$$

Using the results of [14], the solution to problem (8) is written as

$$X_n(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \varphi_n E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}}\left(-\lambda_n x^{\alpha+\beta}\right),$$

where

$$E_{\alpha,m,l}(z) = \sum_{i=0}^{\infty} c_i z^i, \quad c_0 = 1, \quad c_i = \prod_{j=0}^{i-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \quad i \geq 1$$

is the Kilbas–Saigo function [14].

It follows from the results of [16] that the following estimate holds:

$$|X_n(x)| \leq M x^{\alpha-1} |\varphi_n|, \quad 0 < M = \text{const}. \tag{9}$$

Now, we will seek the solution of the problem in the form

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y). \tag{10}$$

Let us find the conditions under which (10) is a regular solution to Problem A. The following theorem is true.

Theorem 2. Let the function $\varphi(y)$, satisfy the following conditions:

1).$\varphi(y) \in C^{2s}[0,1], \quad \varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, j = 0, 1, \ldots, s-1,$

2).$(K(y)\, l(\varphi(y)))^{(j)}(0) = (K(y)\, l(\varphi(y)))^{(j)}(1) = 0, K(y)\, l(\varphi(y)) \in C^{2s}[0,1], \quad j = 0, 1, \ldots, s-1,$

then a solution to problem A exists.

Proof. Series (10) formally satisfies equation (1). Let us show that one can differentiate the series. Considering (9), we have

$$\left|x^{1-\alpha} u(x,y)\right| \leq M \sum_{n=1}^{\infty} |\varphi_n| |Y_n(y)|.$$

Using the Cauchy–Bunyakovsky inequality, one obtains

$$\sum_{n=1}^{\infty} |\varphi_n| |Y_n(y)| \leq \sqrt{\sum_{n=1}^{\infty} \left(\frac{Y_n(y)}{\lambda_n}\right)^2} \sqrt{\sum_{n=1}^{\infty} (\lambda_n \varphi_n)^2}.$$

The convergence of the first factor in the last inequality follows from (7). Consider the second multiplier. We have

$$\varphi_n = \int_0^1 \frac{\varphi(y) Y_n(y) dy}{K(y)} = \frac{1}{\lambda_n} \int_0^1 \varphi(y)\, l(Y_n(y))\, dy.$$

Integrating by parts, we obtain

$$\varphi_n = \frac{1}{\lambda_n} \int_0^1 l(\varphi(y))\, Y_n(y)\, dy. \tag{11}$$

Hence,

$$\lambda_n \varphi_n = \int_0^1 K(y)\, l(\varphi) \frac{Y_n(y)}{K(y)} dy.$$

Consequently, $\lambda_n \varphi_n$ are the Fourier coefficients of the function $K(y)\, l(\varphi)$. Then, from the Bessel inequality, one obtains

$$\sum_{n=1}^{\infty} \lambda_n^2 |\varphi_n|^2 \leq \int_0^1 K(y) \{l(\varphi)\}^2 dy < \infty. \tag{12}$$

From (7) and (12), the convergence of the series (10) and the fulfillment of condition (3) follows. Let us pass to the derivatives. In view of equation (8), we formally have

$$\left|D_{0x}^{\alpha}u\left(x,y\right)\right|\leq\sum_{n=1}^{\infty}\left|D_{0x}^{\alpha}X_{n}\left(x\right)\right|\left|Y_{n}\left(y\right)\right|\leq Mx^{\alpha+\beta-1}\sum_{n=1}^{\infty}\left|\lambda_{n}\varphi_{n}\right|\left|Y_{n}\left(y\right)\right|\leq$$

$$\leq Mx^{\alpha+\beta-1}\sqrt{\sum_{n=1}^{\infty}\left(\lambda_{n}^{2}\varphi_{n}\right)^{2}}\sqrt{\sum_{n=1}^{\infty}\left(\frac{Y_{n}\left(y\right)}{\lambda_{n}}\right)^{2}},\quad x>0. \tag{13}$$

Further, keeping in mind (11), one has

$$\varphi_{n}=\left(\frac{1}{\lambda_{n}}\right)^{2}\int_{0}^{1}K\left(y\right)l\left(\varphi\left(y\right)\right)l\left(Y_{n}\left(y\right)\right)dy=\left(\frac{1}{\lambda_{n}}\right)^{2}\int_{0}^{1}l\left(K\left(y\right)l\left(\varphi\left(y\right)\right)\right)Y_{n}\left(y\right)dy.$$

Now, we apply the Bessel inequality

$$\sum_{n=1}^{\infty}\left(\lambda_{n}^{2}\varphi_{n}\right)^{2}\leq\int_{0}^{1}K\left(y\right)\left\{l\left(K\left(y\right)l\left(\varphi\right)\right)\right\}^{2}dy<\infty. \tag{14}$$

The convergence (13) follows from (7) and (14). So the following functional series

$$D_{0x}^{\alpha}u\left(x,y\right)=\sum_{n=1}^{\infty}D_{0x}^{\alpha}X_{n}\left(x\right)Y_{n}\left(y\right),$$

converges evenly. Uniform convergence of the next series,

$$l\left(u\left(x,y\right)\right)=\sum_{n=1}^{\infty}X_{n}\left(x\right)l\left(Y_{n}\left(y\right)\right)=\frac{1}{K\left(y\right)}\sum_{n=1}^{\infty}\lambda_{n}X_{n}\left(x\right)Y_{n}\left(y\right),$$

is shown by the same way.

Theorem 2 is proved.

3. Uniqueness of the solution

Let us proceed to the proof of the uniqueness of the solution. The following theorem holds.

Theorem 3. If there is a solution to Problem A, then it is unique.

Proof. Let the function $u\left(x,y\right)$ be a solution to Problem A with zero initial and boundary conditions. Consider its Fourier coefficients in terms of the system of eigenfunctions of problem (4):

$$u_{n}\left(x\right)=\int_{0}^{1}\frac{u\left(x,y\right)Y_{n}(y)dy}{K\left(y\right)},\quad n=1,2,....$$

It is rather simple to show that $u_{n}\left(x\right)$ is a solution to the problem

$$\begin{cases}D_{0x}^{\alpha}u_{n}\left(x\right)=-\lambda_{n}x^{\beta}u_{n}\left(x\right),\\ \lim\limits_{x\to0}x^{1-\alpha}u_{n}\left(x\right)=0.\end{cases}$$

This problem has a trivial solution only [14], i.e.,

$$\int_{0}^{1}\frac{u\left(x,y\right)Y_{n}(y)dy}{K\left(y\right)}=0,\quad n=1,2,....$$

Because $\bar{G}\left(y,\xi\right)$ is symmetric continuous function and the following integrals converge:

$$\int_{0}^{1}\bar{G}^{2}\left(y,\xi\right)d\xi<\infty,\quad\int_{0}^{1}\bar{G}^{2}\left(y,\xi\right)dy<\infty,\quad\int_{0}^{1}\int_{0}^{1}\bar{G}^{2}\left(y,\xi\right)dyd\xi<\infty,\quad\lambda_{n}>0,\ \forall n,$$

it follows from Mercer's theorem that

$$\bar{G}\left(y,\xi\right)=\sum_{n=1}^{\infty}\frac{\overline{Y_{n}}\left(y\right)\overline{Y_{n}}\left(\xi\right)}{\lambda_{n}}.$$

Then, we have

$$\frac{u\left(x,y\right)}{\sqrt{K\left(y\right)}}=\int_{0}^{1}\bar{G}\left(y,\xi\right)\left(\sqrt{K\left(\xi\right)}l\left(u\left(x,\xi\right)\right)\right)d\xi=$$

$$\int_0^1 \sum_{n=1}^{\infty} \frac{\overline{Y_n}(y)\,\overline{Y_n}(\xi)}{\lambda_n} \left(\sqrt{K(\xi)} l(u(x,\xi))\right) d\xi =$$

$$\sum_{n=1}^{\infty} \frac{Y_n(y)}{\lambda_n \sqrt{K(y)}} \int_0^1 \frac{Y_n(\xi)\sqrt{K(\xi)}}{\sqrt{K(\xi)}} l(u(x,\xi))\, d\xi.$$

Since the series converges uniformly, the integrating and the summation can be rearranged. Correspondingly, the last expression transforms to the following one

$$\sum_{n=1}^{\infty} \frac{Y_n(y)}{\lambda_n \sqrt{K(y)}} \int_0^1 Y_n(\xi)\, l(u(x,\xi))\, d\xi =$$

$$\sum_{n=1}^{\infty} \frac{Y_n(y)}{\lambda_n \sqrt{K(y)}} \int_0^1 l(Y_n(\xi))\, u(x,\xi)\, d\xi =$$

$$\sum_{n=1}^{\infty} \frac{Y_n(y)}{\sqrt{K(y)}} \int_0^1 \frac{Y_n(\xi)\, u(x,\xi)}{K(\xi)} d\xi = 0.$$

Hence,

$$u(x,y) \equiv 0.$$

Theorem 3 is proved.

4. Conclusion

In this paper, we study the initial boundary value problem for a high-order equation involving fractional derivative in the sense of Riemann–Liouville. The equation has two lines of degeneration $x = 0$ and $y = 0$. The solution is constructed as a series in terms of eigenfunctions of the one-dimensional spectral problem. Sufficient conditions for the expansion of this function in terms of the system of eigenfunctions of the spectral problem were found. The convergence theorems for the series and the uniqueness of the constructed solution of the problem were proved. In this case, various theorems from the theory of differential and integral equations were used. In the future, one can study boundary value problems for inhomogeneous equations, inverse problems with unknown right hand side, problems with nonlocal conditions that have various applications in nanosystems (see [18]). Note that the studied problem has not only theoretical but also applied importance since fractional integro-differential operators are widely used in mathematical models of nanosystem dynamics. In particular, among the mathematical models of transport-diffusion transfer, fractional differential calculus is singled out as a tool for describing transfer processes in complex-structured media [20].

It was noted in [21], that fractional differential equations are convenient for the analysis of frequency characteristics of semiconductor devices in which dispersive transport occurs, because their Fourier transforms give rise to algebraic equations that are much simpler to solve than integral equations derived from equations with variable diffusion coefficient and mobility.

A number of experimental facts testify to the anomalous diffusion of impurities and defects in various materials (see the literature review on this topic in [22]). Equations containing fractional derivatives form the mathematical basis of anomalous self-similarity diffusion [23].

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