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Original article

On the discrete spectrum of the Schrödinger operator using the 2+1 fermionic trimer on the lattice

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ABSTRACT We consider the three-particle discrete Schrödinger operator $H_{\mu,\gamma}(\mathbf{K})$, $\mathbf{K} \in \mathbb{T}^3$, associated with the three-particle Hamiltonian (two of them are fermions with mass 1 and one of them is arbitrary with mass $m = 1/\gamma < 1$), interacting via pair of repulsive contact potentials $\mu > 0$ on a three-dimensional lattice $\mathbb{Z}^3$. It is proved that there are critical values of mass ratios $\gamma = \gamma_1$ and $\gamma = \gamma_2$ such that if $\gamma \in (0, \gamma_1)$, then the operator $H_{\mu,\gamma}(\mathbf{0})$ has no eigenvalues. If $\gamma \in (\gamma_1, \gamma_2)$, then the operator $H_{\mu,\gamma}(\mathbf{0})$ has a unique eigenvalue; if $\gamma > \gamma_2$, then the operator $H_{\mu,\gamma}(\mathbf{0})$ has three eigenvalues lying to the right of the essential spectrum for all sufficiently large values of the interaction energy μ.

KEYWORDS Schrödinger operator, Hamiltonian, contact potential, fermion, eigenvalue, quasi-momentum, invariant subspace, Faddeev operator.

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1. Introduction

The study of few-body systems with contact interaction has a long history and a wide literature throughout the last eight decades, a concise retrospective may be found in [1]. The 2+1 fermionic system is an actual building block for the heteronuclear mixtures with inter-species contact interaction, see [2] for an outlook. For the 2+1 fermionic model, the rigorous construction of the Hamiltonian H_α for $m > m^*$, together with the precise determination of m^* and the proof of the self-adjointness and the semi-boundedness from below of H_α, was done in the work [3] by Correggi, Dell'Antonio, Finco, Michelangeli, and Teta, by means of quadratic form techniques for contact interactions [4]. In [5] the authors had qualified the main features of the spectrum of the Hamiltonian of point interaction for a three-dimensional quantum system consisting of three point-like particles, two identical fermions, plus a third particle of different species, with two-body interaction of zero range. For arbitrary magnitude of the interaction and arbitrary value of the mass parameter (the ratio between the mass of the third particle and that of each fermion) above the stability threshold, the essential spectrum is identified, the discrete spectrum is localized and its finiteness is proved. The existence or absence of bound states is proved in physically relevant regimes of masses.

Throughout physics, stable composite objects are usually formed by way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions that arise from the lattice band structure [6]. The Bose–Hubbard model which is used to describe the repulsive pairs is the theoretical basis for applications. The work [6] exemplifies the important correspondence between the Bose–Hubbard model [7], [8] and atoms in optical lattices, and helps pave the way for many more interesting developments and applications [9]. Stable repulsively bound objects should be viewed as a general phenomenon and their existence will be ubiquitous in cold atoms lattice physics. They give rise also to new potential composites with fermions [10] or Bose–Fermi mixtures [11], and can be formed in an analogous manner with more than two particles [12].

Systems of particles, with zero-range interactions between the pairs of particles, are investigated not only theoretically but also experimentally. Delta-like character of the interaction turns out to be realistic. This is a special case of the *unitary regime*, i.e. the case of negligible interaction range and huge, virtually infinite, scattering length. In this case, unitary gases posses a property of superfluidity [13], and they were intensively studied both experimentally and theoretically [14].

In this paper, we consider the Hamiltonian $\mathrm{H}_{\mu,\gamma}$ for systems of three quantum particles (two of them are fermions with mass 1 and one of them is arbitrary with mass $m = 1/\gamma < 1$) with paired contact repulsive potentials $\mu > 0$ on a three-dimensional lattice $\mathbb{Z}^3$. In the momentum representation, the total three-particle Hamiltonian expands into a direct operator integral (see. [15])

$$\mathrm{H}_{\mu,\gamma} = \int\limits_{\mathbb{T}^3} \oplus H_{\mu,\gamma}(\mathbf{K}) d\mathbf{K}.$$

The fiber operator $H_{\mu,\gamma}(\mathbf{K}) = H_{0,\gamma}(\mathbf{K}) + \mu(V_1 + V_2)$ parametrically depends on the *total quasimomentum* $\mathbf{K} \in \mathbb{T}^3 \equiv \mathbb{R}^3/(2\pi\mathbb{Z}^3)$. It is shown that the essential spectrum of the self-adjoint operator $H_{\mu,\gamma}(\mathbf{K})$ consists of one or two segments, depending on the three-particle quasimomentum $\mathbf{K} \in \mathbb{T}^3$ and the interaction energy $\mu > 0$. Unlike the continuous case, the Schrödinger operator on a lattice can have eigenvalues at the right part of the essential spectrum as well.

The principal results of this paper are given for sufficiently big values of interaction energy $\mu > 0$, i.e., when the two-particle subsystems have bound states with positive energies: there are threshold values of the particle mass ratio γ_1, γ_2 such that if $\gamma \in (0, \gamma_1)$, then the operator $H_{\mu,\gamma}(\mathbf{0})$ has no eigenvalues; if $\gamma \in (\gamma_1, \gamma_2)$, then the operator $H_{\mu,\gamma}(\mathbf{0})$ has a unique eigenvalue; if $\gamma > \gamma_2$, then the operator $H_{\mu,\gamma}(\mathbf{0})$ has three eigenvalues lying to the right of the essential spectrum. Existence of at least one eigenvalue of the three-particle discrete Schrödinger operator $H_\mu(\mathbf{K}) = H_0(\mathbf{K}) - \mu V$ ($\mu \in \mathbb{R}$) for dimensions $d = 1, 2$ was shown in [15] and [12], whose proofs are based on the unboundedness of the norm of the Faddeev operator $\mathbf{T}(\mathbf{K}, z)$ at the lower bound of the essential spectrum $z = \inf(\sigma_{ess}(H_\mu(\mathbf{K})))$. If $d \geq 3$, then the operator $\mathbf{T}(\mathbf{K}, z)$ is also bounded at the edges of the essential spectrum, i.e. in this case, methods for $d = 1, 2$ is not applicable.

In [16], the model operator H_γ^{as} (see (2.6) paper in [16]), associated with three-particle discrete Schrödinger operator on a three-dimensional cubic lattice with pairwise zero-range attractive potentials, is studied, where the family of Friedrichs models with parameters $h_\alpha(\mathbf{k}), \alpha = 1, 2, \mathbf{k} \in \mathbb{T}^3$ is used. The existence of the critical value γ^* of the parameter γ is proved so that if two-particle subsystems have a resonance with zero energy and do not have bound states with negative energy, then H_γ^{as} has an infinite number of eigenvalues, lying to the left of the essential spectrum for $\gamma > \gamma^* \approx 13.607$, and there is no Efimov's effect for $\gamma < \gamma^*$. The similar result holds for the operator we are considering $H_{\mu,\gamma}(\boldsymbol{\pi})$, i.e., at $\gamma > \gamma^*$ and fixed $\mu = \mu_0(\gamma)$, the operator $H_{\mu,\gamma}(\boldsymbol{\pi})$ has an infinite number of eigenvalues to the right of the essential spectrum. "The two-particle branch" of the essential operator spectrum of $H_{\mu,\gamma}(\mathbf{K})$ is shifted to $+\infty$ with the order μ if $\mu \to +\infty$, as a result of which an infinite number of eigenvalues of the operator are "absorbed" by the essential spectrum. Therefore, a natural question arises: whether there are eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{0})$, lying to the right of the essential spectrum for sufficiently large μ, and if so, how many?

In this paper, we prove that the operator $H_{\mu,\gamma}(\mathbf{0}), \mathbf{0} = (0, 0, 0)$, for $\gamma \in (0, \gamma_1)$ ($\gamma_1 \approx 2,937$) has no eigenvalues, but for $\gamma_1 < \gamma < \gamma_2$ ($\gamma_2 \approx 5,396$) has a unique eigenvalue, and for $\gamma > \gamma_2$ has exactly three eigenvalues to the right of the essential spectrum for sufficiently large μ. Physically, this shows the conditions for the system of two fermions (of mass 1), and an arbitrary particle (of mass $m, m < 1$) with pairwise repulsive interaction μ, which is sufficiently large, to have no bound states, one bound state and three bound states, respectively.

Applying the perturbation theory, one can show that the results obtained are preserved for small values $\mathbf{K}$. Note that the problem of finding the number of eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{K})$, which are more z ($z > \tau_{\max,\gamma}(\mu, \mathbf{K})$) reduces to the problem of finding the number of eigenvalues of the Faddeev-type operator $A_{\mu,\gamma}(\mathbf{K}, z)$, which are more 1 (see. (4.2)). Sensitivity of the kernel of the integral operator $A_{\mu,\gamma}(\mathbf{K}, z)$ regarding change $\mathbf{K}$ leads to a change in the number of eigenvalues of the operator $H_{\mu,\gamma}(\mathbf{K})$. Therefore, set the number of eigenvalues for all $\mathbf{K} \in \mathbf{T}^3$ is very difficult.

2. Statement of the problem and formulation of the main result

Let $\mathbb{Z}^3$ is a three-dimensional lattice, $\ell^2[(\mathbb{Z}^3)^d], d = 2, 3$ is a Hilbert space of square integrable functions given on $(\mathbb{Z}^3)^d$ and $\ell^{2,as}[(\mathbb{Z}^3)^d] \subset \ell^2[(\mathbb{Z}^3)^d]$ is a subspace of antisymmetric functions with respect to permutation of the first two coordinates.

We consider a Hamiltonian of a system of three quantum particles (two of them are fermions with mass 1 and one of them is arbitrary with mass $m = 1/\gamma < 1$) that interact through pairwise zero-range repulsive potentials on $\mathbb{Z}^3$. Without a loss of generality, we assume that the first two particles are fermions while the third one is a particle of a different nature.

The Hamiltonian of the system of two arbitrary free particles (a fermion and another particle) on $\mathbb{Z}^3$ in the coordinate representation is associated with the bounded self-adjoint operator $\hat{\mathrm{h}}_{0,\gamma}$ in $\ell^2[(\mathbb{Z}^3)^2]$:

$$\hat{\mathrm{h}}_{0,\gamma} = -\frac{1}{2}\Delta \otimes I - \frac{\gamma}{2} I \otimes \Delta,$$

where Δ is the lattice Laplacian, I is the unity operator in $\ell^2(\mathbb{Z}^3)$, and $\gamma = \dfrac{1}{m}$.

The total Hamiltonian $\hat{\mathrm{h}}_{\mu,\gamma}$ of the system of two arbitrary particles with the zero-range repulsive potential acts in $\ell^2[(\mathbb{Z}^3)^2]$ and is a bounded perturbation of the free Hamiltonian $\hat{\mathrm{h}}_{0,\gamma}$:

$$\hat{\mathrm{h}}_{\mu,\gamma} = \hat{\mathrm{h}}_{0,\gamma} + \mu\hat{v},$$

where $\mu, \mu > 0$, is the interaction energy of two repelling particles (a fermion and another particle), operator $\hat{v}$ describes the zero-range interaction of these particles

$$(\hat{v}\hat{\psi})(\mathbf{x_2}, \mathbf{x}_3) = \delta_{\mathbf{x}_2\mathbf{x}_3}\hat{\psi}(\mathbf{x}_2, \mathbf{x}_3)$$

and $\delta_{\mathbf{x}_2\mathbf{x}_3}$ is the Kronecker symbol. In the space $\ell^{2,as}[(\mathbb{Z}^3)^2]$, there is no two-particle zero-range interaction of fermions (see [15], [17]).

Similarly, the free Hamiltonian $\hat{\mathrm{H}}_{0,\gamma}$ of the system of three particles (two fermions and another particle) on lattice $\mathbb{Z}^3$ is specified in $\ell^{2,as}[(\mathbb{Z}^3)^3]$ by the formula

$$\hat{\mathrm{H}}_{0,\gamma} = -\frac{1}{2}\Delta \otimes I \otimes I - \frac{1}{2} I \otimes \Delta \otimes I - \frac{\gamma}{2} I \otimes I \otimes \Delta.$$

The total Hamiltonian $\hat{\mathrm{H}}_{\mu,\gamma}$ of the system of three particles with pairwise zero-range interactions is a bounded perturbation of the free Hamiltonian $\hat{\mathrm{H}}_{0,\gamma}$:

$$\hat{\mathrm{H}}_{\mu,\gamma} = \hat{\mathrm{H}}_{0,\gamma} + \mu(\hat{\mathrm{V}}_1 + \hat{\mathrm{V}}_2),$$

where

$$(\hat{\mathrm{V}}_1\hat{\psi})(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \delta_{\mathbf{x}_2\mathbf{x}_3}\hat{\psi}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

and

$$(\hat{\mathrm{V}}_2\hat{\psi})(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \delta_{\mathbf{x}_3\mathbf{x}_1}\hat{\psi}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

Let $\mathbb{T}^3$ is a three-dimensional torus and $L_2^{as}[(\mathbb{T}^3)^2] \subset L_2[(\mathbb{T}^3)^3]$ be the Hilbert space of square integrable functions, defined on $(\mathbb{T}^3)^3$ and antisymmetric with respect to permutation of the first two coordinates. Assume that $d\mathbf{p}$ is a unit measure in the torus $\mathbb{T}^3$, that is

$$\int_{\mathbb{T}^3} d\mathbf{p} = 1.$$

The study of spectra of the Hamiltonians $h_{\mu,\gamma}$ and $H_{\mu,\gamma}$ is reduced to studying the spectra of the family of operators $h_{\mu,\gamma}(\mathbf{k}), \mathbf{k} \in \mathbb{T}^3$ and $H_{\mu,\gamma}(\mathbf{K}), \mathbf{K} \in \mathbb{T}^3$, respectively (see [15], [18]).

The two-particle discrete Schrödinger operator $h_{\mu,\gamma}(\mathbf{k}), \mathbf{k} \in \mathbb{T}^3$ acts in $L_2(\mathbb{T}^3)$ by the formula

$$h_{\mu,\gamma}(\mathbf{k}) = h_{0,\gamma}(\mathbf{k}) + \mu v,$$

where

$$(h_{0,\gamma}(\mathbf{k})f)(\mathbf{p}) = \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{p})f(\mathbf{p}), \quad \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{p}) = \varepsilon\big(\mathbf{p}\big) + \gamma\varepsilon\big(\mathbf{k} - \mathbf{p}\big),$$

$$\varepsilon(\mathbf{p}) = 3 - \xi(\mathbf{p}), \;\; \xi(\mathbf{p}) = \sum_{i=1}^{3} \cos p_i, \;\; \mathbf{p} = (p_1, p_2, p_3) \in \mathbb{T}^3, \tag{2.1}$$

$$(vf)(\mathbf{p}) = \int_{\mathbb{T}^3} f(\mathbf{s})d\mathbf{s}.$$

The respective three-particle discrete Schrödinger operator $H_{\mu,\gamma}(\mathbf{K})$ acts in $L_2^{as}[(\mathbb{T}^3)^2]$ by the formula

$$H_{\mu,\gamma}(\mathbf{K}) = H_{0,\gamma}(\mathbf{K}) + \mu(V_1 + V_2),$$

where

$$(H_{0,\gamma}(\mathbf{K})f)(\mathbf{p}, \mathbf{q}) = E_{\mathbf{K},\gamma}(\mathbf{p}, \mathbf{q})f(\mathbf{p}, \mathbf{q}), \quad E_{\mathbf{K},\gamma}(\mathbf{p}, \mathbf{q}) = \varepsilon(\mathbf{p}) + \varepsilon(\mathbf{q}) + \gamma\varepsilon(\mathbf{K} - \mathbf{p} - \mathbf{q}).$$

$$(V_1 f)(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{T}^3} f(\mathbf{p}, \mathbf{s})d\mathbf{s}, \quad (V_2 f)(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{T}^3} f(\mathbf{s}, \mathbf{q})d\mathbf{s}.$$

Let us first introduce the following notation:

$$W = \int_{\mathbb{T}^3} \frac{d\mathbf{s}}{\varepsilon(\mathbf{s})}, \;\; W_1 = \int_{\mathbb{T}^3} \frac{\cos s_1 d\mathbf{s}}{\varepsilon(\mathbf{s})}, \;\; W_{11} = \int_{\mathbb{T}^3} \frac{\cos^2 s_1 d\mathbf{s}}{\varepsilon(\mathbf{s})}, \;\; W_{12} = \int_{\mathbb{T}^3} \frac{\cos s_1 \cos s_2 d\mathbf{s}}{\varepsilon(\mathbf{s})}.$$

The integral W is called the Watson integral and the other integrals W_1, W_{11} and $W_{12}-$ Watson-type integrals (see, for example [20]).

The main result of the paper is the following theorem:

Theorem 2.1. *Let*

$$\gamma_1 = \frac{W}{W_{11}W + 2WW_{12} - 3W_1^2} \approx 2,9368, \qquad \gamma_2 = \frac{1}{W_{11} - W_{12}} \approx 5,3985. \tag{2.2}$$

(i) *Assume that $\gamma \in (0, \gamma_1)$. Then, there exists $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ the operator $H_{\mu,\gamma}(\mathbf{0})$ has no eigenvalues lying to the above of the essential spectrum.*

(ii) *Assume that $\gamma \in (\gamma_1, \gamma_2)$. Then, there exists $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ the operator $H_{\mu,\gamma}(\mathbf{0})$ has a unique eigenvalue lying to the above of the essential spectrum.*

(iii) *Assume that $\gamma \in (\gamma_2, +\infty)$. Then, there exists $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ the operator $H_{\mu,\gamma}(\mathbf{0})$ have three eigenvalues to the above of the essential spectrum.*

Remark 2.2. *The number μ_γ takes on different values in the three cases of the Theorem 2.1.*

3. On the spectrum of the two-particle operator $h_{\mu,\gamma}(\mathbf{k})$

In this section, we study some facts related to the spectrum of the operator $h_{\mu,\gamma}(\mathbf{k})$.

Since v is compact, by Weyl's Theorem [19] for any $\mathbf{k} \in \mathbb{T}^3$, the essential spectrum $\sigma_{ess}(h_{\mu,\gamma}(\mathbf{k}))$ of $h_{\mu,\gamma}(\mathbf{k})$ coincides with the spectrum of $h_{0,\gamma}(\mathbf{k})$, i.e.,

$$\sigma_{ess}(h_{\mu,\gamma}(\mathbf{k})) = [\mathcal{E}_{\min,\gamma}(\mathbf{k}), \mathcal{E}_{\max,\gamma}(\mathbf{k})],$$

where

$$\mathcal{E}_{\min,\gamma}(\mathbf{k}) = \min_{\mathbf{q}\in\mathbb{T}^3} \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{q}) = 3(1+\gamma) - \sum_{i=1}^{3} \sqrt{1 + 2\gamma\cos k_i + \gamma^2},$$

$$\mathcal{E}_{\max,\gamma}(\mathbf{k}) = \max_{\mathbf{q}\in\mathbb{T}^3} \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{q}) = 3(1+\gamma) + \sum_{i=1}^{3} \sqrt{1 + 2\gamma\cos k_i + \gamma^2}.$$

The following Lemma provides an implicit equation for eigenvalues of $h_{\mu,\gamma}(\mathbf{k})$ which is a simple application of the Fredholm determinants theory.

Lemma 3.1. *The number $z \in \mathbb{C} \setminus [\mathcal{E}_{\min,\gamma}(\mathbf{k}), \mathcal{E}_{\max,\gamma}(\mathbf{k})]$ is an eigenvalue of $h_{\mu,\gamma}(\mathbf{k})$ with multiplicity m if and only if z is a zero of the function*

$$\Delta_{\mu,\gamma}(\mathbf{k}, z) = 1 - \mu \int\limits_{\mathbb{T}^3} \frac{d\mathbf{q}}{z - \mathcal{E}_{\mathbf{k},\gamma}(\mathbf{q})} \tag{3.1}$$

with the multiplicity m.

The function $\Delta_{\mu,\gamma}(\mathbf{k}, z)$ is called the *Fredholm determinant* associated to $h_{\mu,\gamma}(\mathbf{k})$.

Note that, the function $\Delta_{\mu,\gamma}(\mathbf{k}, z)$ is the Fredholm determinant of the operator $I - \mu v r_{0,\gamma}(\mathbf{k}, z)$, where $r_{0,\gamma}(\mathbf{k}, z)$ is the resolvent of the operator $h_{0,\gamma}(\mathbf{k})$ and v is the integral operator with the kernel $v(\mathbf{q}, \mathbf{q}') = 1$.

Let us introduce first the following real number:

$$\mu_0(\gamma) = (1+\gamma)\,\frac{1}{W}.$$

Note that this number means harmonic values of the kinetic energies of a fermion and another particle.

Lemma 3.2. *Assume that $\mu > \mu_0(\gamma)$. Then for each $\mathbf{k} \in \mathbb{T}^3$ the operator $h_{\mu,\gamma}(\mathbf{k})$ has a unique simple eigenvalue $z_{\mu,\gamma}(\mathbf{k})$ above the essential spectrum.*

Lemma 3.3. *The eigenvalue $z_{\mu,\gamma}(\mathbf{k}) = z_{\mu,\gamma}(k_1, k_2, k_3)$ is symmetric function with respect to permutation of the variables k_i, k_j, even with respect to $k_i \in [-\pi, \pi]$, and decreasing with respect to $k_i \in [0, \pi], i = 1, 2, 3$.*

Proof. The proof of the lemma follows directly from the properties of the function $\Delta_{\mu,\gamma}(\mathbf{k}, z)$ and assertions of Lemma 3.1. □

Lemma 3.4. *For any $\gamma > 0$ and $\mu > 3(1+\gamma)$, we have the following relations*

$$\mu + 3(1+\gamma) < z_{\mu,\gamma}(\boldsymbol{\pi}) \le z_{\mu,\gamma}(\mathbf{k}) \le z_{\mu,\gamma}(\mathbf{0}) < \mu + 3(1+\gamma) + \frac{9(1+\gamma)^2}{\mu}.$$

Proof. The proof of the Lemma follows from Lemma (3.3) and properties of the function $\Delta_{\mu,\gamma}(\mathbf{k}, z)$. □

Corollary 3.5. *For any $\gamma > 0$, the function $z_{\mu,\gamma}(\mathbf{k})$ has the following asymptotic expansions:*

$$z_{\mu,\gamma}(\mathbf{k}) = \mu + 3(1+\gamma) + O\left(\frac{1}{\mu}\right) \tag{3.2}$$

as $\mu \to \infty$, uniformly $\mathbf{k} \in \mathbb{T}^3$.

4. Essential spectrum of a three-particle operator $H_{\mu,\gamma}(\mathbf{K})$.

For any $\mathbf{K} \in \mathbb{T}^3$, recalling that

$$E_{\min,\gamma}(\mathbf{K}) = \min_{\mathbf{p},\mathbf{q}\in\mathbb{T}^3} E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}), \quad E_{\max,\gamma}(\mathbf{K}) = \max_{\mathbf{p},\mathbf{q}\in\mathbb{T}^3} E_{\mathbf{K},\gamma}(\mathbf{p},\mathbf{q}),$$

$$\tau_{\min,\gamma}(\mu,\mathbf{K}) = \min_{\mathbf{p}\in\mathbb{T}^3}\{z_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})\}, \quad \tau_{\max,\gamma}(\mu,\mathbf{K}) = \max_{\mathbf{p}\in\mathbb{T}^3}\{z_{\mu,\gamma}(\mathbf{K}-\mathbf{p}) + \varepsilon(\mathbf{p})\},$$

where $z_{\mu,\gamma}(\mathbf{p})$ is an eigenvalue of the operator $h_{\mu,\gamma}(\mathbf{p})$ and the essential spectrum of $H_{\mu,\gamma}(\mathbf{K})$ coincides with the union of two segment:

$$\sigma_{ess}(H_{\mu,\gamma}(\mathbf{K})) = [E_{\min,\gamma}(\mathbf{K}), E_{max,\gamma}(\mathbf{K})] \cup [\tau_{\min,\gamma}(\mu,\mathbf{K}), \tau_{\max,\gamma}(\mu,\mathbf{K})]. \tag{4.1}$$

The proof of a similar assertion is given in the paper [18]. Note that $[\tau_{\min,\gamma}(\mu,\mathbf{K}), \tau_{\max,\gamma}(\mu,\mathbf{K})]$ and $[E_{\min,\gamma}(\mathbf{K}), E_{\max,\gamma}(\mathbf{K})]$ are called the "two-particle branch" and the "three-particle branch" of the essential spectrum of $H_{\mu,\gamma}(\mathbf{K})$, respectively.

For fixed $\gamma, \gamma > 0$, we study the discrete spectrum of the operator $H_{\mu,\gamma}(\mathbf{0})$, $\mathbf{0} = (0,0,0)$ for sufficiently large $\mu > 0$. It follows from Lemma 2.4 and the structure of the essential spectrum that (see (4.1)), that the two-particle branch $[\tau_{\min,\gamma}(\mu,\mathbf{0}), \tau_{\max,\gamma}(\mu,\mathbf{0})]$ of the essential spectrum shifts $+\infty$ with order μ at $\mu \to +\infty$.

In what follows we always assume $z \geq \inf \sigma_{ess}(H_{\mu,\gamma}(\mathbf{0})) = \tau_{\max,\gamma}(\mu,\mathbf{0})$.

Discrete spectrum of a three-particle operator $H_{\mu,\gamma}(\mathbf{0})$.

First, we show that the operator $H_{\mu,\gamma}(\mathbf{K})$ has no eigenvalues below the essential spectrum.

Lemma 4.1. *Assume that $\mathbf{K} \in \mathbb{T}^3$. Then for any $\mu > 0$ and $\gamma > 0$ the operator $H_{\mu,\gamma}(\mathbf{K})$ has no eigenvalues below the essential spectrum.*

Proof. Since the operator $V = V_1 + V_2$ is positive by the minimax principle we can conclude that

$$\inf_{\|f\|=1}(H_{\mu,\gamma}(\mathbf{K})f,f) = \inf_{\|f\|=1}[(H_{0,\gamma}(\mathbf{K})f,f) + \mu(Vf,f)] \geq \inf_{\|f\|=1}(H_{0,\gamma}(\mathbf{K})f,f) = E_{\min,\gamma}(\mathbf{K}),$$

leading to $\sigma(H_{\mu,\gamma}(\mathbf{K})) \cap (-\infty, E_{\min,\gamma}(\mathbf{K})) = \emptyset$. $\square$

For any $z > \tau_{\max,\gamma}(\mu,\mathbf{0})$, we define the self-adjoint compact operator of the form

$$(A_{\mu,\gamma}(z)\psi)(\mathbf{p}) = \frac{-\mu}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}} \int_{\mathbb{T}^3} \frac{\psi(\mathbf{s})d\mathbf{s}}{(z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s}))\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}} \tag{4.2}$$

defined in

$$D(A_{\mu,\gamma}(z)) = \left\{\psi \in L_2(\mathbb{T}^3) : \quad \int_{\mathbb{T}^3} \frac{\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}} = 0\right\},$$

where

$$\Lambda_{\mu,\gamma}(\mathbf{p},z) := \Delta_{\mu,\gamma}(-\mathbf{p}, z - \varepsilon(\mathbf{p})), \tag{4.3}$$

and the function $\Delta_{\mu,\gamma}(.,.)$ is given by formula (3.1).

The operator $A_{\mu,\gamma}(z)$ is called the Faddeev-type operator corresponding to the operator $H_{\mu,\gamma}(\mathbf{0})$ (see Remark 4.3 and [21], [22]).

Hence, we found the equivalent equation for the eigenfunctions of the three-particle operator $H_{\mu,\gamma}(\mathbf{0})$.

Lemma 4.2. *The number $z > \tau_{\max,\gamma}(\mu,\mathbf{0})$ is an eigenvalue of the operator $H_{\mu,\gamma}(\mathbf{0})$ if and only if the number 1 is an eigenvalue of the operator $A_{\mu,\gamma}(z)$.*

Proof. Let $z > \tau_{\max,\gamma}(\mu,\mathbf{0})$ is the eigenvalue of the operator $H_{\mu,\gamma}(\mathbf{0})$ and f is the respective eigenfunction, i.e., the equation

$$E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{q})f(\mathbf{p},\mathbf{q}) + \mu\int_{\mathbb{T}^3} f(\mathbf{p},\mathbf{s})d\mathbf{s} + \mu\int_{\mathbb{T}^3} f(\mathbf{s},\mathbf{q})d\mathbf{s} = zf(\mathbf{p},\mathbf{q}) \tag{4.4}$$

has a nonzero solution $f \in L_2^{as}[(\mathbb{T}^3)^2]$. Introducing the notation

$$\varphi(\mathbf{p}) = (V_1 f)(\mathbf{p},\mathbf{q}) = \int_{\mathbb{T}^3} f(\mathbf{p},\mathbf{s})d\mathbf{s}, \tag{4.5}$$

from (4.4) for $z > \tau_{\max,\gamma}(\mu,\mathbf{0})$, we have

$$f(\mathbf{p},\mathbf{q}) = \mu\frac{\varphi(\mathbf{p}) - \varphi(\mathbf{q})}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{q})}. \tag{4.6}$$

Since the function f is antisymmetric, the function φ given by formula (4.5), belongs to the space $L_2(\mathbb{T}^3)$ and satisfies the condition

$$\int_{\mathbb{T}^3} \varphi(\mathbf{p})\, d\mathbf{p} = 0.$$

Substituting the expression (4.6) into (4.5), we obtain that the equation

$$\varphi(\mathbf{p})\bigg(1 - \mu \int_{\mathbb{T}^3} \frac{d\mathbf{s}}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})}\bigg) = -\mu \int_{\mathbb{T}^3} \frac{\varphi(\mathbf{s}) d\mathbf{s}}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})}$$

has a nonzero solution $\varphi \in L_2(\mathbb{T}^3)$. Hence, using notation (3.1) and (4.3), we make sure that $\varphi \in L_2(\mathbb{T}^3)$ is the solution of the equation

$$\varphi(\mathbf{p}) = \frac{-\mu}{\Lambda_{\mu,\gamma}(\mathbf{p},z)} \int_{\mathbb{T}^3} \frac{\varphi(\mathbf{s}) d\mathbf{s}}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})}. \tag{4.7}$$

If we set $\psi(\mathbf{p}) = \sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}\,\varphi(\mathbf{p})$, from (4.7) we have

$$\psi(\mathbf{p}) = \frac{-\mu}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}} \int_{\mathbb{T}^3} \frac{\psi(\mathbf{s}) d\mathbf{s}}{(z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s}))\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}},$$

i.e., $\lambda = 1$ is the eigenvalue of the operator $A_{\mu,\gamma}(z)$ and

$$\int_{\mathbb{T}^3} \frac{\psi(\mathbf{s}) d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}} = 0.$$

Suppose that, for some $z > \tau_{\max,\gamma}(\mu,\mathbf{0})$ the number 1 is the eigenvalue of the operator $A_{\mu,\gamma}(z)$, and $\psi \in D(A_{\mu,\gamma}(z))$ is the corresponding eigenfunction. Then, the function f is given by formula (4.6), where $\varphi(\mathbf{p}) = \psi(\mathbf{p})\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}$, belongs to the space $L_2^{as}[(\mathbb{T}^3)^2]$ and satisfies the equality (4.4). □

Remark 4.3. $a)$ *Note that the relation between eigenfunctions f and ψ, respectively, of $H_{\mu,\gamma}(\mathbf{0})$ and $\mathrm{A}_{\mu,\gamma}(z)$ corresponding to the eigenvalues z and 1 is*

$$f(\mathbf{p},\mathbf{q}) = \mu \frac{(\Lambda_{\mu,\gamma}(\mathbf{p},z))^{-1/2}\,\psi(\mathbf{p}) - (\Lambda_{\mu,\gamma}(\mathbf{q},z))^{-1/2}\,\psi(\mathbf{q})}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{q})}.$$

Therefore, we can say that the operator $A_{\mu,\gamma}(z)$ is the Faddeev-type operator.

$b)$ *A limit operator*

$$\lim_{z \to \tau_{\max,\gamma}(\mu,\mathbf{0})} A_{\mu,\gamma}(z) = A_{\mu,\gamma}(\tau_{\max,\gamma}(\mu,\mathbf{0}))$$

is a compact self-adjoint operator in $L_2(\mathbb{T}^3)$.

For the bounded self-adjoint operator B, acting in the Hilbert space $\mathcal{H}$ and for some $\lambda \in \mathbb{R}$ define a number $n[\lambda, B]$ by

$$n[\lambda, B] := \max\{\dim \mathcal{H}_B(\lambda) : \mathcal{H}_B(\lambda) \subset \mathcal{H};\ (B\varphi,\varphi) > \lambda,\ \ \varphi \in \mathcal{H}_B(\lambda),\ ||\varphi|| = 1\}.$$

If some point of the essential spectrum of the operator B is greater than λ then $n[\lambda, B]$ equals infinity, if $n[\lambda, B]$ is finite, it equals to the number of eigenvalues of the operator B, that are greater than λ (see. example Lemma Glazman [23]).

The known Birman–Schwinger principle (see. [15]) leads to the following lemma.

Lemma 4.4. *Let $\mu > \mu_0(\gamma)$. Then, for any $z \geq \tau_{\max,\gamma}(\mu,\mathbf{0})$ the equality holds*

$$n[z, H_{\mu,\gamma}(\mathbf{0})] = n[1, \mathrm{A}_{\mu,\gamma}(z)].$$

5. On the spectrum of the operator $A_{\mu,\gamma}(z)$.

It is well-known that the three-particle branch $[E_{\min,\gamma}(\mathbf{0}), E_{\max,\gamma}(\mathbf{0})]$ of the essential spectrum of the operator $H_{\mu,\gamma}(\mathbf{0})$ is independent of the parameter $\mu > 0$, and the two-particle branch $[\tau_{\min,\gamma}(\mu,\mathbf{0}), \tau_{\max,\gamma}(\mu,\mathbf{0})]$ of the essential spectrum shifts to $+\infty$, when $\mu \to +\infty$. Therefore, in what follows, we assume that μ is large enough and $z \geq \tau_{\max,\gamma}(\mu,\mathbf{0})$.

Using the equality $\dfrac{1}{1+x} = 1 - x + \dfrac{x^2}{1+x}$, $(x \neq -1)$, and given notation (2.1), we have

$$\frac{1}{z - E_{0,\gamma}(\mathbf{p},\mathbf{s})} = \frac{1}{a(\gamma,z)}\bigg(1 - \frac{(\xi(\mathbf{p}) + \xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s}))}{a(\gamma,z)} + \frac{\zeta(\gamma;\mathbf{p},\mathbf{s})}{a(\gamma,z)}\bigg), \tag{5.1}$$

where

$$a(\gamma, z) = z - 6 - 3\gamma \text{ and } \zeta(\gamma; \mathbf{p}, \mathbf{s}) = \frac{\big(\xi(\mathbf{p}) + \xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})\big)^2}{z - E_{0,\gamma}(\mathbf{p}, \mathbf{s})}.$$

Taking into account the equality (5.1), we represent the operator $A_{\mu,\gamma}(z)$ as a sum

$$A_{\mu,\gamma}(z) = A^{(1)}_{\mu,\gamma}(z) + A^{(2)}_{\mu,\gamma}(z), \tag{5.2}$$

where

$$(A^{(1)}_{\mu,\gamma}(z)\psi)(\mathbf{p}) = \frac{\mu}{a^2(\gamma, z)} \int\limits_{\mathbb{T}^3} \frac{\big(\xi(\mathbf{p}) + \xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s}) - a(\gamma, z)\big)\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p}, z)}\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s}, z)}},$$

$$(A^{(2)}_{\mu,\gamma}(z)\psi)(\mathbf{p}) = -\frac{\mu}{a^2(\gamma, z)} \int\limits_{\mathbb{T}^3} \frac{\zeta(\gamma; \mathbf{p}, \mathbf{s})\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p}, z)}\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s}, z)}}.$$

In what follows, it is shown that the norm of the operator $A^{(2)}_{\mu,\gamma}(z)$ tends to zero as $\mu \to +\infty$ (see Lemma 5.5). Therefore, let us establish the existence of eigenvalues of the operator $A^{(1)}_{\mu,\gamma}(z)$ which are greater 1 for large enough $\mu > 0$.

Let us find the invariant subspaces with respect to $A^{(1)}_{\mu,\gamma}(z)$.

The Hilbert space $L_2(\mathbb{T}^3)$ can be represented as a direct sum

$$L_2(\mathbb{T}^3) = L^o_2(\mathbb{T}^3) \oplus L^e_2(\mathbb{T}^3),$$

where

$$L^o_2(\mathbb{T}^3) = \{\psi \in L_2(\mathbb{T}^3) : \psi(-\mathbf{p}) = -\psi(\mathbf{p})\}, \quad L^e_2(\mathbb{T}^3) = \{\psi \in L_2(\mathbb{T}^3) : \psi(-\mathbf{p}) = \psi(\mathbf{p})\}.$$

Lemma 5.1. *The subspaces $L^e_2(\mathbb{T}^3)$ and $L^o_2(\mathbb{T}^3)$ are invariant under the operators $A_{\mu,\gamma}(z)$, $A^{(1)}_{\mu,\gamma}(z)$ and $A^{(2)}_{\mu,\gamma}(z)$.*

Proof. From the definitions $\Lambda_{\mu,\gamma}(\mathbf{p}, z)$ and $\varepsilon(\mathbf{p})$ it follows that

$$\Lambda_{\mu,\gamma}(-\mathbf{p}, z) = \Delta_{\mu,\gamma}(\mathbf{p}, z - \varepsilon(-\mathbf{p})) = \Delta_{\mu,\gamma}(-\mathbf{p}, z - \varepsilon(\mathbf{p})) = \Lambda_{\mu,\gamma}(\mathbf{p}, z). \tag{5.3}$$

If $\psi \in L^e_2(\mathbb{T}^3)$, then making the change of variable $\mathbf{s} = -\mathbf{q}$, given equalities $E_{\mathbf{0},\gamma}(-\mathbf{p}, -\mathbf{q}) = E_{\mathbf{0},\gamma}(\mathbf{p}, \mathbf{q})$ and (5.3), we get

$$\widetilde{\psi}(-\mathbf{p}) = \big(A_{\mu,\gamma}(z)\psi\big)(-\mathbf{p}) = -\frac{\mu}{\sqrt{\Lambda_{\mu,\gamma}(-\mathbf{p}, z)}} \int\limits_{\mathbb{T}^3} \frac{\psi(\mathbf{s})d\mathbf{s}}{(z - E_{\mathbf{0},\gamma}(-\mathbf{p}, \mathbf{s}))\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s}, z)}} =$$

$$= -\frac{\mu}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p}, z)}} \int\limits_{\mathbb{T}^3} \frac{\psi(\mathbf{q})d\mathbf{q}}{(z - E_{\mathbf{0},\gamma}(-\mathbf{p}, -\mathbf{q}))\sqrt{\Lambda_{\mu,\gamma}(\mathbf{q}, z)}} = \widetilde{\psi}(\mathbf{p}).$$

Therefore, the subspace $L^e_2(\mathbb{T}^3)$ is invariant under $A_{\mu,\gamma}(z)$. Since the operator $A_{\mu,\gamma}(z)$ is self-adjoint, orthogonal complement $L^o_2(\mathbb{T}^3)$ of subspaces $L^e_2(\mathbb{T}^3)$ is also invariant under the operator $A_{\mu,\gamma}(z)$. The other statements are proved similarly. $\square$

Denote by P^o and P^e the space projection operators in $L_2(\mathbb{T}^3)$ into subspaces $L^o_2(\mathbb{T}^3)$ and $L^e_2(\mathbb{T}^3)$, respectively. For $\psi \in L_2(\mathbb{T}^3)$, the following equalities are true

$$(P^o\psi)(\mathbf{p}) = \frac{1}{2}[\psi(\mathbf{p}) - \psi(-\mathbf{p})], \quad (P^e\psi)(\mathbf{p}) = \frac{1}{2}[\psi(\mathbf{p}) + \psi(-\mathbf{p})].$$

From the invariance of subspaces $L^o_2(\mathbb{T}^3)$ and $L^e_2(\mathbb{T}^3)$ with respect to the operator $A^{(1)}_{\mu,\gamma}(z)$, it follows that the projectors P^o and P^e are permutable with operator $A^{(1)}_{\mu,\gamma}(z)$, i.e.,

$$P^o A^{(1)}_{\mu,\gamma}(z) = A^{(1)}_{\mu,\gamma}(z)P^o, \qquad P^e A^{(1)}_{\mu,\gamma}(z) = A^{(1)}_{\mu,\gamma}(z)P^e.$$

Denote by $A^{(1,o)}_{\mu,\gamma}(z)$ the operator restriction $A^{(1)}_{\mu,\gamma}(z)$ to subspace $L^o_2(\mathbb{T}^3)$. Then by definition of the operator $A^{(1,o)}_{\mu,\gamma}(z) = P^o A^{(1)}_{\mu,\gamma}(z)P^o = A^{(1)}_{\mu,\gamma}(z)P^o$ it follows that for any $\psi \in L_2(\mathbb{T}^3)$, it occurs that

$$(A^{(1,o)}_{\mu,\gamma}(z)\psi)(\mathbf{p}) = -\frac{\mu\gamma}{a^2(\gamma, z)\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p}, z)}} \sum_{i=1}^{3} \int\limits_{\mathbb{T}^3} \frac{\sin p_i \sin s_i\, \psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s}, z)}}.$$

By analogous reasoning, one can verify that the restriction $A^{(1,e)}_{\mu,\gamma}(z) = A_{\mu,\gamma}(z) - A^{(1,o)}_{\mu,\gamma}(z)$ of the operator $A_{\mu,\gamma}(z)$ to the subspace $L^e_2(\mathbb{T}^3)$ has the form:

$$(A^{(1,e)}_{\mu,\gamma}(z)\psi)(\mathbf{p}) = \frac{\mu}{a^2(\gamma, z)\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p}, z)}} \int\limits_{\mathbb{T}^3} \Bigg(\sum_{i=1}^{3} (\cos p_i + \cos s_i +$$

$$+\gamma\cos p_i\cos s_i - a(\gamma,z))\Big)\frac{\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}}.$$

Lemma 5.2. *For any $z>\tau_{\max,\gamma}(\mu,\mathbf{0})$, the operator $A^{(1,o)}_{\mu,\gamma}(z)$ is negative, that is*

$$\left(A^{(1,o)}_{\mu,\gamma}(z)\psi,\psi\right)\le 0 \quad \textit{for all} \quad \psi\in L_2^o(\mathbb{T}^3).$$

Proof. Indeed, for any $\psi\in L_2^o(\mathbb{T}^3)$, we have

$$\left(A^{(1,o)}_{\mu,\gamma}(z)\psi,\psi\right) = -\frac{\mu\gamma}{a^2(\gamma,z)}\int\limits_{(\mathbb{T}^3)^2}\frac{\left(\sum\limits_{i=1}^{3}\sin p_i\sin s_i\right)\psi(\mathbf{s})\overline{\psi(\mathbf{p})}d\mathbf{s}d\mathbf{p}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}} =$$

$$= -\frac{\mu\gamma}{a^2(\gamma,z)}\sum_{i=1}^{3}\int\limits_{\mathbb{T}^3}\frac{\sin s_i\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}}\int\limits_{\mathbb{T}^3}\frac{\overline{\sin p_i\psi(\mathbf{p})}d\mathbf{p}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}} =$$

$$= -\frac{\mu\gamma}{a^2(\gamma,z)}\sum_{i=1}^{3}\left|\int\limits_{\mathbb{T}^3}\frac{\sin p_i\psi(\mathbf{p})d\mathbf{p}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}}\right|^2\le 0.$$

□

Let Φ be the one-dimensional subspace spanned by a function

$$\varphi_0(\mathbf{p}) = \frac{c(z)}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}},$$

where $c(z)$ is the normalizing factor, that is

$$\frac{1}{c^2(z)} = \int\limits_{\mathbb{T}^3}\frac{d\mathbf{s}}{\Lambda_{\mu,\gamma}(\mathbf{s},z)}.$$

Denote by Q the subspace projection operator $L_2^e(\mathbb{T}^3)\ominus\Phi$.

Let $B_{\mu,\gamma}(z)$ be the operator restriction $A^{(1,e)}_{\mu,\gamma}(z)$ to the subspace $L_2^e(\mathbb{T}^3)\ominus\Phi$, that is

$$(Q\varphi)(\mathbf{p}) = \varphi(\mathbf{p}) - (\varphi,\varphi_0)\varphi_0(\mathbf{p}), \quad \varphi\in L_2^e(\mathbb{T}^3).$$

Now, using some calculations, we have

$$(B_{\mu,\gamma}(z)\psi)(\mathbf{p}) = (QA^{(1,e)}_{\mu,\gamma}Q\psi)(\mathbf{p}) = \frac{\mu\gamma}{a^2(\gamma,z)\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}}\sum_{i=1}^{3}\int\limits_{\mathbb{T}^3}\varphi_i(\mathbf{p})\varphi_i(\mathbf{s})\frac{\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}},$$

where

$$\varphi_i(p) = c^2(z)b_i(z) - \cos p_i \tag{5.4}$$

and

$$b_i(z) := b_i(\mu,\gamma,z) = \int\limits_{\mathbb{T}^3}\frac{\cos s_i d\mathbf{s}}{\Lambda_{\mu,\gamma}(\mathbf{s},z)}, \quad i=1,2,3.$$

Let

$$b_{ij}(z) := b_{ij}(\mu,\gamma,z) = \int\limits_{\mathbb{T}^3}\frac{\varphi_i(\mathbf{s})\varphi_j(\mathbf{s})d\mathbf{s}}{\Lambda_{\mu,\gamma}(\mathbf{s},z)}, \quad i,j=1,2,3, \tag{5.5}$$

where by functional invariance of $\Lambda_{\mu,\gamma}(\mathbf{p},z)$ regarding the permutation of variables p_i and p_j it follows that

$$b_{11}(z)=b_{22}(z)=b_{33}(z), \quad b_{12}(z)=b_{21}(z)=b_{23}(z)=b_{32}(z)=b_{13}(z)=b_{31}(z).$$

Lemma 5.3. *Let*

$$d(z) := d(\mu,\gamma,z) = \frac{\mu\gamma}{a^2(\gamma,z)}.$$

Then for sufficiently large and positive μ the number

$$\lambda_1(z) = d(z)\big(b_{11}(z)+2b_{12}(z)\big) \tag{5.6}$$

is simple and

$$\lambda_{2,3}(z) = d(z)\big(b_{11}(z)-b_{12}(z)\big) \tag{5.7}$$

is an eigenvalue with the multiplicity two of the operator $B_{\mu,\gamma}(z)$.

Proof. Suppose the equation

$$(B_{\mu,\gamma}(z)\psi)(\mathbf{p}) = \lambda\psi(\mathbf{p})$$

has a nonzero solution $\psi \in L_2^e(\mathbb{T}^3)$. From here,

$$\psi(\mathbf{p}) = \frac{d(z)}{\lambda\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}}\sum_{i=1}^{3} C_i\varphi_i(\mathbf{p}), \tag{5.8}$$

where

$$C_i = \int_{\mathbb{T}^3} \frac{\varphi_i(\mathbf{s})\psi(\mathbf{s})d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}}, \quad i = 1,2,3. \tag{5.9}$$

Substituting the right hand side of the equality (5.8) in (5.9), we obtain a system of equations for C_1, C_2 and C_3:

$$\begin{cases} (d(z)b_{11}(z) - \lambda)\, C_1 + d(z)b_{12}(z)C_2 + d(z)b_{12}(z)C_3 = 0 \\ \\ d(z)b_{12}(z)C_1 + (d(z)b_{11}(z) - \lambda)\, C_2 + d(z)b_{12}(z)C_3 = 0 \\ \\ d(z)b_{12}(z)C_1 + d(z)b_{12}(z)C_2 + (d(z)b_{11}(z) - \lambda)\, C_3 = 0 \end{cases}$$

Determinant $D(\lambda)$ of this system is a third degree polynomial with respect to λ.

Solving the equation $D(\lambda) = 0$, it makes sure that $\lambda_1(z)$ and $\lambda_2(z)$, defined by formulas (5.6) and (5.7), are simple and double zeros, respectively. After elementary calculations, we verify that

$$\psi_1(\mathbf{p}) = \frac{(\varphi_1(\mathbf{p}) + \varphi_2(\mathbf{p}) + \varphi_3(\mathbf{p}))C}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}},$$

is an eigenfunction corresponding to the eigenvalue $\lambda_1(z)$. General view of an element from the subspace of its own functions, corresponding to the double eigenvalue $\lambda_{2,3}(z)$, looks like

$$\psi_2(\mathbf{p}) = \frac{(\varphi_1(\mathbf{p}) - \varphi_3(\mathbf{p}))}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}}C_1 + \frac{(\varphi_2(\mathbf{p}) - \varphi_3(\mathbf{p}))}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}}C_2.$$

□

Lemma 5.4. *Assume that $\mu > 6(1+\gamma)$ and $z \geq \tau_{\max,\gamma}(\mu,\mathbf{0})$. Then the inequalities*

$$\frac{(z_{\mu,\gamma}(\mathbf{p}) - 6 - 6\gamma)(z - 12 - 6\gamma)}{\mu(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))} \leq \frac{1}{\Lambda_{\mu,\gamma}(\mathbf{p},z)} \leq \frac{z\cdot z_{\mu,\gamma}(\mathbf{p})}{\mu(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))} \tag{5.10}$$

hold, where $z_{\mu,\gamma}(\mathbf{p})$ is an eigenvalue of the two-particle operator $h_{\mu,\gamma}(\mathbf{p})$.

Moreover, we obtain the following asymptotics

$$\frac{1}{\Lambda_{\mu,\gamma}(\mathbf{p},\tau_{\max,\gamma}(\mu,\mathbf{0}))} = \frac{\mu}{\varepsilon(\mathbf{p})}\left(1 + O\left(\frac{1}{\mu}\right)\right) \tag{5.11}$$

as $\mu \to +\infty$.

Proof. For all $\mathbf{p} \in \mathbb{T}^3$, by Lemma (3.1), we establish

$$\mu\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}-\mathbf{s})} = \mu\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})} \equiv 1.$$

Observe that

$$\Lambda_{\mu,\gamma}(\mathbf{p},z) = 1 - \mu\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})} =$$

$$= \mu\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})} - \mu\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{z - \varepsilon(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})} =$$

$$= \mu(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p}))\int_{\mathbb{T}^3} \frac{d\mathbf{s}}{[z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})]\,[z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})]}. \tag{5.12}$$

Then, using the assertion $0 \leq \varepsilon(\mathbf{s}) \leq 6$, we get

$$\frac{1}{z_{\mu,\gamma}(\mathbf{p})} \leq \frac{1}{z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})} \leq \frac{1}{z_{\mu,\gamma}(\mathbf{p}) - 6 - 6\gamma}, \tag{5.13}$$

$$\frac{1}{z} \le \frac{1}{z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s})} \le \frac{1}{z-12-6\gamma}. \tag{5.14}$$

(5.10) follows directly from relations (5.13), (5.14) and (5.12).

Now, (5.11) can be obtained as in

$$\frac{1}{z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})} = \frac{1}{z_{\mu,\gamma}(\mathbf{p}) - 3 - 3\gamma}\left(1 - \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})}{z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s})}\right).$$

$$\frac{1}{z - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - \varepsilon(\mathbf{p})} = \frac{1}{z-6-3\gamma}\left(1 - \frac{\xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s}) + \xi(\mathbf{p})}{z - \varepsilon(\mathbf{s}) - \gamma\varepsilon(\mathbf{p}+\mathbf{s}) - \varepsilon(\mathbf{p})}\right).$$

If we take into account the inequalities

$$\mu + 3(1+\gamma) < z_{\mu,\gamma}(\boldsymbol{\pi}) \le z_{\mu,\gamma}(\mathbf{k}) \le z_{\mu,\gamma}(\mathbf{0}) < \mu + 3(1+\gamma) + \frac{9(1+\gamma)^2}{\mu} \le z \tag{5.15}$$

for sufficiently large $\mu > 0$, from (5.12), we have

$$\Lambda_{\mu,\gamma}(\mathbf{p}, \tau_{\max,\gamma}(\mu,\mathbf{0})) = \frac{\mu\left(z - z_{\mu,\gamma}(\mathbf{p}) - \varepsilon(\mathbf{p})\right)}{[z_{\mu,\gamma}(\mathbf{p}) - 3 - 3\gamma]\,[z-6-3\gamma]}\left[1 + O\left(\frac{1}{\mu}\right)\right], \mu \to \infty.$$

Hence, again using the relations (5.15), one can make sure it's true (5.11). □

Lemma 5.5. *Assume that $\gamma > 0$. Then there exists $\mu_\gamma > 0$ such that for any $\mu > \mu_\gamma$ satisfying*

$$\left\|A^{(2)}_{\mu,\gamma}(z)\right\| \le \frac{C}{\mu},$$

which is carried out uniformly $z \ge \tau_{\max,\gamma}(\mu,\mathbf{0})$, C is positive real number depending only on γ.

Proof. Let $\psi \in L_2(\mathbb{T}^3)$ and $\|\psi\| = 1$. Using the inequalities $\xi(\mathbf{p}) \le 3$ and $E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s}) \ge 0$, we get

$$\left|\left(A^{(2)}_{\mu,\gamma}(z)\psi,\psi\right)\right| \le \frac{\mu}{(z-6-3\gamma)^2}\int\limits_{\mathbb{T}^3}\int\limits_{\mathbb{T}^3}\frac{\left(\xi(\mathbf{p}) + \xi(\mathbf{s}) + \gamma\xi(\mathbf{p}+\mathbf{s})\right)^2|\psi(\mathbf{s})||\overline{\psi(\mathbf{p})}|d\mathbf{s}d\mathbf{p}}{(z - E_{\mathbf{0},\gamma}(\mathbf{p},\mathbf{s}))\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}} \le$$

$$\le \frac{\mu(6+3\gamma)^2}{(z-6-3\gamma)^3}\int\limits_{\mathbb{T}^3}\int\limits_{\mathbb{T}^3}\frac{|\psi(\mathbf{s})||\overline{\psi(\mathbf{p})}|d\mathbf{s}d\mathbf{p}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}\sqrt{\Lambda_{\mu,\gamma}(\mathbf{p},z)}} =$$

$$= \frac{\mu(6+3\gamma)^2}{(z-6-3\gamma)^3}\left(\int\limits_{\mathbb{T}^3}\frac{|\psi(\mathbf{s})|d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}}\right)^2. \tag{5.16}$$

Since $z > z_{\mu,\gamma}(\mathbf{p}) + 6 \ge \tau_{\max,\gamma}(\mu,\mathbf{0})$, considering (5.10), if $\mu > 6(1+\gamma)$, we get

$$\left(\int\limits_{\mathbb{T}^3}\frac{|\psi(\mathbf{s})|d\mathbf{s}}{\sqrt{\Lambda_{\mu,\gamma}(\mathbf{s},z)}}\right)^2 \le \left(\int\limits_{\mathbb{T}^3}\sqrt{\frac{z_{\mu,\gamma}(\mathbf{s})z}{\mu(z - z_{\mu,\gamma}(\mathbf{s}) - \varepsilon(\mathbf{s}))}}|\psi(\mathbf{s})|d\mathbf{s}\right)^2 \le$$

$$\le \frac{z^2}{\mu}\int\limits_{\mathbb{T}^3}|\psi(\mathbf{s})|^2 d\mathbf{s}\int\limits_{\mathbb{T}^3}\frac{d\mathbf{s}}{\varepsilon(\mathbf{s})}. \tag{5.17}$$

Since

$$\frac{z}{z-6-3\gamma} \le 2, \quad z \ge \mu + 3(1+\gamma),$$

from (5.16) and (5.17) at $\mu > 6(1+\gamma)$, we have

$$\left|\left(A^{(2)}_{\mu,\gamma}(z)\psi,\psi\right)\right| \le \frac{4\left(6+3\gamma\right)^2 W}{\mu\left(1 - \dfrac{3}{\mu}\right)} \le \frac{C_\gamma}{\mu},$$

where $C_\gamma = 8\left(6+3\gamma\right)^2 W$.

□

6. Proofs of the main results

The following Lemma plays an important role in the proof of the main results.

Lemma 6.1. *Assume that $\gamma > 0$. Then we obtain the following asymptotics:*

$$\lambda_1(\tau_{\max,\gamma}(\mu,\mathbf{0})) = \frac{\gamma}{\gamma_1} + O\Big(\frac{1}{\mu}\Big), \tag{6.1}$$

$$\lambda_{2,3}(\tau_{\max,\gamma}(\mu,\mathbf{0})) = \frac{\gamma}{\gamma_2} + O\Big(\frac{1}{\mu}\Big), \tag{6.2}$$

where γ_1 and γ_2 are defined by formula (2.2).

Proof. Let us prove equality (6.2). Taking into account equalities (5.4), (5.5), (5.11) and (5.15), we have

$$\lambda_{2,3}(\tau_{\max,\gamma}(\mu,\mathbf{0})) = \gamma\Bigg(\frac{\mu}{(z_{\mu,\gamma}(\mathbf{0})-6-3\gamma)^2}\int\limits_{\mathbb{T}^3}\frac{(\cos^2 s_1-\cos s_1\cos s_2)d\mathbf{s}}{\Lambda_{\mu,\gamma}(\mathbf{s},z_{\mu,\gamma}(\mathbf{0}))}\Bigg)+O\Big(\frac{1}{\mu}\Big)=$$

$$=\gamma\int\limits_{\mathbb{T}^3}\frac{(\cos^2 s_1-\cos s_1\cos s_2)d\mathbf{s}}{z_{\mu,\gamma}(\mathbf{0})-\varepsilon(\mathbf{s})-z_{\mu,\gamma}(\mathbf{s})}\Big(1+O\Big(\frac{1}{\mu}\Big)\Big)+O\Big(\frac{1}{\mu}\Big)=$$

$$=\gamma\int\limits_{\mathbb{T}^3}\frac{(\cos^2 s_1-\cos s_1\cos s_2)d\mathbf{s}}{\varepsilon(\mathbf{s})}+O\Big(\frac{1}{\mu}\Big)=\frac{\gamma}{\gamma_2}+O\Big(\frac{1}{\mu}\Big).$$

□

Proof of Theorem 2.1 1. $i)$ Assume that $\gamma\in(0,\gamma_1)$. Then applying Lemma 5.5 and using (5.2), we obtain that there exists $\mu_\gamma>0$ such that for any $\mu>\mu_\gamma$, the operators $A_{\mu,\gamma}(z)$ and $A^{(0)}_{\mu,\gamma}(z)$ have the same number of eigenvalues greater than 1. From Lemma 5.1 and Lemma 5.2, we obtain

$$n\left[1,A^{(1)}_{\mu,\gamma}(z)\right]=n\left[1,A^{(1,o)}_{\mu,\gamma}(z)\right]+n\left[1,A^{(1,e)}_{\mu,\gamma}(z)\right]=n\left[1,A^{(0,e)}_{\mu,\gamma}(z)\right].$$

From the statement of Lemma 5.3, one can conclude that the operator $A^{(1,e)}_{\mu,\gamma}(\tau_{\max,\gamma}(\mu,\mathbf{0}))$ have three eigenvalues $\lambda_1(z),\lambda_{2,3}(z)$ taking into account the multiplicity. Since $0<\gamma<\gamma_1$, the inequalities $\lambda_1(\tau_{\max,\gamma}(\mu,\mathbf{0}))<1$, $\lambda_{2,3}(\tau_{\max,\gamma}(\mu,\mathbf{0}))<1$ are valid for sufficiently large $\mu>0$. By the Birman–Schwinger principle (see Lemma 4.3) the operator $H_{\mu,\gamma}(\mathbf{0})$ has no eigenvalues $z>\tau_{\max,\gamma}(\mu,\mathbf{0})$.

The statements $ii)$ and $iii)$ can be proven similarly.

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