NANOSYSTEMS:
PHYSICS, CHEMISTRY, MATHEMATICS

Bekakra Y., et al. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (1), 5–15.
http://nanojournal.ifmo.ru
DOI 10.17586/2220-8054-2024-15-1-5-15

Original article

Existence and uniqueness theorem for a weak solution of fractional parabolic problem by the Rothe method

Y. Bekakra[1,a], A. Bouziani [1,2,b]

[1]ICOSI Laboratory, Abbes Laghrour University, Khenchela, 04000, Algeria
[2]L'arbi Ben M'hidi University, Oum El Bouagui, 04000, Algeria

[a]youcef.bekakra@univ-khenchela.dz, [b]aefbouziani1963@gmail.com

Corresponding author: Y. Bekakra, youcef.bekakra@univ-khenchela.dz

PACS 02.60−x, 47.11.Bc, 02.30.Jr

ABSTRACT This paper aims to study the existence and uniqueness of a weak solution for the boundary value problem of a time fractional equation involving the Caputo fractional derivative with an integral operator. By utilizing the discretization method, we first derive some a priori estimates for the approximate solutions at the points (x, t_j). We then evaluate the accuracy of the proposed method to demonstrate that the implemented sequence of α-Rothe functions converges in a certain sense, and its limit is the solution (in a weak sense) of our problem. It must be pointed out that the constructed L1 scheme is designed to approximate the Caputo fractional derivative mentioned in the problem.

KEYWORDS weak solution, a priori estimates, Fractional diffusion equation, Rothe's method

ACKNOWLEDGEMENTS The authors would like to thank the reviewer for his valuable comments and suggestions.

FOR CITATION Bekakra Y., Bouziani A. Existence and uniqueness theorem for a weak solution of fractional parabolic problem by the Rothe method. *Nanosystems: Phys. Chem. Math.*, 2024, **15** (1), 5–15.

1. Introduction

In the beginning, fractional calculus was developed as a pure mathematical concept. In recent decades, its use has expanded into a variety of different fields of science, such as physics, mechanics, economics, and engineering (see, for instance, [1–4]).

The time-fractional-order diffusion equations (TFDEs) have attracted many scholars' attention, it is worth mentioning that the crucial importance of this kind of fractional equations is due to its wide use in many real-world applications in the fields of biology [5] , chemistry [6], engineering [7]. It is pointed out that this kind of equation also appears, especially in nanofluid [8–10], nanotechnology [11, 12] and nanophysics [13–15].

In [16], authors studied the following time-fractional diffusion equation:

$$ {}_0^C D_t^\alpha u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) + f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in Q $$

where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative of order $\alpha (0 < \alpha < 1)$. They prove the existence and uniqueness of the solution by using the Lax-Milgram Lemma in some suitable Sobolev spaces. Paper [17] dealt with a time fractional equation on the metric star graph. The authors applied the method of energy integrals to construct Green's matrix-function and discussed applications to nanostructures.

Accordingly, various types of numerical methods have been applied to study this concept. One such method that has drawn the attention of many scholars is the Rothe method. Researchers have exploited and enhanced this technique for study of various differential fractional equations [18–20]. Yang [21] presented a difference scheme for a kind of linear space-time fractional convection-diffusion equation using a finite difference method. Du et al. [22] apply the Rothe method to establish existence, uniqueness, and a priori estimate for a strong solution to an approximate fractional problem.

Assume that $y = y(x, t)$ is a Lipschitz function with respect to t and let G be a bounded domain in multiply connected Ω with a Lipschitz boundary Γ, we consider two-dimensional time fractional equation with an integral operator of the following form:

$$ \begin{aligned} &\partial_{0t}^\alpha y(x, t) - \Delta y = \int_0^t \kappa(x, s, y(x, s)) \mathrm{d}s + f(x) \quad \text{in } Q = G \times (0, T), \quad 0 < \alpha < 1, \qquad (1) \\ &y(x, 0) = 0, \\ &B_1 y = 0 \quad \text{on} \quad \Gamma \times (0, T), \end{aligned} $$

where $\partial_{0t}^{\alpha}u(x,t)$ denotes the Caputo fractional derivative of order α $(0<\alpha<1)$, $f\in L^2(G)$, and B_1 is a given linear operator. Additionally, it is assumed that the kernel $\kappa(x,t,u)$ satisfying the following conditions:

$$\|\kappa(x,t_2,y_2)-\kappa(x,t_1,y_1)\|\leqslant C\left(|t_2-t_1|+\frac{1}{h^{\alpha-1}}\|y_2-y_1\|\right), \quad y_1,y_2\in L_2(G),\quad t_1,t_2\in I=[0,T]. \tag{2}$$

$$u\in\vartheta\Rightarrow\kappa\in L_2(G). \tag{3}$$

Here ϑ is the space defined as follows

$$\vartheta=\left\{\nu;\nu\in W_2^{(k)}(G), B_1\nu=0,\ \text{in the sense of traces}\right\}. \tag{4}$$

In the context of nanomaterials, equation (1) could model the movement of a particle or the diffusion of heat or matter through a medium with non- standard diffusion properties. Details for nanomaterials could be as follows:
Memory and non-locality: Nanomaterials can exhibit memory effects due to their interaction with a non-homogeneous environment, where the past influences the present movement.
Heterogeneous Diffusion: In heterogeneous media, such as nanopores, diffusion is not uniform and is affected by the structure of the material. The function $\kappa(x,s,y(x,s))$ could represent how the heterogeneous medium affects diffusion. It is worth mentioning that the boundary conditions considered in our problem may reflect the interactions of the particle energy with the boundaries of the nanomaterial, which could be reactive surfaces or interfaces with other materials.

First, we introduce the Caputo finite difference formula to discretize the time-fractional derivative of order α [23]:

$$\frac{\partial^{\alpha}y(x,t_j)}{\partial t^{\alpha}}=\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{j-1}\left(z_{j-k}(x)-z_{j-k-1}(x)\right)\sigma_k\quad 0<\alpha<1, \tag{5}$$

where

$$\sigma_k=\left[(k+1)^{1-\alpha}-k^{1-\alpha}\right]\quad k=0,1,\ldots,$$

while $z_j(x)$ denotes the numerical approximation to the exact solution $y(x,t_j)$, $t_j=jh$, $0\leqslant j\leqslant p$, where $h=\dfrac{T}{p}$ is the time step. It is straightforward to check that

$$\sigma_j>0,\quad j=0,1,\ldots,p,$$
$$1=\sigma_0>\sigma_1>\ldots\sigma_p.$$

The organization of the paper is as follows: In Section 2, some preliminary facts regarding fractional calculus and some notations are presented. In Section 3, we obtain a priori estimates. Sections 4 and 5 are devoted to discussing the existence and uniqueness of the weak solution, respectively.

2. Preliminaries

In the first part of this section, we recall some basic background of fractional calculus that we will need in the sequel.

Definition 1. *[24] The Riemann-Liouville fractional integration $I_{0,t}^{\alpha}$ of order $\alpha>0$ of function $\varphi(t)$ is defined as:*

$$I_{0,t}^{\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)\mathrm{d}s\quad(\alpha>0,t>0),$$

where Γ is the well-known Euler Gamma-function.

Definition 2. *[24] The Caputo derivative of fractional order $\alpha\in]0;1[$ of function $u(t)$ is defined as:*

$$\partial_{0,t}^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}u'(s)\mathrm{d}s.$$

Theorem 2.1. *[25] If $x(t)\in C^0[0,T]$ for $T>0$ and $\alpha>0$, then*

$$I_{0,t}^{\alpha}x(0)=0.$$

Theorem 2.2. *[25] If $x(t)\in C^1[0,T]$ and $0<\alpha<1$, then*

$$\partial_{0,t}^{\alpha}I_{0,t}^{\alpha}x(t)=x(t).$$

Lemma 2.1 (Gronwall's lemma [26])**.** *Let $\eta_1,\ldots,\eta_j$ be nonnegative numbers satisfying $\eta_1\leqslant A$, $\eta_i\leqslant A+Bh\sum_{k=1}^{i-1}\eta_k$, $\forall i=2,\ldots,j$, where A, B, and h are positive constants. Then*

$$\eta_i\leqslant A\exp\left[B(i-1)h\right],\quad i=1,2\ldots,j.$$

Theorem 2.3. *If a sequence $\{x_n\}$ in a Hilbert space H is a weakly convergent to $x \in H$, then the following statements hold:*

$$\partial_{0,t}^{\alpha} x_n \rightharpoonup \partial_{0,t}^{\alpha} x \quad in\ H, \tag{6}$$

$$I_{0,t}^{\alpha} x_n \rightharpoonup I_{0,t}^{\alpha} x \quad in\ H. \tag{7}$$

For the proof of our theorem, we need the following lemma.

Lemma 2.2. *[26] A sequence $\{x_n\}$ in Hilbert space H converges weakly to $x \in H$ implying that for any bounded linear functional g defined on H we have*

$$g(x_n) \to g(x).$$

Proof of Theorem 2.3 (6) The caputo derivative is a bounded linear functional. Consequently, from

$$x_{n_k} \rightharpoonup x,$$

it follows according to the preceding lemma that

$$\partial_{0,t}^{\alpha} x_{n_k} \to \partial_{0,t}^{\alpha} x.$$

Finally, the strong convergence implies weak convergence and this proof is completed.
(7) The proof is the same as for the previous case.

Lemma 2.3. *[27] Any absolutely continuous function $v\,(t)$ on $[0,T]$ satisfies the inequality*

$$\left(v, \partial_{0,t}^{\alpha} v\right) \geqslant \frac{1}{2}\, \partial_{0,t}^{\alpha} \|v\|^2, \qquad 0<\alpha<1.$$

Lemma 2.4. *[27] Let a nonnegative absolutely continuous function $y(t)$ satisfy the inequality*

$$\partial_{0,t}^{\alpha} y(t) \le c_1 y(t) + c_2(t), \quad 0<\alpha\le 1,$$

for a. a. t in $[0,T]$, where $c_1>0$ and $c_2(t)$ is an integrable nonnegative function on $[0,T]$. Then

$$y(t) \le y(0) E_{\alpha}\left(c_1 t^{\alpha}\right) + \Gamma(\alpha) E_{\alpha,\alpha}\left(c_1 t^{\alpha}\right) D_{0t}^{-\alpha} c_2(t)$$

where $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \Gamma(\alpha n+1)$ *and* $E_{\alpha,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\mu)}$ *are the Mittag-Leffler functions.*

The following assertions are presented at the end of this section. Let $\|.\|$ and $(.)$ denote $L_2(G)$-norm and $L_2(G)$-inner product, respectively. By $W_2^{(k)}(G)$ we denote the usual Sobolev space. We denote the bilinear form corresponding to the operator $-\Delta u$ by $b_{\zeta}(.,.)$:

$$b_{\zeta}(u,v) = \sum_{i=1}^{N} \int_G \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} dx.$$

Definition 3. *[26] The form $b_{\zeta}(v,u)$ is called $\vartheta - elliptic$ if a constant η can be found such that*

$$b_{\zeta}(v,v) \geqslant \eta \|v\|^2_{W_2^{(k)}(G)} \quad \forall v \in \vartheta. \tag{8}$$

In $L_2\,(I,\vartheta)$, the scalar product is provided by:

$$(y_1, y_2)_{L_2(I,H)} = \int_I (y_1(t), y_2(t))_H \ \mathrm{d}t,$$

and, consequently, the norm is given by:

$$\|y\|^2_{L_2(I,\vartheta)} = \int_I \|y(t)\|^2_{\vartheta} \ \mathrm{d}t. \tag{9}$$

3. Weak Formulation and a priori estimates

Substituting (5) into (1) , we get

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \left(z_{j-k}(x) - z_{j-k-1}(x)\right) \sigma_k - \Delta z_j(x) = h\left(\kappa_0 + \kappa_1 + \ldots \kappa_{j-1}\right) + f, \quad 0<\alpha<1. \tag{10}$$

With simplification by omitting the dependence of $z_j(x)$ on x, (10) can be rewritten as follows:

$$-\Gamma(2-\alpha)\Delta z_j + \frac{z_j - z_{j-1}}{h^{\alpha}} = -\left(\sum_{k=1}^{j-1} \frac{z_{j-k} - z_{j-k-1})}{h^{\alpha}} \sigma_k\right)$$
$$+\Gamma(2-\alpha) h\left(\kappa_0 + \kappa_1 + \ldots \kappa_{j-1}\right) + \Gamma(2-\alpha) f, \quad 0<\alpha<1,$$

where

$$\kappa_j(x) = \kappa\left(x, jh, z_j(x)\right), \quad j=1,\ldots,p.$$

Hence, the integral identities i.e. the corresponding weak formulation has the following form:

$$\begin{aligned}
&\Gamma(2-\alpha)b_\zeta(z_j,v)+\frac{1}{h^\alpha}\left(z_j-z_{j-1},v\right)\\
&=\left(-\frac{1}{h^\alpha}\sum_{k=1}^{j-1}\left(z_{j-k}-z_{j-k-1}\right)w_k+\Gamma(2-\alpha)h\left(\kappa_0+\kappa_1+\ldots\kappa_{j-1}\right)+\Gamma(2-\alpha)f,v\right)\\
&\forall v\in\vartheta,\quad 0<\alpha<1.
\end{aligned} \tag{11}$$

Thus, we can construct the α-Rothe function $y_1(t)=y_1(x,t)$ defined in the intervals $I_j=[t_{j-1},t_j]$, $j=1,\ldots,p$, by:

$$y_1(t)=z_{j-1}+\frac{z_j-z_{j-1}}{h^\alpha}\left(t-t_{j-1}\right)^\alpha\quad 0<\alpha<1.$$

In the same way, we get, for the divisions $d_n, n=2,3\ldots$, with

$$h_n=\frac{T}{2^{n-1}p}$$

the α-Rothe sequence

$$\{y_n(t)\} \tag{12}$$

of functions:

$$y_n(t)=z_{j-1}^n+\frac{z_j^n(x)-z_{j-1}^n(x)}{h_n^\alpha}\left(t-t_{j-1}^n\right)^\alpha\quad 0<\alpha<1. \tag{13}$$

Now, we are in a position to establish some a priori estimates.

Lemma 3.1. *There exist two constants c_1, c_2 independent of α and j such that*

$$\|z_j\|\leqslant c_1,\quad j=1,\ldots,p, \tag{14}$$

$$\|Z_j\|\leqslant c_2,\quad j=1,\ldots,p, \tag{15}$$

with

$$Z_j(x)=\frac{z_j(x)-z_{j-1}(x)}{h^\alpha}.$$

Proof. First, since y belongs to the Lipschitz class with respect to t and because of$z_0=0$, it produces

$$\|z_j\|\leqslant\|z_1\|+\|z_2-z_1\|+\ldots+\|z_j-z_{j-1}\|\leqslant ljh\leqslant lT=c_1,$$

where l is the Lipschitz constant.

Next, putting $v=Z_1$ in the first integral identity (11), we get

$$\Gamma(2-\alpha)b_\zeta(z_1,Z_1)+\frac{1}{h^\alpha}\left(z_1,Z_1\right)=\Gamma(2-\alpha)\left(h\kappa_0+f,Z_1\right),\quad 0<\alpha<1.$$

Hence, due to the Schwartz inequality and the fact that $b_\zeta(z_1,Z_1)=\dfrac{1}{h^\alpha}b_\zeta(z_1,z_1)\geqslant 0$ we obtain that

$$\|Z_1\|\leqslant\Gamma(2-\alpha)(h\|\kappa_0\|+\|f\|). \tag{16}$$

Subtracting the integral identities (11) written for $j=1$ from that written for $j=2$, we obtain

$$\Gamma(2-\alpha)b_\zeta(z_2-z_1,v)+(Z_2-Z_1,v)=(-\sigma_1Z_1+\Gamma(2-\alpha)h\kappa_1+f,v)\quad\forall v\in\vartheta.$$

Putting $v=Z_2$, we get

$$\|Z_2\|\leqslant(\sigma_0-\sigma_1)\|Z_1\|+\Gamma(2-\alpha)h\|\kappa_1\|. \tag{17}$$

Similarly, one obtains

$$\|Z_3\|\leqslant(\sigma_0-\sigma_1)\|Z_2\|+(\sigma_1-\sigma_2)\|Z_1\|+\Gamma(2-\alpha)h\|\kappa_2\|.$$

Following this procedure, one comes to the estimation

$$\|Z_j\|\leqslant\sum_{k=1}^{j-1}(\sigma_{k-1}-\sigma_k)\|Z_{j-k}\|+\Gamma(2-\alpha)h\|\kappa_{j-1}\|. \tag{18}$$

Adding up (16)–(18) yields

$$\|Z_j\|\leqslant\Gamma(2-\alpha)\psi_0\|f\|+\Gamma(2-\alpha)h\left(\psi_0\|\kappa_0\|+\psi_1\|\kappa_1\|+\ldots+\psi_{j-1}\|\kappa_{j-1}\|\right) \tag{19}$$

with

$$\psi_m=(\sigma_0-\sigma_1)^{j-(m+1)}+\sum_{k=1}^{j-(m+2)}(\sigma_0-\sigma_1)^{j-k-(m+2)}(\sigma_k-\sigma_{k-1})(j-k-(m+1)),$$
$$\text{for all}\ \ m=0,1,\ldots,j-1.$$

Now, we can easily proof that $\psi_0 < \psi_1 < \ldots < \psi_{j-1} = 1$, so (19) yields

$$\|Z_j\| \leqslant \Gamma(2-\alpha)\left[\|f\| + h\left(\|\kappa_0\| + \|\kappa_1\| + \ldots + \|\kappa_{j-1}\|\right)\right]. \tag{20}$$

In the other hand, by using the fact that $z_0 = 0$, we obtain from (2) that

$$\begin{aligned}\|\kappa_1\| &\leqslant \|\kappa_0\| + \|\kappa_1 - \kappa_0\| \\ &\leqslant \|\kappa_0\| + C\left(h + \frac{\|z_1\|}{h^{\alpha-1}}\right) \\ &= \|\kappa_0\| + Ch\left(1 + \|Z_1\|\right).\end{aligned}$$

Then,

$$\begin{aligned}\|\kappa_2\| &\leqslant \|\kappa_0\| + \|\kappa_2 - \kappa_0\| \\ &\leqslant \|\kappa_0\| + C\left(2h + \frac{\|z_2\|}{h^{\alpha-1}}\right) \\ &\leqslant \|\kappa_0\| + Ch\left(2 + \frac{\|z_2 - z_1\|}{h^{\alpha}} + \frac{\|z_1\|}{h^{\alpha}}\right) \\ &= \|\kappa_0\| + Ch\left(2 + \|Z_1\| + \|Z_2\|\right) \\ &= \|\kappa_0\| + Ch\left(2 + \|Z_1\| + \|Z_2\|\right).\end{aligned}$$

Hence, using the analogous arguments, we can conclude that

$$\begin{aligned}\|\kappa_{j-1}\| &\leqslant \|\kappa_0\| + \|\kappa_{j-1} - \kappa_0\| \\ &\leqslant \|\kappa_0\| + Ch\left(j - 1 + \|Z_1\| + \|Z_2\| + \ldots + \|Z_{j-1}\|\right).\end{aligned} \tag{21}$$

Inserting these results into (20), we obtain

$$\begin{aligned}\|Z_j\| &\leqslant \Gamma(2-\alpha)\left[\|f\| + jh\|\kappa_0\| + Ch^2\left(1 + 2 + \ldots + j - 1\right)\right] \\ &+ Ch^2\left[(j-1)\|Z_1\| + (j-2)\|Z_2\| + \ldots + \|Z_{j-1}\|\right].\end{aligned}$$

Now, $(j-1)\,h < ph = T$. Thus, we have

$$\begin{aligned}\|Z_j\| &\leqslant \Gamma(2-\alpha)\left[\|f\| + T\|\kappa_0\| + CT^2\left(1 + 2 + \ldots + j - 1\right)\right] \\ &+ CTh\left(\|Z_1\| + \|Z_2\| + \ldots + \|Z_{j-1}\|\right) \quad j = 1, 2, \ldots p.\end{aligned}$$

Using Lemma 2.1 , we get, finally

$$\begin{aligned}\|Z_j\| &\leqslant \Gamma(2-\alpha)\left(\|f\| + T\|\kappa_0\| + CT^2\right)e^{CT(j-1)h} \\ &\leqslant \Gamma(2-\alpha)\left(\|f\| + T\|\kappa_0\| + CT^2\right)e^{CT^2}, \quad j = 1, 2, \ldots p.\end{aligned}$$

Then (15) holds with $c_2 = \Gamma(2-\alpha)\left(\|f\| + T\|K_0\| + CT^2\right)e^{CT^2}$, and so the proof is complete. □

Remark 1. *The estimates* (14)*,* (15) *are independent of h, consequently, they remain valid for any division* $d_n, n = 1, 2, \ldots$*. Thus, we have*

$$\left\|z_j^n\right\| \leqslant c_1, \tag{22}$$

$$\left\|Z_j^n\right\| \leqslant c_2, \tag{23}$$

for all $n = 1, 2, \ldots,$ *and* $1 \leqslant j \leqslant 2^{n-1}p$*, where*

$$Z_j^n(x) = \frac{z_j^n(x) - z_{j-1}(x)}{h_n^{\alpha}}. \tag{24}$$

Recall that the space ϑ is determined by (4). Taking into account (22) and using (8), we arrive at the following corollary:

Corollary 3.1. *There exists a constant* c_3 *such that*

$$\left\|z_j^n\right\|_V \leqslant c_3, \tag{25}$$

for all $n = 1, 2, \ldots,$ *and* $1 \leqslant j \leqslant 2^{n-1}p$*.*

Lemma 3.2. $\left\|\kappa_j^n\right\|$ *is uniformly bounded with respect to* j *and* n *as well, where* $\kappa_j^n(x) = \kappa\left(x, jh_n, z_j^n(x)\right)$*.*

Proof. For the division d_n, due to (21), one obtains

$$\left\|\kappa_j^n\right\| \leqslant \|\kappa_0^n\| + Ch_n\left(j + \|Z_1^n\| + \|Z_2^n\| + \ldots + \left\|Z_j^n\right\|\right).$$

Taking into account (23), we come to the inequality

$$\left\|\kappa_j^n\right\| \leqslant \|\kappa_0^n\| + CT + CTc_2 = c_4. \tag{26}$$

□

4. Existence of weak solution

This section is devoted to proving the existence of a weak solution to our problem. For this purpose, let's start with the following lemma:

Lemma 4.1. *The α-Rothe sequence (12) admits a subsequence $\{y_{n_k}(t)\}$ weakly convergent in $L_2(I,\vartheta)$ to function $y \in L_2(I,\vartheta)$, we write*

$$y_{n_k} \rightharpoonup y \quad in \;\; L_2(I,\vartheta). \tag{27}$$

Proof. Since $L_2(I,\vartheta)$ is a Hilbert space it is sufficient to show that the α-Rothe sequence (12) is bounded in this space. Keeping in mind that

$$0 \leqslant \left(\frac{t-t_{j-1}^n}{h_n}\right)^{\alpha} \leqslant 1 \quad \text{in } \; I_j^n,$$

and by (13), (25), we obtain, for arbitrary $t \in I$,

$$\begin{aligned} \|y_n(t)\|_{\vartheta} &= \left\| z_{j-1}^n \left(1 - \frac{\left(t-t_{j-1}^n\right)^{\alpha}}{h_n^{\alpha}}\right) + z_j^n \frac{\left(t-t_{j-1}^n\right)^{\alpha}}{h_n^{\alpha}} \right\|_{\vartheta} \\ &\leqslant \left\| \left(1 - \frac{\left(t-t_{j-1}^n\right)^{\alpha}}{h_n^{\alpha}}\right) z_{j-1}^n \right\|_{\vartheta} + \left\| \frac{\left(t-t_{j-1}^n\right)^{\alpha}}{h_n^{\alpha}} z_j^n \right\|_{\vartheta} \\ &\leqslant \left(1 - \frac{\left(t-t_{j-1}^n\right)^{\alpha}}{h_n^{\alpha}}\right) c_3 + \frac{\left(t-t_{j-1}^n\right)^{\alpha}}{h_n^{\alpha}} c_3 = c_3. \end{aligned}$$

From (9), we get, for $n = 1, 2, \ldots$

$$\|y_n(t)\|_{L_2(I,\vartheta)}^2 = \int_0^T \|y_n(t)\|_{\vartheta}^2 \, \mathrm{d}t \leqslant c_3^2 T.$$

Thus, the α-Rothe sequence (12) is bounded in $L_2(I,\vartheta)$, and the proof is finished. □

Remark 2. *[26] Note that besides* (27) *it holds that*

$$y_{n_k} \to y \quad in \quad C(I, L_2(G). \tag{28}$$

Now, taking into account (23) and using the same reasoning as in the proof of the preceding corollary, we arrive at the following corollary:

Corollary 4.1. *There exists a subsequence $\left\{\mathbb{Y}_{n_k}^{\alpha}(t)\right\}$and a function $\mathbb{Y}^{\alpha}$ such that*

$$\mathbb{Y}_{n_k}^{\alpha} \rightharpoonup \mathbb{Y}^{\alpha} \quad in \;\; L_2\left(I, L_2(G)\right), \tag{29}$$

where

$$\mathbb{Y}_n^{\alpha}(t) : I \to L_2(G), \quad n = 1, 2, \ldots$$

defined by

$$\mathbb{Y}_n^{\alpha}(t) = \begin{cases} \Gamma(\alpha+1) Z_1^{n,\alpha} & for \quad t = 0, \\ \Gamma(\alpha+1) Z_j^{n,\alpha} & for \quad t \in \widetilde{I}_j^n = (t_{j-1}^n, t_j^n], \; j = 1, \ldots, 2^{n-1}p. \end{cases} \tag{30}$$

Lemma 4.2.

$$\partial_{0,t}^{\alpha} y(t) = \mathbb{Y}^{\alpha}(t) \;\; in \;\; L_2\left(I, L_2(G)\right) \quad for\, a.\, a.\; t \in I. \tag{31}$$

Proof. It follows from (13) and (30) that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{Y}_{n_k}^{\alpha}(t)(s) \mathrm{d}s = y_{n_k}(t). \tag{32}$$

Recalling the definition of the Riemann-Liouville integral, the left-hand side of (32) is nothing but $I_{0,t}^{\alpha} \mathbb{U}_{n_k}^{\alpha}(t)$, hence (32) becomes

$$I_{0,t}^{\alpha} \mathbb{Y}_{n_k}^{\alpha}(t) = y_{n_k}(t). \tag{33}$$

Applying the Caputo derivative $\partial_{0,t}^{\alpha}$ to the both sides of (33), we get by Theorem 2.2

$$\partial_{0,t}^{\alpha} y_{n_k}(t) = \mathbb{Y}_{n_k}^{\alpha}(t). \tag{34}$$

Applying now (6) for (27), we obtain in view of (29), (34), and by the uniqueness of the weak limit that

$$\partial_{0,t}^{\alpha} y(t) = \mathbb{Y}^{\alpha}(t) \;\; in \;\; L_2(G) \quad \text{for a.a. } t \in I.$$

The proof is finished. □

Corollary 4.2. *The function $y(t)$ verifies*

$$y(0) = 0 \quad \text{in } C\left(I, L_2(G)\right), \tag{35}$$

$$y \in AC\left(I, L_2(G)\right). \tag{36}$$

Proof. From (7), (27), and (33), following the same reasoning as in the proof of (31), we get

$$I_{0,t}^{\alpha}\mathbb{Y}^{\alpha}(t) = y(t).$$

Hence, by Theorem 2.1, we complete the proof. □

Lemma 4.3. *[26] If*

$$y_{n_k} \rightharpoonup y \quad \text{in } L_2(I, \vartheta),$$

then also

$$\widetilde{y}_{n_k} \rightharpoonup y \quad \text{in } L_2(I, \vartheta), \tag{37}$$

where $\{\widetilde{y}_n(t)\}$ is the sequence defined by:

$$\widetilde{y}_n(t) = \begin{cases} z_1^n \;\; for\; t = 0 \\ z_j^n \;\; in \;\; I_j^n = (t_{j-1}^n, t_j^n],\; j = 1, \ldots, p. \end{cases} \tag{38}$$

Let

$$w(t) = w(x,t) = \kappa(x, t, y(x,t)). \tag{39}$$

Since $y \in \vartheta$ for a. a. $t \in I$, we have, by (3) $w \in L_2(G)$ for a.a. $t \in I$. Consequently, we obtain the following corollary.

Corollary 4.3.

$$\lim_{n_k \to \infty} \widetilde{w}_{n_k}(t) = w(t) \;\; \text{in } \; L_2(G) \;\; \text{for a.a. } t \in I, \tag{40}$$

where

$$\widetilde{w}_n(t) : I \to L_2(G), \quad n = 1, 2, \ldots$$

is defined by

$$\widetilde{w}_n(t) = \begin{cases} \kappa_0^n \quad\; for \quad t = 0, \\ \kappa_{j-1}^n \quad in \quad \widetilde{I}_j^n = (t_{j-1}^n, t_j^n],\; j = 1, \ldots, 2^{n-1}p. \end{cases}$$

Proof. Let $\varepsilon > 0$ be given. We have to show that

$$\|w(t) - \widetilde{w}_{n_k}(t)\| \leqslant \varepsilon \quad \text{for a.a. } t \in I \quad \text{if } n_k > n_0(\varepsilon). \tag{41}$$

By (2) we obtain for fixed division d_{n_k} and for arbitrary $t \in \widetilde{I}_{j+1}^{n_k}$

$$\begin{aligned} \|w(t) - \widetilde{w}_{n_k}(t)\| &= \left\|\kappa(x,t,y(t)) - \kappa\left(x, jh_{n_k}, z_j^{n_k}\right)\right\| \\ &\leqslant C\left(|t - jh_{n_k}| + \frac{1}{h_{n_k}^{\alpha-1}}\left\|y(t) - z_j^{n_k}\right\|\right) \leqslant C\left[h_{n_k} + \|y(t) - y\left(jh_{n_k}\right)\| + \right. \\ &\left. + \left\|y\left(jh_{n_k}\right) - z_j^{n_k}\right\|\right]. \end{aligned}$$

Choosing $\eta = \dfrac{\varepsilon}{3C}$, we get the desired result due to (28) and (36). □

Remark 3. (40)*,* (41) *imply that the function $w(t)$ is Bochner integrable as an abstract function from I into $L_2(G)$. Consequently, we can define the following functions*

$$\mathbb{W}(t) = \int_0^t w(s)ds, \tag{42}$$

$$\mathbb{W}_n(t) = \int_0^t \widetilde{w}_n(s)ds. \tag{43}$$

The proof of the following corollary is essentially the same as that in [26]. For the sake of convenience we give the proof.

Corollary 4.4. *Define*

$$\widetilde{\mathbb{W}}_n(t) = \begin{cases} h_n\kappa_0^n \;\; for \quad t = 0, \\ h_n\left(\kappa_0^n + \ldots + \kappa_{j-1}^n\right) \;\; in\; \widetilde{I}_j^n, \end{cases} \tag{44}$$

then

$$\lim_{n_k \to \infty} \widetilde{\mathbb{W}}_{n_k} = \mathbb{W} \quad in \;\; L_2(I, L_2(G)). \tag{45}$$

Proof. First, from (41) we immediately obtain

$$\lim_{n_k\to\infty} \mathbb{W}_{n_k} = \mathbb{W} \quad \text{in } \ L_2(I, L_2(G)). \tag{46}$$

From (43), (44), we get for $t \in \widetilde{I}_j^n$:

$$\begin{aligned} \mathbb{W}_n(t) - \widetilde{\mathbb{W}}_n(t) &= \int_0^{h_n} \widetilde{w}_n(s)\mathrm{d}s + \ldots + \int_{(j-2)h_n}^{(j-1)h_n} \widetilde{w}_n(s)\mathrm{d}s + \\ &+ \int_{(j-1)h_n}^{t} \widetilde{w}_n(s)\mathrm{d}s - \widetilde{\mathbb{W}}_n(t) = h_n\left(\kappa_0^n + \ldots + \kappa_{j-2}^n\right) + \\ &+ \left[t-(j-1)h_n\right]\kappa_{j-1}^n - h_n\left(\kappa_0^n + \ldots + \kappa_{j-1}^n\right) = (t - jh_n)\,\kappa_{j-1}^n. \end{aligned}$$

Hence, for $t \in \widetilde{I}_j^n$ it holds that

$$\left\|\mathbb{W}_n(t) - \widetilde{\mathbb{W}}_n(t)\right\| \leqslant h_n \left\|\kappa_{j-1}^n\right\|.$$

Combining (46), (26), we complete the proof of Corollary 3.4. □

We're now prepared to demonstrate in which senses the function $y(t)$ satisfies the given equation (1). In view of (24), we can rewrite the integral identities (11) for the division d_n in the following way:

$$\begin{aligned} &\Gamma(2-\alpha)b_\zeta(z_j^n, v) + \left(\sum_{k=0}^{j-1} Z_{j-k}^n \sigma_k, v\right) \\ &= \Gamma(2-\alpha)\left(h_n\left(\kappa_0^n + \kappa_1^n + \ldots \kappa_{j-1}^n\right) + \Gamma(2-\alpha)f, v\right) \quad \forall v \in \vartheta, \quad 0<\alpha<1. \end{aligned} \tag{47}$$

Let $v(t)$ be a fixed abstract function from $L_2(I, \vartheta)$. Using (30), (38) and (44) and defining the abstract function $f(t)$ from I into $L_2(G)$ by $f(t) = f \quad \forall t \in I$, we transform (47) to the form

$$\Gamma(2-\alpha)b_\zeta(\widetilde{y}_n, v) + \frac{1}{\Gamma(\alpha+1)}\left(\sum_{l=0}^{j-1}\sigma_l \mathbb{Y}_n^\alpha, v\right) = \Gamma(2-\alpha)\left(\widetilde{\mathbb{W}}_n, v\right) + \Gamma(2-\alpha)\,(f, v) \quad \text{for a.a. t} \in I. \tag{48}$$

Taking into consideration the indices n_k from (27) only and integrating (48) with n replaced by n_k between the limits $t = 0$ and $t = T$, we get

$$\begin{aligned} &\Gamma(2-\alpha)\int_0^T b_\zeta(\widetilde{y}_{n_k}, v)dt + \frac{1}{\Gamma(\alpha+1)}\sum_{l=0}^{j-1}\sigma_l\int_0^T \left(\mathbb{Y}_{n_k}^\alpha, v\right)dt \\ &= \Gamma(2-\alpha)\int_0^T \left(\widetilde{\mathbb{W}}_{n_k}, v\right)dt + \Gamma(2-\alpha)\int_0^T (f, v)\,dt. \end{aligned} \tag{49}$$

Hence,

$$v \in L_2(I, \vartheta) \Rightarrow v \in L_2(I, L_2(G)).$$

Thus, (29), (31) imply

$$\lim_{n_k\to\infty}\frac{1}{\Gamma(\alpha+1)}\int_0^T\left(\sum_{l=0}^{j-1}\sigma_l\mathbb{Y}_{n_k}^\alpha, v\right)dt = \frac{1}{\Gamma(\alpha+1)}\sum_{l=0}^{j-1}\sigma_l\int_0^T(\partial_{0t}^\alpha y, v)\,dt.$$

Similarly, by (45), one obtains

$$\lim_{n_k\to\infty}\Gamma(2-\alpha)\int_0^T\left(\widetilde{\mathbb{W}}_{n_k}, v\right)dt = \Gamma(2-\alpha)\int_0^T(\mathbb{W}, v)\,dt,$$

where $\mathbb{W}(t) = \displaystyle\int_0^t \kappa(x, s, y(x, s))ds$, by (39), (42). Finally, for $v \in L_2(I, \vartheta)$ fixed,

$$\int_0^T b_\zeta(y, v)dt$$

is a bounded linear functional in $L_2(I, \vartheta)$. Thus,

$$\lim_{n_k\to\infty}\Gamma(2-\alpha)\int_0^T b_\zeta(\widetilde{y}_{n_k}, v)dt = \Gamma(2-\alpha)\int_0^T b_\zeta(y, v)dt$$

holds by (37). Consequently, for $n_k \to \infty$, (49) gives one

$$\int_0^T b_\zeta(y, v)dt + \frac{1}{\Gamma(\alpha+1)}\sum_{l=0}^{j-1}\sigma_l\int_0^T(\partial_{0t}^\alpha y, v)\,dt = \Gamma(2-\alpha)\int_0^T(\mathbb{W}, v)\,dt + \Gamma(2-\alpha)\int_0^T(f, v)\,dt. \tag{50}$$

Due to the fact that function $v \in L_2(I,\vartheta)$ is chosen arbitrarily, (50) holds for every $v \in L_2(I,\vartheta)$. In this weak sense, equation (1) is fulfilled. Thus, the function $y(t)$ satisfies

$$y \in L_2(I,\vartheta) \cap AC\left(I, L_2(G)\right), \tag{51}$$

$$\partial_{0t}^{\alpha} y(t) \in L_2\left(I, L_2(G)\right), \tag{52}$$

$$y(0) = 0 \;\text{ in }\; C\left(I, L_2(G)\right), \tag{53}$$

$$\begin{aligned}&\int_0^T b_\zeta(y,v)dt + \frac{1}{\Gamma(\alpha+1)}\sum_{l=0}^{j-1}\sigma_l\int_0^T \left(\partial_{0t}^{\alpha} y, v\right)dt\\ &= \Gamma(2-\alpha)\int_0^T (\mathbb{W},v)\,dt + \Gamma(2-\alpha)\int_0^T (f,v)\,dt \quad \forall v \in L_2(I,\vartheta).\end{aligned} \tag{54}$$

Definition 4. *Function $y(t)$ satisfying* (51) – (54) *is called weak solution to our problem.*

5. Uniqueness

To establish uniqueness, we assume the existence of two distinct weak solutions y_1 and y_2 and come to a contradiction. Denote their difference by

$$y^* = y_1 - y_2.$$

Therefore, to prove the uniqueness, it suffices to show that $y^* = 0$.

By (51) – (54), $y^*(t)$ satisfies

$$\begin{aligned}&y^* \in L_2(I,\vartheta),\\ &y^* \in AC\left(I, L_2(G)\right),\\ &\partial_{0,t}^{\alpha} y^* \in L_2\left(I, L_2(G)\right),\\ &y^*(0) = 0 \;\text{ in }\; C\left(I, L_2(G)\right),\end{aligned}$$

$$\begin{aligned}&\Gamma(2-\alpha)\int_0^T b_\zeta(y^*,v)\mathrm{d}t + \int_0^T \left(\sum_{l=1}^{j-1}\sigma_l \partial_{0t}^{\alpha} y^*, v\right)\mathrm{d}t\\ &= \Gamma(2-\alpha)\int_0^T \left(\int_0^t \left[\kappa\left(x,s,y_2(s)\right) - \kappa\left(x,s,y_1(s)\right)\right]\mathrm{d}s, v\right)\mathrm{d}t \quad \forall v \in L_2(I,\vartheta).\end{aligned} \tag{55}$$

According to (2), we have

$$\begin{aligned}&\left\|\int_0^t \left[\kappa\left(x,s,y_2(s)\right) - \kappa\left(x,s,y_1(s)\right)\right]\right\|\mathrm{d}s\\ &\leqslant \int_0^t \left\|\kappa\left(x,s,y_2(s)\right) - \kappa\left(x,s,y_1(s)\right)\right\|\mathrm{d}s\\ &\leqslant \int_0^t \frac{C}{h^{\alpha-1}}\left\|y_2(s) - y_1(s)\right\|\mathrm{d}s = \frac{C}{h^{\alpha-1}}\int_0^t \|y^*(s)\|\mathrm{d}s \leqslant C\int_0^t \|y^*(s)\|\mathrm{d}s.\end{aligned} \tag{56}$$

Let us divide the interval I into a finite number of subintervals of lengths l. The function $y(t)$ belongs to $C\left(I, L_2(G)\right)$ hence the function $\|y(t)\|$ is continuous in the interval $[0,l]$. Consequently, $\|y(t)\|$ attains its maximum on this interval at a certain point $t_1 \in [0,l]$. Let

$$\max_{t\in[0,l]} \|y^*(t)\| = \left\|y^*\left(t_1\right)\right\|. \tag{57}$$

For $v(t)$ in (55), let us choose the function

$$v\left(t\right) = \begin{cases} y^*\left(t\right) \;\text{ for }\; t \in [0,t_1],\\ 0 \quad \text{for }\; t \in (t_1,T].\end{cases}$$

Thus, we obtain

$$\begin{aligned}&\Gamma(2-\alpha)\int_0^{t_1} b_\zeta(y^*,y^*)\mathrm{d}t + \sum_{l=1}^{j-1}\sigma_l\int_0^T \left(y^*, \partial_{0t}^{\alpha} y^*\right)\mathrm{d}t\\ &= \Gamma(2-\alpha)\int_0^{t_1}\left(y^*, \int_0^t \left[\kappa\left(x,s,y_2(s)\right) - \kappa\left(x,s,y_1(s)\right)\right]\mathrm{d}s\right)\mathrm{d}t.\end{aligned} \tag{58}$$

$\vartheta-$ellipticity of the form $b_\zeta(v,y)$ gives us that

$$\Gamma(2-\alpha)\int_0^{t_1} b_\zeta(y^*,y^*)\mathrm{d}t \geqslant 0. \tag{59}$$

Additionally, according to Lemma 2.3, one has

$$\int_0^{t_1}\left(y^*,\sum_{l=1}^{j-1}\sigma_l\partial_{0t}^{\alpha}y\right)\mathrm{d}t \geqslant \frac{\sum_{l=1}^{j-1}\sigma_l}{2}\int_0^{t_1}\partial_{0t}^{\alpha}\|y\|^2 = c_5\partial_{0t}^{\alpha}\|y(t_1)\|^2, \tag{60}$$

where c_5 is a constant. (56) yields

$$\begin{aligned}&\left\|\Gamma(2-\alpha)\int_0^{t_1}\left(y^*,\int_0^t\left[\kappa\left(x,s,y_2(s)\right)-\kappa\left(x,s,y_1(s)\right)\right]\mathrm{d}s\right)\mathrm{d}t\right\|\\ &\leqslant \Gamma(2-\alpha)\int_0^{t_1}\|y^*(t_1)\|\left\|\int_0^t\left[\kappa\left(x,s,y_2(s)\right)-\kappa\left(x,s,y_1(s)\right)\right]\right\|\mathrm{d}s\\ &\leqslant \Gamma(2-\alpha)\int_0^l\|y^*(t_1)\|\left(C\int_0^l\|y^*(s)\|\mathrm{d}s.\right)\leqslant Cl^2\|y^*(t_1)\|^2.\end{aligned} \tag{61}$$

Combining (58) – (61) , we can get

$$\partial_{0t}^{\alpha}\|y^*(t_1)\|^2 \leqslant c_6\|y^*(t_1)\|^2,$$

where c_6 is a constant. Using Lemma 2.4, we obtain with regard to (35),

$$\|y^*(t_1)\| = 0.$$

Formula (57) then implies

$$y^*(t) = 0 \ \text{ in } L_2(G) \quad \forall t\in[0,l]. \tag{62}$$

Performing the same consideration in the interval $[l,2l]$ with the function

$$v\left(t\right) = \begin{cases} y^*\left(t\right) \ \text{ for } t\in[0,t_2],\\ 0 \quad \text{for } t\ \in(t_2,T],\end{cases}$$

where t_2 is a point at which $\|y(t)\|$ attains its maximum in the interval $[l,2l]$, and using the just obtained result (62), we obtain that

$$y^*(t) = 0 \quad \text{in } L_2(G) \quad \forall t\in[l,2l].$$

After a finite number of steps we thus come to the conclusion that

$$y^*(t) = 0 \quad \text{in } L_2(G) \quad \forall t\in I.$$

Hence, the uniqueness is proved.

6. Conclusion and future scope

Two objectives have been successfully reached in this study. Firstly, a new function has been constructed using the method of discretization for a time fractional equation. Secondly, the proposed method has been validated by demonstrating the convergence of the α-Rothe sequence to the unique weak solution of our problem. As we've seen, TFDE is closely linked to various nanoscience phenomena, consequently, the authors are optimistic that the results obtained can be extended to a wide range of nanoscience applications, thus offering a promising avenue for further exploration (Non-Fourier conduction, confinement effects, Phonon-Phonon and Phonon-Defect Interactions...).

References

[1] Carpinteri A., Mainardi F. *Fractals and Fractional Calculus in Continuum Mechanics*. Springer, Berlin, 1997.

[2] Tien D. Fractional stochastic differential equations with applications to finance. *Journal of Mathematical Analysis and Applications*, 2013, **397**(1), P. 334–348.

[3] Kenneth S.M., Betram R. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, NewYork, 1993.

[4] West B.J., Grigolini P. *Applications of Fractional Calculus in Physics*. World Scientific, Singapore, 1998.

[5] Vineet K.S., Sunil K., Mukesh K.A., Brajesh K.S. Two-dimensional time fractional-order biological population model and its analytical solution. *Egyptian Journal of Basic and Applied Sciences*, 2014, **1**(1), P. 71–76.

[6] Ahmad H., Hossein J., Pranay G., Vernon Ariyan. Solving Time-Fractional Chemical Engineering Equations. *Thermal Science*, 2020, **24**(1), P. 157–164.

[7] Khan N.A., Asmat A., Amir M. Approximate solution of time-fractional chemical engineering equations: a comparative study. *International Journal of Chemical Reactor Engineering*, 2010, **8**(1). Article A19.

[8] Aman S., Khan I., Ismail Z., Salleh M.Z. Tlili I. A new Caputo time fractional model for heat transfer enhancement of water based graphene nanofluid. *An application to solar energy. Results in Physics*, 2018, **9**, P. 1352–1362.

[9] Aman S., Khan I., Ismail Z., Salleh M.Z. Applications of fractional derivatives to nanofluids: exact and numerical solutions. *Mathematical Modelling of Natural Phenomena*, 2018, **13**(1), P. 1–12.

[10] Hasin F., Ahmad Z., Ali F., Khan N., Khan I. A time fractional model of Brinkman-type nanofluid with ramped wall temperature and concentration. *Advances in Mechanical Engineering*, 2022, **14**(5), 168781322210960.

[11] Baleanu D., Guvenc Z.B., Machado J.A.T. *New Trends in Nanotechnology and Fractional Calculus Applications*. Springer, Netherlands, 2009.

[12] Zada L., Rashid N., Samia S.B. An efficient approach for solution of fractional order differential-difference equations arising in nanotechnology. *Applied Mathematics E-Notes*, 2020, P. 297–307.

[13] Aldea A., Barsan V. *Trends in nanophysics*. Springer, Berlin Heidelberg, 2010.
[14] Lobo R.F., Pinheiro M.J. *Advanced Topics in Contemporary Physics for Engineering: Nanophysics, Plasma Physics, and Electrodynamics*. CRC Press, 2022.
[15] Uchaikin V.V. *Fractional derivatives for physicists and engineers* Berlin: Springer, 2013.
[16] Nazdar A.A., Davood R. Identifying an unknown time-dependent boundary source in time-fractional diffusion equation with a non-local boundary condition. *Journal of Computational and Applied Mathematics*, 2019, **355**, P. 36–50.
[17] Sobirov Z.A., Rakhimov K.U., Ergashov R.E. Green's function method for time-fractional diffusion equation on the star graph with equal bonds. *Nanosystems: Physics, Chemistry, Mathematics*, 2021, **12**(3), P. 271–278.
[18] Bahuguna D., Jaiswal A. Application of Rothe's method to fractional differential equations. *Malaya Journal of Matematik*, 2019, **7**(3), P. 399–407.
[19] Zeng B. Existence for a class of time-fractional evolutionary equations with applications involving weakly continuous operator. *Fractional Calculus and Applied Analysis*, 2023, **26**, P. 172–192.
[20] Raheem A., Bahuguna D. Rothe's method for solving some fractional integral diffusion equation. *Applied Mathematics and Computation*, 2014, **236**, P. 161–168.
[21] Yang Z. A finite difference method for fractional partial differential equation. *Applied Mathematics and Computation*, 2009, **215**(2), P. 524–529.
[22] Du G., Lu C., juang Y. Rothe's method for solving multi-term Caputo-Katugampola fractional delay integral diffusion equations. *Mathematical Methods in the Applied Siences*, 2023, **45**(12), P. 7426–7442.
[23] Lin Y., Xu C. Finite difference/spectral approximations for the time-fractional diffusion equation. *Journal of Computational Physics*, 2007, **225**, P. 1533–1552.
[24] Kilbas A.A., Srivastava H.M., Trujillo J.J. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam. 2006.
[25] Li C., Weihua D. Remarks on fractional derivatives, *Applied Mathematics and Computation*, 2007, **187**(2), P. 777–784.
[26] Rektorys.K. *The Method of Discretization in Time and Partial Differential Equations* D. Reidel Publishing Company, Dordrecht. 1982.
[27] Alikhanov A.A. A priori estimates for solutions of boundary value problems for fractional-order equations. *Differential Equations*, 2010, **46**(5), P. 660–666.

Submitted 29 August 2023; revised 12 January 2024; accepted 14 January 2024

Information about the authors:

Y. Bekakra – Department of Mathematics and Informatics, ICOSI Laboratory, Abbes Laghrour University, Khenchela, 04000, Algeria; ORCID 0000-0001-5672-4348; youcef.bekakra@univ-khenchela.dz

A. Bouziani – Department of Mathematics and Informatics, ICOSI Laboratory, Abbes Laghrour University, Khenchela, 04000, Algeria; Department of Mathematics, L'arbi Ben M'hidi University, Oum El Bouagui, 04000, Algeria; ORCID 0000-0002-1216-8033; aefbouziani1963@gmail.com

Conflict of interest: the authors declare no conflict of interest.