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Original article

On solutions to nonlinear integral equation of the Hammerstein type and its applications to Gibbs measures for continuous spin systems

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ABSTRACT The paper deals with the problem of constructing kernels of Hammerstein-type equations whose positive solutions are not unique. This problem arises from the theory of Gibbs measures, and each positive solution of the equation corresponds to one translation-invariant Gibbs measure. Also, the problem of finding kernels for which the number of positive solutions to the equation is greater than one is equivalent to the problem of finding models which has phase transition. In these articles, the number of solutions corresponding to the constructed kernels does not exceed 3, and in turn, it only gives us a chance to check the existence of phase transitions. The constructed kernels in this paper are more general than the kernels in the above-mentioned papers and except for checking phase transitions, it allows us to classify the set of Gibbs measures.

KEYWORDS Generalized SOS model, spin values, Cayley tree, gradient Gibbs measure, periodic boundary law

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1. Introduction

The Hammerstein equation covers a large variety of areas and is of much interest to a wide audience due to the fact that it has applications in numerous areas. Several problems arise in differential equations (ordinary and partial), for instance, elliptic boundary value problems whose linear parts possess Green's function can be transformed into the Hammerstein integral equations. Equations of the Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see e.g., [1]).

The Hammerstein equation is a mathematical model that describes the response of a system to an input signal. It is commonly used in control systems engineering and signal processing.

Let $G(u(t))$ represent the static nonlinearity component of the system, which is a function of the input signal, and $H(u(t))$ represent the dynamic linear component of the system, which is a linear function of the input signal. Then the equation is typically represented as: $y(t) = G(u(t)) + H(u(t))$, where $y(t)$ and $u(t)$ represent the output and input signal to the system at time t.

The Hammerstein equation allows one modeling of systems that exhibit both linear and nonlinear behavior, making it useful in various applications.

In statistical mechanics, the Gibbs measure is a probability distribution that describes the equilibrium state of a system. It is commonly used to describe systems with countable spin values, where the spins can take on a finite or countably infinite number of discrete values. However, there are also models with uncountable spin values, where the spins can take uncountably many values, such as in continuous spin systems. In these cases, the Gibbs measure needs to be adapted to handle the uncountable nature of the spin values.

The definition of the Gibbs measure for models with uncountable spin values often involves considering the Hamiltonian or energy function that governs the system. The Gibbs measure assigns a probability to each spin configuration based on its energy, such that spin configurations with lower energy have higher probabilities.

Specifically, for a given spin configuration, the probability assigned by the Gibbs measure is proportional to the exponential of the negative energy of that configuration. The proportionality constant is chosen such that the total probability assigned by the Gibbs measure sums to 1 over all possible spin configurations. Note that all the above-mentioned and additional information can be found in References [2–4].

In recent years, increasing attention was given to models with a *uncountable* many spin values on particle systems (especially, the Cayley tree). By this time, models with four interactions with the set of spin values $[0,1]$ were considered, and the following results were achieved: In [5] splitting Gibbs measures on the Cayley tree of order k are described by solutions to a special nonlinear integral equation of Hammerstain's type. The existence of positive solutions to the nonlinear integral equation of Hammerstain's type and a sufficient condition of the uniqueness of the equation given in [6]. Also, in [7–10], results on non-uniqueness of positive solutions to the equation with non-degenerate kernels are given and for the case of degenerate kernel can be seen in [11,12]. Finally, in [13,14] translation-invariant Gibbs measures for models with four competing interactions are described by solutions to the Lyapunov integral equation. In this paper, we construct more general kernels which gives us a chance to check the existence of phase transitions and to classify the set of Gibbs measures.

2. Special equation of Hammerstein type

Let Ω be a set. A nonlinear integral equation of Hammerstein type on Ω is one of the form

$$u(x)+\int_{\Omega}K(x,y)f(l(y))dy=h(x) \tag{2.1}$$

where dy stands for a $\sigma-$ finite measure on the measure space Ω; the mappings l and K are measurable on Ω and $\Omega\times\Omega$ respectively, f is a real-valued function defined on $\mathbb{R}$ and is in general nonlinear and h is given function on Ω. If we define the operator A by

$$Av(x)=\int_{\Omega}K(x,y)v(y)dy, \tag{2.2}$$

and N_f to be the Nemystkii operator associated with f :

$$\tilde{N}_f l(x)=f(l(x)) \tag{2.3}$$

then integral equation (2.1) can be transformed to functional equation form as follows:

$$l+A\tilde{N}_f l=0, \tag{2.4}$$

where without the loss of generality, we have taken $h\equiv 0$ (see detail in [15]).

We consider a special equation of Hammerstein type, i.e.

$$\int_0^1 K(x,y)l^k(y)dy=l(x),\ k\in\mathbb{N}. \tag{2.5}$$

Here and below, $K:[0,1]^2\to(0,+\infty)$.

Actually, the last equation is very sufficient in the Gibbs measure theory. In the present paper, we give short information about relation between the last equation and the Gibbs measures. Detailed information can be found in references [5,8].

Let $\Gamma^k=(V,L)$ be a Cayley tree and we consider models where the spin takes values in the set $[0,1]$, and is assigned to the vertices of the tree. For $A\subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A:A\to[0,1]$. For $J\in R\setminus\{0\}$ and $\xi:(u,v)\in[0,1]^2\to\xi_{uv}\in R$ we define a Hamiltonian

$$H(\sigma)=-J\sum_{\langle x,y\rangle\in L}\xi_{\sigma(x)\sigma(y)}, \tag{2.6}$$

which is the generalization of some classic models such as Ising, Potts, ... Here and below, the notation $\langle x,y\rangle$ means the nearest neighbor vertices.

Let $h(t,x):=h_{t,x},(t,x)\in[0,1]\times V$ be real mapping of $x\in V\setminus\{x^0\}$, where x^0 is a root of the tree. Now, we define the family of probability measures $\{\mu^{(n)}\}_{n\in\mathbb{N}}$ on Ω_{V_n}, i.e.,

$$\mu^{(n)}(\sigma_n)=Z_n^{-1}\exp\left(-\beta H(\sigma_n)+\sum_{x\in W_n}h_{\sigma(x),x}\right), \tag{2.7}$$

Here, as before, $\sigma_n:x\in V_n\mapsto\sigma(x)$ and Z_n is the partition function.

If the family of probability measures $\{\mu^{(n)}\}_{n\in\mathbb{N}}$ is compatible then by Kolmogorov's extension theorem, there exists a unique measure μ on Ω_V such that $\mu\left(\{\sigma|_{V_n}=\sigma_n\}\right)=\mu^{(n)}(\sigma_n)$ for all $n\in\mathbb{N}$. The measure μ is called *splitting Gibbs measure* corresponding to Hamiltonian (2.6)

The following result describes a new condition which is equivalent to the condition of compatibility.

Proposition 1. *[5] The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (2.7) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{tu}) f(u, y) du}{\int_0^1 \exp(J\beta\xi_{0u}) f(u, y) du}. \tag{2.8}$$

Here, and below $f(t, x) = \exp(h_{t,x} - h_{0,x}),\ t \in [0, 1]$ and $du = \lambda(du)$ is the Lebesgue measure.

Let $f(t, x)$ do not depend on the vertices of the Cayley tree. Then the last equation (2.8) has strongly positive solution if and only if the equation (2.5) has strongly positive solution in $\mathcal{M}_0 = \{f \in C^+[0, 1] : f(0) = 1\}$, where $C^+[0, 1]$ is the set of all positive continuous functions on $[0, 1]$ (see [6]).

Let $f\left(\frac{1}{3}\right) = g\left(\frac{2}{3}\right) = c$ and denote that

$$\varphi_1(t) = \begin{cases} f(t) & \text{if } t \in \left[0, \frac{1}{3}\right] \\ c & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ g(t) & \text{if } t \in \left[\frac{2}{3}, 1\right] \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} g(1-t) & \text{if } t \in \left[0, \frac{1}{3}\right] \\ c & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ f(1-t) & \text{if } t \in \left[\frac{2}{3}, 1\right] \end{cases}.$$

Also, for $f_1\left(\frac{1}{3}\right) = g_1\left(\frac{2}{3}\right) = c_1$, we define functions:

$$\psi_1(u) = \begin{cases} f_1(u) & \text{if } u \in \left[0, \frac{1}{3}\right] \\ c_1 & \text{if } u \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ g_1(u) & \text{if } u \in \left[\frac{2}{3}, 1\right] \end{cases} \quad \text{and} \quad \psi_2(u) = \begin{cases} g_1(1-u), & \text{if } u \in \left[0, \frac{1}{3}\right] \\ c_1, & \text{if } u \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ f_1(1-u), & \text{if } u \in \left[\frac{2}{3}, 1\right] \end{cases}.$$

By using these functions, we define a degenerate kernel:

$$\tilde{K}(t, u) = \varphi_1(t)\psi_1(u) + \varphi_2(t)\psi_2(u). \tag{2.9}$$

Then equation (2.5) can be written as

$$f(t) = \int_0^1 (\varphi_1(t)\psi_1(u) + \varphi_2(t)\psi_2(u)) f^k(u) du. \tag{2.10}$$

Namely,

$$f(t) = \varphi_1(t) \int_0^1 \psi_1(u) f^k(u) d(u) + \varphi_2(t) \int_0^1 \psi_2(u) f^k(u) d(u) = f(t).$$

Put

$$C_1 = \int_0^1 \psi_1(u) f^k(u) d(u), \qquad C_2 = \int_0^1 \psi_2(u) f^k(u) du.$$

Consequently, by taking into account $f(t) = C_1\varphi_1(t) + C_2\varphi_2(t)$, we obtain

$$C_i = \int_0^1 \psi_i(u)(C_1\varphi_1(u) + C_2\varphi_2(u))^k du, \quad i \in \{1, 2\}.$$

The last equality is equivalent to

$$\begin{cases} C_1 = \begin{pmatrix} k \\ 0 \end{pmatrix} C_1^k \alpha_1 + \begin{pmatrix} k \\ 1 \end{pmatrix} C_1^{k-1} C_2 \alpha_2 \quad + \ldots. \quad + \begin{pmatrix} k \\ k \end{pmatrix} C_2^k \alpha_{k+1} \\ C_2 = \begin{pmatrix} k \\ 0 \end{pmatrix} C_2^k \alpha_1 + \begin{pmatrix} k \\ 1 \end{pmatrix} C_2^{k-1} C_1 \alpha_2 \quad + \ldots. \quad + \begin{pmatrix} k \\ k \end{pmatrix} C_1^k \alpha_{k+1}. \end{cases} \tag{2.11}$$

Here and below,

$$\alpha_1=\int_0^1\psi_1(u)\varphi_1^k(u)du,\quad \alpha_2=\int_0^1\psi_1(u)\varphi_1^{k-1}(u)\varphi_2(u)du,\ ...\ \alpha_{k+1}=\int_0^1\psi_1(u)\varphi_2^{k+1}(u)du,$$

$$\beta_1=\int_0^1\psi_2(u)\varphi_1^k(u)du,\quad \beta_2=\int_0^1\psi_2(u)\varphi_1^{k-1}(u)\varphi_2(u)du,\ ...\ \beta_{k+1}=\int_0^1\psi_2(u)\varphi_2^{k+1}(u)du. \tag{2.12}$$

Let $x=\dfrac{C_1}{C_2}$, then the system of equations (2.11) can be written as

$$x=\frac{\begin{pmatrix}k\\0\end{pmatrix}\alpha_1x^k+\begin{pmatrix}k\\1\end{pmatrix}\alpha_2x^{k-1}+\begin{pmatrix}k\\2\end{pmatrix}\alpha_3x^{k-2}+\ ...\ +\begin{pmatrix}k\\k-1\end{pmatrix}\alpha_kx+\begin{pmatrix}k\\k\end{pmatrix}\alpha_{k+1}}{\begin{pmatrix}k\\0\end{pmatrix}\alpha_{k+1}x^k+\begin{pmatrix}k\\1\end{pmatrix}\alpha_kx^{k-1}+\begin{pmatrix}k\\2\end{pmatrix}\alpha_{k-1}x^{k-2}+\ ...\ +\begin{pmatrix}k\\k-1\end{pmatrix}\alpha_2x+\begin{pmatrix}k\\k\end{pmatrix}\alpha_1}.$$

Namely,

$$Q_k(x):=\begin{pmatrix}k\\0\end{pmatrix}\alpha_{k+1}x^{k+1}+\left(\begin{pmatrix}k\\1\end{pmatrix}\alpha_k-\begin{pmatrix}k\\0\end{pmatrix}\alpha_1\right)x^k+\ ...$$

$$...+\left(\begin{pmatrix}k\\k\end{pmatrix}\alpha_1-\begin{pmatrix}k\\k-1\end{pmatrix}\alpha_k\right)x-\begin{pmatrix}k\\k\end{pmatrix}\alpha_{k+1}. \tag{2.13}$$

Now, we can conclude the following result:

Proposition 2. *Finding positive solutions of equation (2.5) with the kernel (2.9) is equivalent to finding positive roots of the polynomial $Q_k(x)$.*

In the next sections, we consider the positive roots of the polynomial $Q_k(x)$ in special cases.

3. Solutions to the equation (2.5) with the kernel (2.9) for the case $k=2$.

In this section, we study positive solutions to equation (2.5) with the kernel (2.9) for the case $k=2$.
Let $k=2$ then the system of equations (2.11) can be rewritten as

$$\begin{cases} C_1=C_1^2\alpha_1+2C_1C_2\alpha_2+C_2^2\alpha_3\\ C_2=C_1^2\beta_1+2C_1C_2\beta_2+C_2^2\beta_3. \end{cases} \tag{3.1}$$

By (2.12) and the construction of kernel one obtains:

$$\alpha_1=\int_0^1\psi_1(u)\varphi_1^2(u)d(u)=\int_0^{\frac13}f_1(u)f^2(u)d(u)+\int_{\frac13}^{\frac23}c_1c^2d(u)+\int_{\frac23}^1g_1(u)g^2(u)du=$$

$$=\int_0^{\frac13}g_1(1-u)g^2(1-u)d(u)+\int_{\frac13}^{\frac23}c_1c^2d(u)+\int_{\frac23}^1f_1(1-u)f^2(1-u)du=\int_0^1\psi_2(u)\varphi_2^2(u)d(u)=\beta_3.$$

Similarly,

$$\alpha_2=\int_0^1\psi_1(u)\varphi_1(u)\varphi_2(u)d(u)=\int_0^{\frac13}f_1(u)f(u)g(1-u)d(u)+\int_{\frac13}^{\frac23}c_1c^2d(u)+\int_{\frac23}^1g_1(u)g(u)f(1-u)du=$$

$$=\int_0^{\frac13}g_1(1-u)f(u)g(1-u)d(u)+\int_{\frac13}^{\frac23}c_1c^2d(u)+\int_{\frac23}^1f_1(1-u)g(u)f(1-u)d(u)=\int_0^1\psi_2(u)\varphi_1(u)\varphi_2(u)d(u)=\beta_2$$

and

$$\alpha_3=\int\limits_0^1\psi_1(u)\varphi_2^2(u)d(u)=\int\limits_0^{\frac13}f_1(u)g^2(1-u)d(u)+\int\limits_{\frac13}^{\frac23}c_1c^2d(u)+\int\limits_{\frac23}^1 g_1(u)f^2(1-u)d(u)=$$

$$=\int\limits_0^{\frac13}g_1(1-u)f^2(u)d(u)+\int\limits_{\frac13}^{\frac23}c_1c^2d(u)+\int\limits_{\frac23}^1 f_1(1-u)g^2(u)d(u)=\int\limits_0^1\psi_2(u)\varphi_1^2(u)d(u)=\beta_1.$$

Hence,

$$\alpha_1=\beta_3,\quad \alpha_2=\beta_2,\quad \alpha_3=\beta_1. \tag{3.2}$$

By taking into account $x=\dfrac{C_1}{C_2}$ and the equality (2.13), we have

$$Q_2(x)=\alpha_3x^3+(2\alpha_2-\alpha_1)x^2+(\alpha_1-2\alpha_2)x-\alpha_3=0.$$

Thus, we obtain

$$Q_2(x)=(x-1)(a_3x^2+(a_3+2a_2-a_1)x+a_3)=0.$$

Put

$$D:=(2a_2-a_1-a_3)(2a_2-a_1+3a_3).$$

It is easy to check $a_1+a_3\geq 2a_2$. Therefore, the sign of D is the same as the sign of the expression $a_1-2a_2-3a_3$.

Proposition 3. *Let $k=2$ and $\tilde{K}(t,u)$ be the kernel which defined in (2.9). Then the following statements hold:*

(1) *If $a_1<2a_2+3a_3$ then there is unique positive solution of (2.10);*

(2) *If $a_1=2a_2+3a_3$ then there are exactly two positive solutions of (2.10);*

(3) *If $a_1>2a_2+3a_3$ then there are exactly three positive solutions of (2.10).*

In the language of Gibbs measure theory, we obtain:

Theorem 1. *Let $k=2$ and $\tilde{K}(t,u)$ be the function of the Hamiltonian (2.6). Then the following assignments hold:*

(1) *If $a_1<2a_2+3a_3$ then there is unique translation-invariant Gibbs measure of the model (2.6);*

(2) *If $a_1=2a_2+3a_3$ then there are exactly two translation-invariant Gibbs measures of the model (2.6);*

(3) *If $a_1>2a_2+3a_3$ then there are exactly three translation-invariant Gibbs measures of the model (2.6).*

4. Solutions to the equation (2.5) with the kernel (2.9) for the case $k=3$.

Now, we consider positive solutions to equation (2.5) with the kernel (2.9) for the case $k=3$. Then the system of equations (2.11) can be written as

$$\begin{cases} C_1=C_1^3\alpha_1+3C_1^2C_2\alpha_2+3C_1C_2^2\alpha_3+C_2^3\alpha_4\\[2mm] C_2=C_1^3\beta_1+3C_1^2C_2\beta_2+3C_1C_2^2\beta_3+C_2^3\beta_4\end{cases} \tag{4.1}$$

and

$$\alpha_1=\int\limits_0^1\psi_1(u)\varphi_1^3(u)d(u)=\int\limits_0^{\frac13}f_1(u)f^3(u)du+\int\limits_{\frac13}^{\frac23}c_1c^3du+\int\limits_{\frac23}^1 g_1(u)g^3(u)du=\beta_4.$$

Analogously,

$$\alpha_2=\int\limits_0^1\psi_1(u)\varphi_1^2(u)\varphi_2(u)du=\int\limits_0^{\frac13}f_1(u)f^2(u)g(1-u)du+\int\limits_{\frac13}^{\frac23}c_1c^3du+\int\limits_{\frac23}^1 g_1(u)g^2(u)f(1-u)du=\beta_3,$$

$$\alpha_3=\int\limits_0^1\psi_1(u)\varphi_1(u)\varphi_2^2(u)du=\int\limits_0^{\frac13}f_1(u)f(u)g^2(1-u)du+\int\limits_{\frac13}^{\frac23}c_1c^3du+\int\limits_{\frac23}^1 g_1(u)g(u)f^2(1-u)du=\beta_2,$$

$$\alpha_4=\int\limits_0^1\psi_1(u)\varphi_2^3(u)du=\int\limits_0^{\frac13}f_1(u)g^3(1-u)du+\int\limits_{\frac13}^{\frac23}c_1c^3du+\int\limits_{\frac23}^1 g_1(u)f^3(1-u)du=\beta_1.$$

As a result, one obtains

$$\alpha_1 = \beta_4, \quad \alpha_2 = \beta_3, \quad \alpha_3 = \beta_2, \quad \alpha_4 = \beta_1. \tag{4.2}$$

Consequently, the polynomial, which is defined in (2.13), is

$$Q_3(x) = \alpha_4 x^4 + (3\alpha_3 - \alpha_1)x^3 + (\alpha_1 - 3\alpha_3)x - \alpha_4 = 0.$$

Namely,

$$Q_3(x) = (x-1)(x+1)\big(\alpha_4 x^2 + (3\alpha_3 - \alpha_1)x + \alpha_4\big) = 0.$$

The discriminant of the polynomial $\alpha_4 x^2 + (3\alpha_3 - \alpha_1)x + \alpha_4$ is

$$D := (3\alpha_3 - \alpha_1 - 2\alpha_4)(3\alpha_3 - \alpha_1 + 2\alpha_4).$$

Proposition 4. *Let $k = 3$ and $\tilde{K}(t,u)$ be the kernel which defined in (2.9). Then the following statements hold:*

(1) *If $|3\alpha_3 - \alpha_1| > 2\alpha_4$ then there is unique positive solution of (2.10);*

(2) *If $|3\alpha_3 - \alpha_1| = 2\alpha_4$ then there are exactly two positive solutions of (2.10);*

(3) *If $|3\alpha_3 - \alpha_1| < 2\alpha_4$ then there are exactly three positive solutions of (2.10).*

In the language of Gibbs measure theory, we can conclude the following theorem.

Theorem 2. *Put $k = 3$ and $\tilde{K}(t,u)$ is the function of the Hamiltonian (2.6). Then the following assignments hold:*

(1) *If $|3\alpha_3 - \alpha_1| > 2\alpha_4$ then there is unique translation-invariant Gibbs measure of the model (2.6);*

(2) *If $|3\alpha_3 - \alpha_1| = 2\alpha_4$ then there are exactly two translation-invariant Gibbs measures of the model (2.6);*

(3) *If $|3\alpha_3 - \alpha_1| < 2\alpha_4$ then there are exactly three translation-invariant Gibbs measures of the model (2.6).*

5. Solutions to the equation (2.5) with the kernel (2.9) for the case $k = 4$.

In the Gibbs measure theory, it is more interesting to construct Hamiltonian which has more than three translation-invariant Gibbs measures. In this section, we show that the Hamiltonian which corresponds to our kernel has more than three Gibbs measures. For the case $k = 4$, the system of equations (2.11) can be written as

$$\begin{cases} C_1 = C_1^4\alpha_1 + 4C_1^3C_2\alpha_2 + 6C_1^2C_2^2\alpha_3 + 4C_1C_2^3\alpha_4 + C_2^4\alpha_5 \\[2mm] C_2 = C_1^4 + 4C_1^3C_2\beta_2 + 6C_1^2C_2^2\beta_3 + 4C_1C_2^3\beta_4 + C_2^4\beta_5 \end{cases}$$

and

$$\alpha_1 = \int_0^1 \psi_1(u)\varphi_1^4(u)du = \int_0^{\frac{1}{3}} f_1(u)f^4(u)du + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^4du + \int_{\frac{2}{3}}^1 g_1(u)g^4(u)du = \beta_5,$$

$$\alpha_2 = \int_0^1 \psi_1(u)\varphi_1^3(u)\varphi_2(u)du = \int_0^{\frac{1}{3}} f_1(u)f^3(u)g(1-u)du + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^4du + \int_{\frac{2}{3}}^1 g_1(u)g^3(u)f(1-u)du = \beta_4,$$

$$\alpha_3 = \int_0^1 \psi_1(u)\varphi_1^2(u)\varphi_2^2(u)du = \int_0^{\frac{1}{3}} f_1(u)f^2(u)g^2(1-u)du + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^4du + \int_{\frac{2}{3}}^1 g_1(u)g^2(u)f^2(1-u)du = \beta_3,$$

$$\alpha_4 = \int_0^1 \psi_1(u)\varphi_1(u)\varphi_2^3(u)du = \int_0^{\frac{1}{3}} f_1(u)f(u)g^3(1-u)du + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^4du + \int_{\frac{2}{3}}^1 g_1(u)g(u)f^3(1-u)du = \beta_2,$$

$$\alpha_5 = \int_0^1 \psi_1(u)\varphi_2^4(u)du = \int_0^{\frac{1}{3}} f_1(u)g^4(1-u)du + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^4du + \int_{\frac{2}{3}}^1 g_1(u)f^4(1-u)du = \beta_1.$$

Hence,

$$\alpha_1 = \beta_5, \quad \alpha_2 = \beta_4, \quad \alpha_3 = \beta_3, \quad \alpha_4 = \beta_2, \quad \alpha_5 = \beta_1. \tag{5.1}$$

By taking into account $x = \dfrac{C_1}{C_2}$ and the equality (5.1), one obtains

$$Q_4(x) = \alpha_5 x^5 + (4\alpha_4 - \alpha_1)x^4 + (6\alpha_3 - 4\alpha_2)x^3 + (4\alpha_2 - 6\alpha_3)x^2 + (\alpha_1 - 4\alpha_4)x - \alpha_5 = 0.$$

Namely,

$$Q_4(x) = (x-1)\big(\alpha_5 x^4 + (\alpha_5 + 4\alpha_4 - \alpha_1)x^3 + (\alpha_5 + 4\alpha_4 - \alpha_1 + 6\alpha_3 - 4\alpha_2)x^2 + (\alpha_5 + 4\alpha_4 - \alpha_1)x + \alpha_5\big) = 0.$$

After short calculations, we have (for $x \neq 1$)

$$\frac{Q_4(x)}{x^2(x-1)} = \alpha_5 y^2 + (\alpha_5 + 4\alpha_4 - \alpha_1)y + (4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2),$$

where $y = x + \dfrac{1}{x}$. The discriminant of the right hand side of the last equality is

$$D = (\alpha_5 + 4\alpha_4 - \alpha_1)^2 - 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2).$$

Thus, we can give the following result:

Proposition 5. *Let $k = 4$ and $\tilde{K}(t,u)$ be the kernel which defined in (2.9). Then the following statements hold:*

(1) *If $(\alpha_5 + 4\alpha_4 - \alpha_1)^2 < 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2)$ then there is unique positive solution of (2.10);*

(2) *If $(\alpha_5 + 4\alpha_4 - \alpha_1)^2 = 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2)$ then there are at most three positive solutions of (2.10);*

(3) *If $(\alpha_5 + 4\alpha_4 - \alpha_1)^2 > 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2)$ then there are at most five positive solutions of (2.10).*

In the Gibbs measure theory, we obtain the following theorem.

Theorem 3. *Put $k = 4$ and let $\tilde{K}(t,u)$ be the function of the Hamiltonian (2.6). Then the following assignments hold:*

(1) *If $(\alpha_5 + 4\alpha_4 - \alpha_1)^2 < 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2)$ then there is unique translation-invariant Gibbs measure of the model (2.6);*

(2) *If $(\alpha_5 + 4\alpha_4 - \alpha_1)^2 = 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2)$ then there are at most three translation-invariant Gibbs measures of the model (2.6);*

(3) *If $(\alpha_5 + 4\alpha_4 - \alpha_1)^2 > 4\alpha_5(4\alpha_4 - \alpha_5 - \alpha_1 + 6\alpha_3 - 4\alpha_2)$ then there are at most five translation-invariant Gibbs measures of the model (2.6).*

6. Conclusion

The central concept in understanding phase transitions on the Cayley trees is the notion of cluster decomposition. In the absence of phase transitions, there is a unique Gibbs measure that describes the equilibrium behavior of the system. However, in the presence of a phase transition, multiple Gibbs measures can coexist. The existence of multiple Gibbs measures is often associated with the breaking of symmetry in the system. This symmetry breaking can manifest itself in different ways, such as the appearance of multiple stable states or the coexistence of different phases (e.g. [3]).

Mathematically, the presence of multiple Gibbs measures can be established by studying the behavior of the system under different boundary conditions or by considering the behavior of relevant observables. The occurrence of non-uniqueness in the limit of large system size indicates the existence of phase transitions and the coexistence of multiple Gibbs measures. Also, the properties of the different Gibbs measures can also provide insights into the nature of the phase transition. For example, the entropy of the Gibbs measures can exhibit non-analytic behavior or the correlation length may diverge at the critical point (see [2] and [4]).

In short, the existence of multiple Gibbs measures is closely related to the presence of phase transitions on the Cayley trees. These phase transitions are characterized by breaking of symmetry and the coexistence of different equilibrium states. The identification and characterization of different Gibbs measures play a crucial role in understanding the critical phenomenon and the behavior of the system near the phase transition point. In this paper, we showed that under certain conditions (see Theorems 1,2,3) there exist multiple Gibbs measures of the model (2.6) (this model is a generalization of Ising, Potts, SOS, ...) on the Cayley trees of order two, three and four.

Now, we will give short information about the novelty of our paper. Firstly, in [5], authors consider the model (2.6) on the Cayley trees and prove that there is not any phase transition on the Cayley tree of order $k = 1$. For the case $k \geq 2$, models in which phase transitions exist are constructed in [8] and the generalization of the constructed model is studied in [7, 9]. In [6], the problem of finding translation-invariant (periodic with period two) Gibbs measures of the model (2.6) was reduced to finding positive fixed points of the nonlinear operator of Hammerstein type and the problem of the existence of fixed points of this operator considered in [10, 15, 16] ([13, 14]). But, from the Gibbs measure theory, it is interesting to study fixed points of the nonlinear operator of the Hammerstein type with degenerate kernel and all the above works corresponding to non-degenerate kernels. Then works for degenerate kernels considered in [11] and [12] but in these papers, the fixed point of the operator with the degenerate kernel does not correspond to any Gibbs measure. In the present paper, each fixed point of the operator with a degenerate kernel is corresponding to one Gibbs measure.

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