NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Kucharov R.R., et al. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (1), 31–36.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2024-15-1-31-36

Original article

Non-compact perturbation of the spectrum of multipliers given by a special form

Ramziddin R. Kucharov[1,2,a,b], Tillohon M. Tuxtamurodova[2,c]

[1]Tashkent International University of Financial Management and Technology, Tashkent, Uzbekistan
[2]National University of Uzbekistan, Tashkent, Uzbekistan

[a]r.kucharov@tift.uz, [b]ramz3364647@yahoo.com, [c]mirzayevatilloxon13@gmail.com

Corresponding author: R. R. Kucharov, r.kucharov@tift.uz, ramz3364647@yahoo.com

ABSTRACT In this paper, the change of the spectrum of multiplier $H_0 f(x,y) = k_0(x,y) f(x,y)$ for perturbation with non-compact partially integral operators is studied. In addition, the existence of resonance is investigated in the model $H = H_0 - (\gamma_1 T_1 + \gamma_2 T_2)$.

KEYWORDS essential spectrum, discrete spectrum, lower bound of the essential spectrum, non-compact partial integral operator, resonance with zero energy.

FOR CITATION Kucharov R.R., Tuxtamurodova T.M. Non-compact perturbation of the spectrum of multipliers given by a special form. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (1), 31–36.

1. Introduction

Self-adjoint partial integral operators appear in the theory of discrete Schrödinger operators. The study of the theory of elasticity [1], continuum mechanics [2–4], aerodynamics [5] and other problems leads to the problem of spectral analysis of the partial integral operators. In 1969, Uchiyama [6,7] obtained the first results on the finiteness of the discrete spectrum of N-particle Hamiltonians with $N > 2$. He found sufficient conditions for the finiteness of the number of discrete eigenvalues for the energy operators. In 1971, Zhislin [8] assumed the total charge of the system to be less than -1 and proved that the discrete spectrum of the energy operators is finite in the symmetry spaces of negative atomic ions of molecules with any mass of nucleus and infinitely heavy nuclei.

Let $\mathcal{H}$ be a separable Hilbert space and the operator $H_0 : \mathcal{H} \to \mathcal{H}$ be self-adjoint and have only essential spectrum ($\sigma(H_0) = \sigma_{ess}(H_0)$), i.e. the operator H_0 lacks the discrete spectrum ($\sigma_{disc}(H_0) = \emptyset$). Let's assume that the operator H_0 is perturbed by the self-adjoint operator T, i.e. consider the operator $H_0 + \varepsilon T$, $\varepsilon > 0$. The main questions in the theory of perturbation of spectra are as follows:

1) How is the structure of the spectrum of the operator $H_0 + \varepsilon T$ related to the spectrum of the original (unperturbed) operator H_0?

2) What are the properties of the spectrum as a function of $\varepsilon > 0$?

Let H_0 be a multiplier in $L_2(\Omega)$ ($\Omega \subset \mathbb{R}^m$ – compact): $H_0 f(x) = u(x) f(x)$, where $u(x)$ is a given real valued continuous function on Ω, $T : L_2(\Omega) \to L_2(\Omega)$ is a linear self adjoint compact operator. The operator $H_0 + \varepsilon T$, $\varepsilon > 0$, is an operator in the Friedrichs model. It is known for such an operator that $\sigma_{ess}(H_0 + \varepsilon T) = \sigma(H_0)$ [9]. In addition, a number of methods have been developed [10–12] to investigate the existence of an eigenvalue in the discrete spectrum $\sigma_{disc}(H_0 + \varepsilon T)$ and to study the finiteness (infiniteness) of the discrete spectrum $\sigma_{disc}(H_0 + \varepsilon T)$. If the operator T is non-compact, then there is no general way to study the spectrum of the perturbed operator $H_0 + \varepsilon T$. In [13,14] , a method is proposed for studying the spectrum of the operator $H_0 + \varepsilon T : L_2(\Omega_1 \times \Omega_2) \to L_2(\Omega_1 \times \Omega_2)$ ($\Omega_1 \subset \mathbb{R}^{m_1}$, $\Omega_2 \subset \mathbb{R}^{m_2}$ are nonempty compact sets), when H_0 is a multiplier defined by a continuous function $k_0(x,y)$ on $\Omega_1 \times \Omega_2$ and $T = T_1 + T_2$ is a linear bounded self adjoint operator with partial integrals in $L_2(\Omega_1 \times \Omega_2)$, i.e. T_1, T_2 are partially integral operators (PIO). It should be stressed that T_1 and T_2 with a non-zero kernel are not compact. In [13] it is proved that $\sigma_{ess}(H_0 + \varepsilon T) = \sigma(H_0 + \varepsilon T_1) \cup \sigma(H_0 + \varepsilon T_2)$, in the case when the kernels of T_1 and T_2 are continuous functions.

Consider the multiplier H_0, given by the function $h_0(x,y)$, having the following form: $h_0(x,y) = u(x) + \omega(x,y) + v(x)$, and PIO T_1, T_2 with kernels identically equal to one.

Let the multiplier H_0 be perturbed by a non-compact operator $T = \gamma_1 T_1 + \gamma_2 T_2$, where $\gamma_1 > 0$, $\gamma_2 > 0$. The purpose of the work is to apply the method from [13] to study the structure of the essential spectrum of the operator $H_0 - (\gamma_1 T_1 + \gamma_2 T_2)$ and to study the existence of resonance in the model $H = H_0 - (\gamma_1 T_1 + \gamma_2 T_2)$.

We denote by $\sigma(\cdot)$, $\sigma_{ess}(\cdot)$ and $\sigma_{disc}(\cdot)$, respectively, the spectrum, the essential spectrum and the discrete spectrum of the self-adjoint operators.

The number

$$E_{\min}(H) = \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$$

is called the bound edge (or the lower edge) of the essential spectrum of the operator H.

2. Non-compact perturbation of the essential spectrum

Let $\Omega_1 = [0,1]^{\nu_1} \subset \mathbb{R}^{\nu_1}$ and $\Omega_2 = [0,1]^{\nu_2} \subset \mathbb{R}^{\nu_2}$ $(\nu_1, \nu_2 \in \mathbb{N})$. In the space $L_2(\Omega_1 \times \Omega_2)$, let us consider a linear bounded self-adjoint operator H of the form

$$H = H_0 - (\gamma_1 T_1 + \gamma_2 T_2), \ \gamma_1 > 0, \ \gamma_2 > 0, \tag{1}$$

where H_0 is the multiplier given by the continuous real valued function $k_0(x,y)$, i.e. $H_0 f(x,y) = k_0(x,y)f(x,y)$, and the operators $T_1, \ T_2$ in the space $L_2(\Omega_1 \times \Omega_2)$ are defined by the following formulas:

$$T_1 f(x,y) = \int\limits_{\Omega_1} f(s,y)d\mu_1(s), \ T_2 f(x,y) = \int\limits_{\Omega_2} f(x,t)d\mu_2(t),$$

where $\mu_j(\cdot)$ is the Lebesgue measure on $\Omega_j, \ j = 1,2$.

It is known that $\sigma(H_0) = [k_0^{\min}, k_0^{\max}] \subset \sigma_{ess}(H)$, where $k_0^{\min} = \min k_0(x,y), \ k_0^{\max} = \max k_0(x,y)$, and $\sigma_{ess}(H) = \sigma(W_1) \cup \sigma(W_2)$, where $W_k = H_0 - \gamma_k T_k, \ k = 1,2$ (see. [13]).

Assume that $k_0(x,y) = u(x) + v(y)$, where $u(x)$ and $v(y)$ are real valued continuous functions on Ω_1 and Ω_2, respectively. Then the operator H (1) will be unitarily equivalent to the operator $H_1 \otimes E + E \otimes H_2$, where $H_1, \ H_2$ are operators in the Friedrichs model, E is the identity operator and "$\otimes$" means the tensor product of operators [13]. Using the spectral properties of the tensor product of operators [15,16], it can be argued that for all positive values of the parameters γ_1 and γ_2, the operator H has at most one eigenvalue outside the essential spectrum and $E_{\min}(H) \leq 0$. The eigenvalue $\lambda \in \sigma_{disc}(H)$ of the operator H is simple and $\lambda < E_{\min}(H)$.

Suppose that $k_0(x,y) = u(x)v(y)$, where $u(x)$ and $v(y)$ are non-negative continuous functions on Ω_1 Ω_2, respectively, and $0 \in Ran(u) \cap Ran(v)$. Then $E_{\min}(H) < 0$ and the operator H has at most one eigenvalue below the lower edge of the essential spectrum. The eigenvalue $\lambda < E_{\min}(H)$ of the operator H is simple [9,10].

Let $\omega(x,y)$ is a non-negative continuous function on $\Omega_1 \times \Omega_2$ and $0 \in Ran(\omega)$. Assume that $u(\theta) = v(\theta) = 0$ and $\omega(x,\theta) = \omega(\theta,y) = 0, \ x \in \Omega_1, \ y \in \Omega_2$, where the zero element in the corresponding linear space is denoted by θ.

Let the multiplier in (1) be given by the function

$$h_0(x,y) := k_0(x,y) = u(x) + \omega(x,y) + v(y).$$

Here, we study the spectral properties of the operator:

$$H = H_0 - (\gamma_1 T_1 + \gamma_2 T_2), \ \gamma_1, \gamma_2 > 0, \tag{2}$$

in the case

$$H_0 f(x,y) = h_0(x,y)f(x,y)$$

and under the following assumptions: the following integrals exist and are finite

$$\int\limits_{\Omega_1} \frac{ds}{u(s)}, \quad \int\limits_{\Omega_2} \frac{dt}{v(t)}.$$

For each $\beta \in \Omega_2$, we define the self-adjoint operator $H_1(\beta) : L_2(\Omega_1) \to L_2(\Omega_1)$ in the Friedrichs model:

$$H_1(\beta)\varphi(x) = h_0(x,\beta)\varphi(x) - \gamma_1 \int\limits_{\Omega_1} \varphi(s)ds.$$

Similarly, for each $\alpha \in \Omega_1$, we define the operator $H_2(\alpha) : L_2(\Omega_2) \to L_2(\Omega_2)$ in the Friedrichs model:

$$H_2(\alpha)\psi(y) = h_0(\alpha,y)\psi(y) - \gamma_2 \int\limits_{\Omega_2} \psi(t)dt.$$

Let's put $M_1(\beta) = \max\limits_{x \in \Omega_1} h_0(x,\beta), \ M_2(\alpha) = \max\limits_{y \in \Omega_2} h_0(\alpha,y)$.

By Weyl's theorem [1] on the compact perturbation of the essential spectrum, we have $\sigma_{ess}(H_k(\xi)) = [0, M_k(\xi)], \ \xi \in \Omega_j, \ j \neq k, \ j,k = 1,2$.

Lemma 2.1. [18] *The number* $\lambda \in \mathbb{R} \backslash [0, M_1(\beta)]$ *is the eigenvalue of the operator* $H_1(\beta)$*if and only if* $\Delta_1(\beta;\gamma_1,\lambda) = 0$, *where*

$$\Delta_1(\beta;\gamma_1,\lambda) = 1 - \gamma_1 \int\limits_{\Omega_1} \frac{ds}{h_0(s,\beta) - \lambda}.$$

Let's define the function $h_1(\beta)$ on Ω_2 by the formula

$$h_1(\beta) = \int\limits_{\Omega_1} \frac{ds}{h_0(s,\beta)}.$$

The function $h_1(\beta)$ is continuous on the set of Ω_2 and $h_1(\beta) > 0$, $\beta \in \Omega_2$.

We define [20] non-positive and continuous functions $\pi_1(y)$ on Ω_2 and $\pi_2(x)$ on Ω_1 using the following equalities

$$\pi_1(y) = \inf_{\|\varphi\|=1}(H_1(y)\varphi,\varphi),\ \ y\in\Omega_2,\ \ \pi_2(x) = \inf_{\|\psi\|=1}(H_2(x)\psi,\psi),\ \ x\in\Omega_1.$$

Let's put $\pi_j^{\min} = \min\limits_{\xi\in\Omega_k}\pi_j(\xi),\ \ \pi_j^{\max} = \max\limits_{\xi\in\Omega_k}\pi_j(\xi),\ \ j\neq k,\ \ j,k=1,2,\ \ h_0^{\max} = \max\limits_{(x,y)\in\Omega_1\times\Omega_2} h_0(x,y).$

Proposition 2.1. *The following conditions hold for the operators W_1 and W_2*
a) $\sigma(W_1) = [\pi_1^{\min},\pi_1^{\max}]\cup\sigma(H_0)$;
b) $\sigma(W_2) = [\pi_2^{\min},\pi_2^{\max}]\cup\sigma(H_0)$.

Proof. a) In [13], the equality $\sigma(W_1) = \sigma(H_0)\cup\sigma_1$ is proven, where

$$\sigma_1 = \{\lambda\in\rho(H_0) : \Delta_1(\beta_0;\lambda,\gamma) = 0 \ \text{ for some } \ \beta_0\in\Omega_2\}.$$

Let $\pi_1(\beta_0) < 0$ for some $\beta_0\in\Omega_2$. Then, due to the minimax principle, solution $\lambda_0(\beta_0)$ of the equation $\Delta_1(\beta_0;\gamma_1,\lambda) = 0$, is defined using continuous function $\pi_1(\beta_0)$. i.e. $\lambda_0(\beta_0) = \pi_1(\beta_0)$. Therefore, $\pi_1(\beta_0)\in\sigma_1$. If $\pi_1(\beta) < 0$ for all $\beta\in\Omega_2$, then $\lambda(\beta) = \pi_1(\beta)\in\sigma_1, \sigma_1 = [\pi_1^{\min},\pi_1^{\max}]$ and $\sigma(W_1) = \sigma(H_0)\cup\sigma_1 = [0,h_0^{\max}]\cup[\pi_1^{\min},\pi_1^{\max}]$. If $\pi_1(\beta_0) = 0$ for some $\beta_0\in\Omega_2$, then $\pi_1^{\max} = 0$. Hence, we obtain $\sigma(W_1) = \sigma(H_0)\cup\sigma_1 = [0,h_0^{\max}]\cup[\pi_1^{\min},\pi_1^{\max}] = [\pi_1^{\min},h_0^{\max}]$.
The equality $\sigma(W_2) = [0,h_0^{\max}]\cup[\pi_2^{\min},\pi_2^{\max}]$ is proved similarly.

Proposition 2.2. *If $\gamma_1\leq h_1^{-1}(\theta)$, then $\sigma(H_1(\beta)) = \sigma_{ess}(H_1(\beta))$ for all $\beta\in\Omega_2$.*
Proof. Since

$$h_0(x,y) = u(x) + \omega(x,y) + v(y) \geq u(x),\ \ x\in\Omega_1,\ \ y\in\Omega_2,$$

then

$$H_1(\beta)\geq H_1(\theta),\ \ \beta\in\Omega_2. \tag{3}$$

However, $E_{\min}(H_1(\theta)) = 0$ and

$$\Delta_1(\theta;\gamma_1,\lambda) = 1 - \gamma_1\int\limits_{\Omega_1}\frac{ds}{u(s)-\lambda}.$$

The function $\Delta_1(\lambda) = \Delta_1(\theta;\gamma_1,\lambda)$ on $(-\infty,0)$ is strictly decreasing, while $\lim\limits_{\lambda\to-\infty}\Delta_1(\lambda) = 1$ and $\lim\limits_{\lambda\to 0-}\Delta_1(\lambda) = 1-\gamma_1 h_1(\theta)\geq 0$. Hence, one obtains that $\Delta_1(\lambda) = \Delta_1(\theta;\gamma_1,\lambda) > 0$ for $\lambda\in(-\infty,0)$. Then, according to Lemma 2.1, $\sigma_{disc}(H_1(\theta)) = \emptyset$, i.e. $\sigma(H_1(\theta)) = [0,M_1(\theta)]$. It follows from (3) that

$$\inf_{\|\varphi\|=1}(H_1(\beta)\varphi,\varphi)\geq\inf_{\|\varphi\|=1}(H_1(\theta)\varphi,\varphi) = 0,\ \ \beta\in\Omega_2.$$

However, $0\in\sigma(H_1(\beta)),\ \ \beta\in\Omega_2$ and consequently $\inf\limits_{\|\varphi\|=1}(H_1(\beta)\varphi,\varphi) = E_{\min}(H_1(\beta)) = 0,\ \ \beta\in\Omega_2$. Hence, due to the minimax principle [1], it follows that $\sigma_{disc}(H_1(\beta)) = \emptyset$, for all $\beta\in\Omega_2$.
Now we define the function $h_2(\alpha)$ on Ω_1 by the following formula

$$h_2(\alpha) = \int\limits_{\Omega_2}\frac{dt}{h_0(\alpha,t)}.$$

Obviously, the function $h_2(\alpha)$ is continuous in Ω_1 and $h_2(\alpha) > 0,\ \ \alpha\in\Omega_1$.
Just as in proposition 2.2, it is proved that if $\gamma_2\leq h_2^{-1}(\theta)$, then $\sigma(H_2(\alpha)) = \sigma_{ess}(H_2(\alpha))$ for all $\beta\in\Omega_2$.
Hence, due to Proposition 2.2, the following theorem is proved:
Theorem 2.1. *a) if $\gamma_1\leq h_1^{-1}(\theta)$, then $\sigma(W_1) = \sigma(H_0) = [0,h_0^{\max}]$;*
b) if $\gamma_2\leq h_2^{-1}(\theta)$, then $\sigma(W_2) = \sigma(H_0) = [0,h_0^{\max}]$.

We define the sets $\mathcal{D}_0\subset\Omega_2$ and $\mathcal{D}_1\subset\Omega_2$:

$$\mathcal{D}_0 = \{\beta\in\Omega_2 : \gamma_1\leq h_1^{-1}(\beta)\},\ \ \mathcal{D}_1 = \Omega_2\setminus\mathcal{D}_0.$$

Lemma 2.2. *a) If $\mathcal{D}_0 = \Omega_2$ (i.e. $\mathcal{D}_1 = \emptyset$), then $\pi_1(t)\equiv 0$;*
b) if $\mathcal{D}_0\neq\emptyset$ $\mathcal{D}_1\neq\emptyset$, then $\pi_1^{\min} < \pi_1^{\max} = 0$;
c)if $\mathcal{D}_0 = \emptyset$, then $\pi_1^{\min} < \pi_1^{\max} < 0$.
Proof. Obviously, for every fixed $\beta\in\Omega_2$ and $\gamma_1 > 0$, the function $\Delta_1(\lambda) = \Delta_1(\beta;\gamma_1,\lambda)$ is strictly decreasing on $(-\infty,0)$ and

$$\lim_{\lambda\to-\infty}\Delta_1(\lambda) = 1 \ \text{ and } \ \lim_{\lambda\to 0-}\Delta_1(\lambda) = 1-\gamma_1 h_1(\beta).$$

a) Let $\mathcal{D}_0 = \Omega_2$. Then $1-\gamma_1 h_1(\beta)\geq 0$ for all $\beta\in\Omega_2$. Due to monotonicity of the function $\Delta_1(\lambda)$ for $(-\infty,0)$ we have $\Delta_1(\beta;\gamma_1,\lambda) > 0$ for all $\lambda\in(-\infty,0)$ for each $\beta\in\Omega_2$. Hence, by virtue of Lemma 2.1, we obtain $\sigma(H_1(\beta)) = \sigma_{ess}(H_1(\beta)),\ \ \beta\in\Omega_2$. Then, by the minimax principle $\pi_1(t) = 0$ for all $t\in\Omega_2$.

b) Let $\mathcal{D}_0 \neq \emptyset$. Then there exists a point $\beta_0 \in \mathcal{D}_0 \subset \Omega_2$, such that,

$$\lim_{\lambda\to 0-} \Delta_1(\beta_0;\gamma_1,\lambda) = 1-\gamma_1 h_1(\beta_0) \geq 0,$$

i.e. $\Delta_1(\beta_0;\gamma_1,\lambda) \geq 0$ on $(-\infty,0)$. Hence, due to the lemma 2.1, we obtain that $\sigma(H_1(\beta_0)) = \sigma_{ess}(H_1(\beta_0))$. Therefore, we have $\pi_1(\beta_0) = 0$. Since $\pi_1(t) \leq 0,\ t \in \Omega_2$, we have $\pi_1^{\max} = \pi_1(\beta_0) = 0$. If $\mathcal{D}_1 \neq \emptyset$, then there exists $\beta_1 \in \mathcal{D}_1 \subset \Omega_2$ such that

$$\lim_{\lambda\to 0-} \Delta_1(\beta_1;\gamma_1,\lambda) = 1-\gamma_1 h_1(\beta_1) < 0.$$

Hence, the equation $\Delta_1(\beta_1;\gamma_1,\lambda) = 0$ on $(-\infty,0)$ has unique solution $\lambda_0 < 0$. By Lemma 2.1, the number λ_0 is an eigenvalue of the operator $H_1(\beta_1)$. Hence, following the minimax principle, we obtain that $\pi_1(\beta_1) = \lambda_0 < 0$, i.e. $\pi_1^{\min} \leq \pi_1(\beta_1) < 0$.

c) Let $\mathcal{D}_0 = \emptyset$. Then $\mathcal{D}_1 = \Omega_2$ and therefore for each $\beta \in \Omega_2$, we have

$$\lim_{\lambda\to 0-} \Delta_1(\beta;\gamma_1,\lambda) = 1-\gamma_1 h_1(\beta) < 0.$$

Due to the monotonicity of the function $\Delta_1(\beta;\gamma_1,\lambda)$ on $(-\infty,0)$ there is a negative number $\lambda = \lambda(\beta)$ such that $\Delta_1(\beta;\gamma_1,\lambda(\beta)) = 0$, i.e. the number $\lambda(\beta)$ is the eigenvalue of the operator $H_1(\beta)$. Then, by the minimax principle, we obtain that $\pi_1(\beta) = \lambda(\beta),\ \beta \in \Omega_2$. It follows from the continuity of the function $\pi_1(t)$ on Ω_2 that $\pi_1^{\max} < 0$.

We prove that $\pi_1^{\min} < \pi_1^{\max}$. Let's assume the opposite: let $\pi_1^{\min} = \pi_1^{\max}$. Then the solutions λ_0, $\lambda_0 < 0$, and λ_1, $\lambda_1 < 0$ of the equations $\Delta_1(\theta;\gamma_1,\lambda) = 0$ and $\Delta_1(\beta;\gamma_1,\lambda) = 0,\ \beta \in \Omega_2, \beta \neq \theta$ coincide, i.e.

$$\Delta_1(\theta;\gamma_1,\lambda_0) = \Delta_1(\beta;\gamma_1,\lambda_0) = 0.$$

Therefore, we obtain

$$\int\limits_{\Omega_1} \frac{h_0(s,\beta)-u(s)}{(u(s)-\lambda_0)(h_0(s,\beta)-\lambda_0)}ds = 0. \tag{4}$$

However, $h_0(s,\beta) - u(s) \geq 0,\ \ s \in \Omega_2$ and the function

$$F_1(s,\beta) = \frac{h_0(s,\beta)-u(s)}{(u(s)-\lambda_0)(h_0(s,\beta)-\lambda_0)}$$

is non-negative continuous on Ω_2 and distinct from a constant. Hence, in accordance with the property of the Lebesgue integral, we obtain that $\int\limits_{\Omega_2} F_1(s,\beta)ds > 0$. This contradicts equality (4). Therefore, $\pi_1^{\min} \neq \pi_2^{\max}$.

We put:

$h_j^{\min} = \min\limits_{\xi\in\Omega_k} h_j(\xi)$ and $h_j^{\max} = \max\limits_{\xi\in\Omega_k} h_j(\xi),\ \ j=1,2,\ \ k=1,2,\ \ j\neq k.$

Lemma 2.2 implies the proof the theorem

Theorem 2.2.*a)if* $\gamma_1 > (h_1^{\min})^{-1}$, *then* $\pi_1^{\max} < 0$;
b) if $(h_1^{\max})^{-1} < \gamma_1 \leq (h_1^{\min})^{-1}$, *then* $\pi_1^{\min} < 0\ \ \pi_1^{\max} = 0$;
c) if $\gamma_1 \leq (h_1^{\max})^{-1}$, then $\pi_1(t) = 0$.

A similar theorem is true for the function $\pi_2(x)$.

Corollary 2.1. *If* $\gamma_1 \leq (h_1^{\max})^{-1}\ \ \ \gamma_2 \leq (h_2^{\max})^{-1}$, *then* $\sigma_{ess}(H) = \sigma(H_0)$.
Proof. For the essential spectrum of the operator H, the equality holds (see. [13])

$$\sigma_{ess}(H) = \sigma(W_1) \cup \sigma(W_2),$$

where $W_k = H_0 - \gamma_k T_k,\ \ k = 1,2$. Hence, by Theorem 2.2 and Proposition 2.1, we obtain

$$\sigma_{ess}(H) = \sigma(H_0) = [0, h_0^{\max}].$$

Corollary 2.2 *Let in* (1) $\gamma_1 = h_1^{-1}(\theta)$ *and* $\gamma_2 = h_2^{-1}(\theta)$. *Then* $\sigma_{ess}(H) = \sigma(H_0)$.
Proof. Consider PIO V, defined by the equality

$$V = H_0 - (h_1^{-1}(\theta)T_1 + h_2^{-1}(\theta)T_2).$$

For $\gamma_1 = h_1^{-1}(\theta)$, one has

$$\lim_{\lambda\to 0-} \Delta_1(\theta;\gamma_1,\lambda) = 1 - h_1^{-1}(\theta) \lim_{\lambda\to 0-} \int\limits_{\Omega_1} \frac{ds}{h_0(s,\theta)-\lambda} = 0.$$

From the monotonicity of the function $\Delta_1(\theta;h_1^{-1}(\theta),\lambda)$ on $(-\infty,0)$ we obtain that $\Delta_1(\theta;h_1^{-1}(\theta),\lambda) > 0$ on $(-\infty,0)$, i.e. $\sigma(H_0 - h_1^{-1}(\theta)T_1) = \sigma(H_0)$. Similarly, it is shown that $\sigma(W_2) = \sigma(H_0)$. Hence, $\sigma_{ess}(V) = \sigma(W_1)\cup\sigma(W_2) = \sigma(H_0)$.

3. Zero-energy resonance of the operator H

It is said that, the operator $H_1(\theta)$ (operator $H_2(\theta)$))) has a resonance with zero energy [19] if the number 1 is the eigenvalue of the integral operator $H_1 : L_2(\Omega_1) \to L_2(\Omega_1)$ ($H_2 : L_2(\Omega_2) \to L_2(\Omega_2)$), where

$$H_1\varphi(x) = \gamma_1 \int\limits_{\Omega_1} \frac{\varphi(s)ds}{u(s)}, \quad H_2\psi(y) = \gamma_2 \int\limits_{\Omega_1} \frac{\psi(t)dt}{v(t)}$$

and at least one corresponding eigenfunction $\varphi_0(x)$ (eigenfunction $\psi_0(y)$) satisfies the condition $\varphi_0(\theta) \neq 0$ ($\psi_0(\theta) \neq 0$).

Theorem 3.1. *Let $\gamma_1 = h_1^{-1}(\theta)$. Then:*
a) operator $H_1(\theta)$ has a resonance with zero energy;
b) for all $\beta \in \Omega_2,\ \beta \neq 0$ operator $H_1(\beta)$ has no negative eigenvalue..

Proof. a) Let $\varphi_0(x) \equiv 1$. Then $V_1\varphi_0 = \gamma_1 h_1(\theta) = \varphi_0(x)$, i.e. the equation $V_1\varphi = \varphi$ has a solution φ_0 from $C(\Omega_1)$ and $\varphi_0(0) \neq 0$.

b) If $\gamma_1 = h_1^{-1}(\theta)$, then the condition of Proposition 2.2 is satisfied, and therefore $\sigma(H_1(\beta)) = \sigma_{ess}(H_1(\beta))$ for all $\beta \in \Omega_2$, i.e. there is no negative eigenvalue for the operators $H_1(\beta),\ \beta \in \Omega_2$.

Example. Let $\Omega_1 = \Omega_2 = [0, 1]$ and

$$u(x) = v(x) = x^{1/2},\ \omega(x, y) = \left(1 - \cos\frac{\pi}{2}x\right)\left(1 - \cos\frac{\pi}{2}y\right).$$

We have

$$\int\limits_0^1 \frac{ds}{u(s)} = \int\limits_0^1 \frac{dt}{v(t)} = 2.$$

The function $h_1(x)$ strictly decreases on [0,1], and hence, $h_1^{\min} = h_1(1)$ and $h_1^{\max} = h_1(0) = 2$. It is obvious that $\dfrac{1}{u(x)} \notin L_2(\Omega_1)$, i.e. $\dfrac{1}{u(x)} \in L_1(\Omega_1) \setminus L_2(\Omega_1)$. Hence, for $\gamma_1 = \dfrac{1}{2}$ the operator $H_1(0)$ has a resonance with zero energy and for all $\beta \in (0, 1]$ operator $H_1(\beta)$ has no negative eigenvalue.

4. Conclusion

Our main goals are the description of the essential spectrum of the operator H and studying its spectral properties. This work differs from the work of other scientists because we choose the special form of the multiplier H_0 and the non-compactness of the partial integral operators T_1 and T_2 takes place. To summarize, we applied the method of [13] for the description of the essential spectrum. Additionally, we mainly used the minimax principle from [9] to prove the theorems and found the exact description of the essential spectrum proved by conditioning the parameters γ_1 and γ_2.

References

[1] Vekua I.N., *New Methods for Solving Elliptic Equations*. OGIZ, Moscow Leningrad, 1948 [inRussian].
[2] Aleksandrov V.M. and Kovalenko E.V. A class of integral equations of mixed problems of continuum mechanics. *Sov. Phys., Dokl.*, 1980, **25**, P. 354.
[3] Aleksandrov V.M. and Kovalenko E.V. Contact interaction between coated bodies with wear. *Sov. Phys., Dokl.*, 1984, **29**, P. 340.
[4] Manzhirov A.V. On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology, *J. Appl. Math. Mech.*, 1985, **49**, P. 777.
[5] Kalitvin A.S. *Linear Operators with Partial Integrals* Izd. Voronezh. Gos. Univ., Voronezh, 2000 [in Russian].
[6] Uchiyama J. Finiteness of the number of discrete eigenvalues of the Schrödinger operator for a three particle system. *Publ. Res. Inst.,Math. Sci.*, 1969,**5**(1), P. 51–63.
[7] Uchiyama J. Corrections to finiteness of the number of discrete eigenvalues of the Schrödinger operator for a three particle system. *Publ. Res. Inst. Math. Sci.*, 1970, **6**(1), P. 189–192.
[8] Zhislin G.M. On the finiteness of the discrete spectrum of the energy operator of negative atomic and molecular ions. *Theor. Math. Phys.*, 1971, **7**, P. 571–578.
[9] Reed M., Simon B. *Methods of Modern Mathematical Physics*. Analysis of Operators, Acad. Press, New York, 1982, **4**.
[10] Friedrichs K.O. "Uber die Spectralzerlegung eines Integral Operators. *Math. Ann.*, 1938, **115**, P. 249–272.
[11] Ladyzhenskaya O.A. and Faddeev L.D. To the theory of perturbations of the continuous spectrum. *Dokl. Akad. Nauk SSSR*, 1958, **6**(120), P. 1187–1190.
[12] Eshkabilov Yu.Kh. On infinity of the discrete spectrum of operators in the Friedrichs model. *Siberian Adv. Math.*, 2012, **22**, P. –12.
[13] Eshkabilov Yu.Kh. On a discrete three-particle Schrödinger operator in the Hubbard model. *Theor. Math. Phys.*, 2006, **149**(2), P. 1497–1511.
[14] Eshkabilov Yu.Kh. On the discrete spectrum of partially integral operators. *Siberian Adv. Math.*, 2013, **23**, P. 227–233.
[15] Ichinose T. Spectral properties of tensor products of linear operators:I. *Trans.Amer.Math.Soc.*, 1978, **235**, P. 75–113.
[16] Ichinose T. Spectral properties of tensor products of linear operators:II. *Trans.Amer.Math.Soc.*, 1978, **237**, P. 223–254.
[17] Eshkabilov Yu.Kh., Kucharov R.R. On the finiteness of negative eigenvalues of a partially integral operator. *Siberian Adv. Math.*. 2015, **25**(3), P. 179–190.
[18] Eshkabilov Yu.Kh., Kucharov R.R. Essential and discrete spectra of the three-particle Schrodinger operator on a lattice. *Theor.Math.Phys.*, 2012, **170**(3), P. 341–353.

[19] Albeverio S., Lakaev S.N., Muminov Z.I. On the number of eigenvalues of a model operator associated to a system of three-particles on lattices. *Russ. J. of Math. Phys.*, 2007, **14**(4), P. 377–387.

[20] Kucharov R.R., Tuxtamurodova T.M., Arzikulov G.P. Essential spectrum of one partial integral operator with degenerate kernel. *Vestnik NUUz*, 2023, **1**, P. 85–96.

Submitted 30 December 2023; revised 13 January 2024; accepted 15 January 2024

Information about the authors:

Ramziddin R. Kucharov – Tashkent International University of Financial Management and Technology; National University of Uzbekistan, Mathematics, Tashkent, 4, 100174, Uzbekistan; ORCID 0000-0002-0728-9340; r.kucharov@tift.uz; ramz3364647@yahoo.com

Tillohon M. Tuxtamurodova – National University of Uzbekistan, Mathematics, Tashkent, 4, 100174, Uzbekistan; ORCID 0009-0002-1721-880X; mirzayevatilloxon13@gmail.com

Conflict of interest: the authors declare no conflict of interest.